Ten Erdős-style problems on primes, irrationality, and divisibility

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Abstract

This note records ten proposed problems at the interface of Beatty sequences, primes, irrationality, and divisibility. Each section gives a clean problem statement, nearby literature, a comparison with numbered Erdős problems from ErdősProblems.com, and elementary or standard proofs where they are available. The label "Erdős #n" always refers to the numbered item on ErdősProblems.com; the proposed problems below are not claimed to be original Erdős problems. The purpose is to separate what appears to be genuinely new from what is already covered by known theorems or by natural variants of classical questions.

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1 Notation and a short map to Erdős problems

For an irrational number $\alpha > 1$, write

$$\mathcal{B}_\alpha = \{\lfloor n\alpha \rfloor : n \geq 1\}.$$

Let $P^+(m)$ denote the largest prime factor of $m > 1$, with any harmless convention for $m = 1$, and let $\Omega(n)$ be the number of prime factors of n, counted with multiplicity. For $x \in \mathbb{R}$, $\|x\|$ denotes distance to the nearest integer.

The closest numbered Erdős problems are summarized below. The phrase "Erdős #n" is used deliberately; the website itself is cited in the bibliography.

Proposed problem	Closest numbered Erdős problems	Main connection
Large prime factors of $\lfloor \alpha p \rfloor$	Erdős #972	Beatty primes: is $\lfloor \alpha p \rfloor$ prime infinitely often?
Pairwise coprime Beatty subsequences	Erdős #972, Erdős #535	Beatty primes give a lower bound; gcd-pattern restrictions are in the spirit of #535.
Primitive subsets of Beatty sequences	Erdős #164	Primitive sets and divisibility antichains.
Distinct-prime quotient chains in Beatty sequences	Erdős #164, Erdős #675	Divisibility chains, primitive sets, and sparse divisibility constraints.
Uniform divisibility of $\lfloor n\alpha \rfloor$ by small squarefree d	Erdős #675, Erdős #972	Irrational rotations plus divisibility by prescribed moduli.
Bohr-prime quotient-free sets	Erdős #675, Erdős #240	Avoiding divisibility by a thin prime set; comparison with prime-generated semigroups.
Prime reciprocal subset sums modulo one	Erdős #317, Erdős #1, Erdős #47	Unit fractions, signed reciprocal sums, and subset-sum spacing.
Prime Cantor constants	Erdős #69, Erdős #251, Erdős #252	Irrationality of prime/arithmetic-function series.
Prime divisors of continued-fraction denominators	Erdős #251, Erdős #240	Prime divisors appearing in naturally generated integer sequences.
Pairwise non-coprime Beatty families	Erdős #535	Extremal gcd configurations.

2 Large prime factors of prime-indexed Beatty numbers

Problem 1 (Large prime factors of $\lfloor \alpha p \rfloor$)**.** *Let $\alpha > 1$ be irrational. Define*

$$M_\alpha(X) = \max_{\substack{p \le X \\ p \text{ prime}}} P^+(\lfloor \alpha p \rfloor).$$

Is it true that

$$M_\alpha(X) \ge X^{1-o(1)}?$$

Equivalently, does every fixed irrational $\alpha > 1$ have infinitely many primes p for which $\lfloor \alpha p \rfloor$ has a prime factor larger than $p^{1-\varepsilon}$ for every fixed $\varepsilon > 0$?

Comparison with Erdős problems

The direct ancestor is Erdős #972: for irrational $\alpha > 1$, are there infinitely many primes p such that $\lfloor p\alpha \rfloor$ is prime? The large-prime-factor problem above is a deliberate weakening of Erdős #972: if $\lfloor \alpha p \rfloor$ is prime for infinitely many p, then $M_\alpha(X) \gg_\alpha X$ along an unbounded sequence of X.

The same page records Vinogradov's theorem that $\{p\alpha\}$ is uniformly distributed modulo one for irrational α, and hence infinitely many primes occur in the Beatty sequence $\mathcal{B}_\alpha$ itself. This is different from asking whether $\lfloor \alpha p \rfloor$ is prime: in the first question the index is arbitrary and the value is prime; in the second the index is prime and the value is required to have a large prime factor.

Nearby literature

There is now a substantial literature on primes and friable or k-free numbers in Beatty sequences. Banks and Shparlinski proved asymptotic formulae for primes in Beatty sequences of finite type and for prime divisors arising in Beatty sequences [16, 15]. Harman studied Beatty primes in short intervals [17]. Baker and Zhao studied gaps between Beatty primes using Maynard-type ideas [18]. Baker proved asymptotic formulae for smooth numbers in Beatty sequences [19]. Çam Çelik studied k-free values of integer parts $\lfloor \alpha p \rfloor$ [20]. These papers do not seem to settle the large-prime-factor question above, because the variable is constrained to be prime before taking the integer part.

Heuristic

A random integer of size $\asymp X$ has a prime factor exceeding $X^{1-\varepsilon}$ with probability roughly

$$\sum_{X^{1-\varepsilon} < q \le X} \frac{1}{q} \sim \varepsilon.$$

Since there are $\asymp X/\log X$ primes $p \le X$, the heuristic predicts many such p. The obstruction is that the sequence $\lfloor \alpha p \rfloor$ is neither a polynomial sequence nor a linear form in two independent variables; it is a prime-indexed Beatty sequence.

A stronger but probably much harder form

The natural Hardy–Littlewood prediction is

$$\#\{p \le X : \lfloor \alpha p \rfloor \text{ is prime}\} \sim c_\alpha \frac{X}{(\log X)^2},$$

with a positive local constant c_α. This is essentially Erdős #972 with the index constrained to be prime rather than arbitrary.

3 Pairwise coprime subsequences of irrational Beatty sequences

Problem 2 (Pairwise coprime Beatty subsequences)**.** *Let*

$$C_\alpha(X) = \max\{|A| : A \subseteq \mathcal{B}_\alpha \cap [1, X],\ \gcd(a, b) = 1 \textit{ for all } a \neq b \in A\}.$$

Is

$$C_\alpha(X) = \left(\frac{1}{\alpha} + o(1)\right) \pi(X)$$

for every irrational $\alpha > 1$?

Comparison with Erdős problems

The Beatty-prime input is again Erdős #972, but the extremal-gcd aspect is closer to Erdős #535, which asks for large subsets of $[N]$ constrained by pairwise gcd patterns. Problem 2 is a Beatty analogue of the elementary fact that the largest pairwise coprime subset of $[1, X]$ has size $\pi(X)+1$: choose 1 and all primes, and no larger set is possible.

First bounds

Proposition 1. *For every $S \subseteq [1, X]$ with pairwise coprime elements,*

$$|S| \leq \pi(X) + 1.$$

Consequently $C_\alpha(X) \leq \pi(X) + 1$.

Proof. Discard 1 if it occurs. Each remaining element $s > 1$ has at least one prime divisor p_s. Pairwise coprimality implies that the primes p_s are all distinct. Since $p_s \leq s \leq X$, this injects $S \setminus \{1\}$ into the primes up to X. □

Proposition 2. *For every irrational $\alpha > 1$,*

$$C_\alpha(X) \geq \left(\frac{1}{\alpha} + o(1)\right) \pi(X).$$

Proof. A positive integer m belongs to $\mathcal{B}_\alpha$ if and only if

$$\left\{\frac{m}{\alpha}\right\} > 1 - \frac{1}{\alpha}.$$

By Vinogradov's theorem, $\{p/\alpha\}$ is uniformly distributed modulo one as p runs over the primes. Hence

$$\#\{p \leq X : p \text{ prime},\ p \in \mathcal{B}_\alpha\} = \left(\frac{1}{\alpha} + o(1)\right) \pi(X).$$

These primes form a pairwise coprime subset of $\mathcal{B}_\alpha \cap [1, X]$. □

The gap between the lower bound in Proposition 2 and the general upper bound in Proposition 1 is the point of the problem. A positive answer says that composites in $\mathcal{B}_\alpha$ cannot improve the main term beyond the Beatty primes.

Nearby literature

For ordinary intervals, the complementary extremal problem of forbidding many pairwise coprime integers was resolved asymptotically and, for large n, exactly by Ahlswede and Khachatrian [12]. Blinovsky and subsequent authors studied variants with restricted prime divisors [14]. These results do not immediately transfer to Beatty sequences because divisibility by a fixed prime in $\mathcal{B}_\alpha$ is governed by an irrational rotation rather than by a residue class of density exactly $1/p$ with periodic structure.

4 Primitive subsets of irrational Beatty sequences

Problem 3 (Primitive Beatty subsequences)**.** *A set* $A \subseteq \mathbb{N}$ *is primitive if no element of* A *divides another. Define*

$$P_\alpha(X) = \max\{|A| : A \subseteq \mathcal{B}_\alpha \cap [1, X],\ A \text{ primitive}\}.$$

Does the limit

$$\lambda(\alpha) = \lim_{X\to\infty} \frac{P_\alpha(X)}{X}$$

exist? If it exists, determine it.

The immediate construction is

$$A = \mathcal{B}_\alpha \cap (X/2, X],$$

which is primitive and has size

$$\left(\frac{1}{2\alpha} + o(1)\right) X.$$

For sparse Beatty sequences, however, one might be able to add many smaller elements without creating divisibility relations, so the naive value $1/(2\alpha)$ may be false.

Comparison with Erdős problems

The direct ancestor is Erdős #164, which asks whether, among all primitive sets A, the sum

$$\sum_{n\in A} \frac{1}{n \log n}$$

is maximized by the primes. Erdős proved convergence of this sum for primitive sets, and the extremal conjecture has now been resolved. Problem 3 replaces the weighted infinite problem by a finite density problem inside an aperiodic sparse set.

Classical interval case

Proposition 3. *The largest primitive subset of* $[1, X]$ *has size* $\lceil X/2 \rceil$.

Proof. The interval $(X/2, X]$ is primitive, giving the lower bound. For the upper bound, write every integer $m \le X$ uniquely as $m = 2^j u$ with u odd. For each odd $u \le X$, the integers

$$u, 2u, 4u, \ldots, 2^j u \le X$$

form a divisibility chain. These chains partition $[1, X]$. A primitive set contains at most one element from each chain. The number of odd $u \le X$ is $\lceil X/2 \rceil$. □

The proof breaks in $\mathcal{B}_\alpha$ because the intersection of $\mathcal{B}_\alpha$ with the dyadic chains is irregular: many chains may be empty, and a primitive set can potentially use several different chains below $X/2$ while losing few elements above $X/2$.

Nearby literature

Besides the work surrounding Erdős #164, the finite problem belongs to the general theory of divisor posets and maximum antichains. For subsets of random or pseudorandom sets of integers, one expects connections with sparse divisor graphs, but the deterministic Beatty sequence has arithmetic correlations that are not captured by independent random models.

5 Distinct-prime quotient chains in Beatty sequences

Problem 4 (Prime-quotient chains in Beatty sequences)**.** *Let $L_\alpha(X)$ be the largest k for which there exist*

$$b_0 \mid b_1 \mid \cdots \mid b_k, \qquad b_i \in \mathcal{B}_\alpha \cap [1, X],$$

such that each quotient b_i/b_{i-1} is prime and the k quotient primes are distinct. Is

$$L_\alpha(X) = (1 + o(1)) \frac{\log X}{\log \log X}$$

for every irrational $\alpha > 1$?

Comparison with Erdős problems

This is a chain analogue of the primitive-set theme in Erdős #164. It also resembles the divisibility restrictions in Erdős #675, where one studies sets generated or avoided by prescribed divisibility conditions. The new feature is that all vertices of the chain must lie in the aperiodic set $\mathcal{B}_\alpha$.

Universal upper bound

Proposition 4. *For every $\alpha > 1$,*

$$L_\alpha(X) \le (1 + o(1)) \frac{\log X}{\log \log X}.$$

Proof. If $b_i/b_{i-1} = q_i$ with distinct primes q_i, then

$$q_1 q_2 \cdots q_k \le b_k \le X.$$

The least product of k distinct primes is the kth primorial $2 \cdot 3 \cdot 5 \cdots p_k$. By the prime number theorem,

$$\log(2 \cdot 3 \cdot 5 \cdots p_k) = \vartheta(p_k) = (1 + o(1))k \log k.$$

Thus $k \log k \le (1 + o(1)) \log X$, which is equivalent to the claimed upper bound. □

The proposed problem asks whether the trivial primorial obstruction is the only asymptotic obstruction even after imposing membership in $\mathcal{B}_\alpha$.

Possible variants

One may first ask for chains with unrestricted prime quotients, or for chains with squarefree quotients. Requiring distinct primes removes the artificial possibility of long powers of 2 and makes the upper bound match the natural primorial scale.

6 Uniform small-prime divisibility of $\lfloor n\alpha \rfloor$

Problem 5 (Uniform divisibility in Beatty sequences)**.** *For squarefree d, define*

$$A_\alpha(N; d) = \#\{n \le N : d \mid \lfloor n\alpha \rfloor\}.$$

For each fixed d, equidistribution gives

$$A_\alpha(N;d) = \frac{N}{d} + o(N).$$

How large can $Y = Y(N)$ be so that, uniformly for all squarefree $d \le Y$,

$$A_\alpha(N;d) = \frac{N}{d} + o\left(\frac{N}{d}\right)?$$

Comparison with Erdős problems

The fixed-d statement is an elementary rotation-theoretic counterpart of the equidistribution phenomenon cited in Erdős #972. The uniform-in-d problem is closer to the divisibility questions in Erdős #675, where arithmetic structure is tested simultaneously against many moduli or divisibility constraints.

A proof for badly approximable α up to the square-root barrier

Theorem 1. *If α is badly approximable, then for every $\varepsilon > 0$,*

$$A_\alpha(N;d) = \frac{N}{d} + o\left(\frac{N}{d}\right)$$

uniformly for all squarefree $d \le N^{1/2-\varepsilon}$. More quantitatively,

$$A_\alpha(N;d) = \frac{N}{d} + O_\alpha(d \log N).$$

Proof. The condition $d \mid \lfloor n\alpha \rfloor$ is equivalent to $\{n\alpha/d\}$ lying in the interval $[0, 1/d)$, apart from endpoint conventions that affect only $O(1)$ terms. Therefore

$$A_\alpha(N;d) = \#\{n \le N : \{n\alpha/d\} \in [0, 1/d)\}.$$

If α is badly approximable, then there is $c(\alpha) > 0$ such that

$$\left|\alpha - \frac{a}{b}\right| \ge \frac{c(\alpha)}{b^2}$$

for all rationals a/b. It follows that α/d is badly approximable with constant $\gg_\alpha d^{-1}$, and the partial quotients of α/d are $O_\alpha(d)$. The standard discrepancy bound for rotations with bounded partial quotients gives

$$D_N(\{n\alpha/d\}) = O_\alpha(d \log N).$$

Applying this to the interval $[0, 1/d)$ yields

$$A_\alpha(N;d) = \frac{N}{d} + O_\alpha(d \log N).$$

If $d \le N^{1/2-\varepsilon}$, then $d \log N = o(N/d)$, proving the relative asymptotic. □

The exponent $1/2$ is natural for this proof because the discrepancy error is roughly $d \log N$. Improving beyond it would require cancellation beyond ordinary one-dimensional discrepancy of rotations.

Nearby literature

The proof uses standard discrepancy estimates for rotations; see Kuipers–Niederreiter [21] or Drmota–Tichy [22]. The Beatty-prime literature listed in Section 1 often requires much stronger exponential-sum estimates for finite-type irrationals.

7 Bohr-prime quotient-free sets

Problem 6 (Quotient-free sets for Bohr primes)**.** *Fix irrational α and $0 < \eta < 1/2$. Let*

$$\mathcal{P}_{\alpha,\eta} = \{p \text{ prime} : \|\alpha p\| < \eta\}.$$

Let $F_{\alpha,\eta}(N)$ be the maximum size of a set $A \subseteq [1, N]$ such that there are no $a, b \in A$ with

$$b = ap$$

for some $p \in \mathcal{P}_{\alpha,\eta}$. Does the limit

$$\lim_{N\to\infty} \frac{F_{\alpha,\eta}(N)}{N}$$

exist? If so, determine it as a function of η and possibly of the Diophantine type of α.

Comparison with Erdős problems

Erdős #675 studies sets forced or forbidden by divisibility through prescribed families of integers, including squarefree and coprime structures. Erdős #240 concerns multiplicative semigroups generated by an infinite set of primes and asks for gap properties. Problem 6 combines these: the forbidden ratios are primes chosen by the Bohr condition $\|\alpha p\| < \eta$.

First facts

Vinogradov's theorem gives

$$\#\{p \le X : p \in \mathcal{P}_{\alpha,\eta}\} = (2\eta + o(1))\pi(X).$$

So the forbidden prime set has relative density 2η among primes.

Proposition 5. *For every irrational α and $0 < \eta < 1/2$,*

$$F_{\alpha,\eta}(N) \ge \left(\frac{1}{2} + o(1)\right) N.$$

Proof. Let

$$A_N = \{n \le N : \Omega(n) \text{ is even}\}.$$

Multiplication by a prime changes the parity of $\Omega(n)$. Hence A_N contains no pair a, ap for any prime p, and in particular no pair with $p \in \mathcal{P}_{\alpha,\eta}$. Since the Liouville function $\lambda(n) = (-1)^{\Omega(n)}$ has average $o(1)$, the two parity classes of $\Omega(n)$ both have density $1/2$. □

When η is small, only a sparse positive-density subset of primes is forbidden, so one might expect the independent-set density to exceed $1/2$ substantially. Even proving existence of the limit appears nontrivial.

Graph interpretation

The problem is the independence number of the graph on $[1, N]$ with edges $n \sim np$ for $p \in \mathcal{P}_{\alpha,\eta}$. If all primes are forbidden, the graph is bipartite according to the parity of $\Omega(n)$, and the lower bound $1/2$ is sharp. For a positive-density subset of primes, the graph is a subgraph of that bipartite graph, and the extremal density should encode how the selected primes generate components of the divisibility graph.

8 Prime reciprocal subset sums modulo one

Problem 7 (Prime reciprocal covering radius)**.** *Let*

$$\mathcal{S}_N = \left\{ \sum_{p \le N} \frac{\varepsilon_p}{p} \pmod 1 : \varepsilon_p \in \{0, 1\} \right\}.$$

Define the covering radius

$$\Delta(N) = \sup_{\xi \in \mathbb{R}/\mathbb{Z}} \min_{s \in \mathcal{S}_N} \|\xi - s\|.$$

Is

$$\Delta(N) = 2^{-(1+o(1))\pi(N)}?$$

Equivalently, is the trivial cardinality lower bound essentially sharp?

Comparison with Erdős problems

This sits between several numbered Erdős problems. Erdős #317 asks about very small nonzero signed sums of reciprocals $1/k$. Erdős #47 and the surrounding unit-fraction problems ask for arithmetic structure forced by sums of reciprocals. Erdős #1 concerns subset sums and distinctness phenomena. Problem 7 is a metric version: do the $2^{\pi(N)}$ subset sums of reciprocal primes cover the circle almost as well as random points?

Trivial lower bound and distinctness

Proposition 6. *The subset sums in $\mathcal{S}_N$ are distinct modulo one. Consequently*

$$\Delta(N) \ge 2^{-\pi(N)-1}.$$

Proof. Suppose two subset sums are congruent modulo one. Their difference has the form

$$\sum_{p \in T} \frac{\sigma_p}{p} \in \mathbb{Z}, \qquad \sigma_p \in \{-1, 1\},$$

where T is a finite set of primes. If T is nonempty, multiply by $Q = \prod_{p \in T} p$. Reducing modulo any fixed $q \in T$, all terms except the q-term vanish, leaving

$$\sigma_q \frac{Q}{q} \not\equiv 0 \pmod q,$$

because $q \nmid Q/q$. This contradicts integrality. Hence T is empty and the sums are distinct.

There are $2^{\pi(N)}$ distinct points on the circle. Any m points have covering radius at least $1/(2m)$, giving the stated lower bound. ∎

Nearby literature

The problem is related to subset sums modulo q and additive bases in finite groups, where the exact set of denominators plays a decisive role; see for example Deshouillers and Freiman [24]. It is also adjacent to modern work on Erdős unit-fraction problems, beginning with Croot and continuing through the results and references collected under Erdős #47 [23, 2]. However, the present problem seems closer to a pseudorandomness question for the set of reciprocal primes modulo one than to the existence of an exact unit-fraction representation.

9 Prime Cantor constants and irrationality exponents

Problem 8 (Prime Cantor constants)**.** *Let p_n be the nth prime and*

$$Q_n = p_1 p_2 \cdots p_n.$$

For an infinite set $S \subseteq \mathbb{N}$, define

$$x_S = \sum_{n \in S} \frac{1}{Q_n}.$$

Determine the irrationality exponent $\mu(x_S)$ in terms of the gap structure of S. In particular, for

$$x_{\mathbb{N}} = \sum_{n \geq 1} \frac{1}{p_1 p_2 \cdots p_n},$$

is $\mu(x_{\mathbb{N}}) = 2$?

Comparison with Erdős problems

This belongs to the irrationality-series family represented by Erdős #69, Erdős #251, and Erdős #252. Erdős #69 asks about irrationality of a series involving $\omega(n)$; Erdős #251 asks about series involving the primes p_n; Erdős #252 asks about series involving divisor sums. Problem 8 is more structured: the denominators are the prime primorials Q_n, so it is a Cantor-series problem with prime bases.

Irrationality and a first lower bound for μ

Proposition 7. *If S is infinite, then x_S is irrational.*

Proof. Let

$$x_S^{(N)} = \sum_{\substack{n \in S \\ n \leq N}} \frac{1}{Q_n}.$$

Then $Q_N x_S^{(N)}$ is an integer, while

$$0 < Q_N(x_S - x_S^{(N)}) = \sum_{\substack{n \in S \\ n > N}} \frac{Q_N}{Q_n} \to 0$$

along $N \in S$ tending to infinity. If $x_S = a/b$ were rational, the fractional parts of $Q_N a/b$ could take only the values $0, 1/b, \ldots, (b-1)/b$. They cannot be positive and tend to zero. Thus x_S is irrational. □

Proposition 8. *Let* $S = \{s_1 < s_2 < \cdots\}$. *Then*

$$\mu(x_S) \geq \max\left(2, \limsup_{j\to\infty} \frac{\log Q_{s_{j+1}}}{\log Q_{s_j}}\right).$$

Proof. Dirichlet's theorem gives $\mu(x_S) \geq 2$ for every irrational real number. For the second lower bound, approximate x_S by the partial sum

$$r_j = \sum_{i\leq j} \frac{1}{Q_{s_i}} = \frac{a_j}{Q_{s_j}}.$$

The tail satisfies

$$0 < x_S - r_j < \frac{2}{Q_{s_{j+1}}}$$

for all sufficiently large j, because the denominators grow at least geometrically. Thus there are infinitely many rational approximations with denominator at most Q_{s_j} and error $\ll Q_{s_{j+1}}^{-1}$. This gives the stated limsup lower bound. □

Since $\log Q_n = \vartheta(p_n) \sim n \log n$, the lower bound reads roughly

$$\mu(x_S) \geq \max\left(2, \limsup_{j} \frac{s_{j+1}\log s_{j+1}}{s_j \log s_j}\right).$$

Sparse S therefore produces large irrationality exponents, while dense S leaves open the expected generic value 2.

Nearby literature

This is a special Cantor series. General irrationality and irrationality-measure results for Cantor series were developed by Hančl and Tijdeman [25], Marques [27], and related work. The prime primorial denominators give far more arithmetic structure than a general Cantor expansion, but they also make the approximating denominators unusually transparent.

10 Prime divisors of continued-fraction denominators

Problem 9 (Prime support of convergent denominators)**.** *Let*

$$\alpha = [a_0; a_1, a_2, \ldots]$$

be irrational, and let q_n be the denominator of the nth convergent. Define

$$\omega_\alpha(N) = \#\{p \text{ prime} : p \mid q_n \textit{ for some } 1 \leq n \leq N\}.$$

If α has bounded partial quotients, how fast must $\omega_\alpha(N)$ tend to infinity?

Comparison with Erdős problems

This is not an exact restatement of a numbered Erdős problem, but it is close in spirit to Erdős #251, where prime-indexed sequences occur inside irrationality questions, and to Erdős #240, where one studies integers whose prime divisors lie in a prescribed infinite prime set. Here the sequence is generated dynamically by the continued-fraction recurrence

$$q_{n+1} = a_{n+1}q_n + q_{n-1}.$$

The qualitative question has a standard proof

Theorem 2. *If α has bounded partial quotients, then infinitely many primes divide the sequence $(q_n)_{n\geq 1}$. Equivalently,*

$$\omega_\alpha(N) \to \infty.$$

Proof sketch. Suppose only finitely many primes divide all q_n. Let S be this finite set enlarged by all primes at most M, where M bounds the partial quotients a_n. Then q_n and $a_{n+1}q_n$ are S-units up to sign, and the recurrence gives infinitely many solutions to

$$q_{n+1} - q_{n-1} = a_{n+1}q_n.$$

Since a_{n+1} takes only finitely many values, some value a occurs infinitely often, giving infinitely many solutions in S-units to the linear equation

$$X - Y = aZ.$$

After dividing by one variable, this is an S-unit equation of the form

$$x + y = 1$$

inside a finitely generated multiplicative group. The S-unit theorem gives only finitely many nondegenerate solutions. Degenerate solutions are impossible here because consecutive convergent denominators are positive and strictly increasing for large n. This contradiction proves that the prime support is infinite. □

The proof is qualitative and ineffective for the desired lower bounds. It suggests the sharper problem:

$$\text{find an explicit } f_M(N) \to \infty \text{ such that } \omega_\alpha(N) \geq f_M(N)$$

for every irrational α with $a_n \leq M$.

Quadratic irrationals

If α is quadratic irrational, the sequence (q_n) is eventually governed by a linear recurrence. For $\alpha = (1+\sqrt{5})/2$, the denominators are Fibonacci numbers. Primitive-divisor theorems for Lucas and Lehmer sequences, including the theorem of Bilu–Hanrot–Voutier [32], then give much stronger information: new prime divisors must occur regularly. Problem 9 asks for analogous lower bounds without periodicity.

Nearby literature

The qualitative proof uses the finiteness of S-unit equations, in the form developed by Evertse, Schlickewei, Schmidt and others [28, 29]. The quadratic case connects to Zsigmondy-type primitive divisor theorems and Lucas sequence theory [30, 31, 32].

11 Pairwise non-coprime families inside Beatty sequences

Problem 10 (Pairwise non-coprime Beatty values)**.** *Let*

$$I_\alpha(N) = \max\{|A| : A \subseteq [1, N],\ \gcd(\lfloor m\alpha \rfloor, \lfloor n\alpha \rfloor) > 1 \textit{ for all distinct } m, n \in A\}.$$

Is

$$I_\alpha(N) = \left(\frac{1}{2} + o(1)\right) N$$

for every irrational $\alpha > 1$? Are all near-extremal examples essentially obtained by forcing all $\lfloor n\alpha \rfloor$ to be divisible by one fixed prime?

Comparison with Erdős problems

This is the closest new problem to Erdős #535, which asks for large sets under prescribed pairwise gcd constraints. It is also a Beatty analogue of an old extremal question: how large can a subset of $[1, N]$ be if every two elements have gcd greater than 1?

Lower bound

Proposition 9. *For every irrational $\alpha > 1$,*

$$I_\alpha(N) \geq \left(\frac{1}{2} + o(1)\right) N.$$

Proof. Take

$$A = \{n \leq N : 2 \mid \lfloor n\alpha \rfloor\}.$$

For $m, n \in A$, the two Beatty values have common divisor 2. Equidistribution of $n\alpha$ modulo 2 gives $|A| = (1/2 + o(1))N$. □

Ordinary interval analogue

For the ordinary interval $[1, N]$, the extremal construction is all even numbers, of size $\lfloor N/2 \rfloor$. More generally, Ahlswede and Khachatrian studied the maximum size of a subset of $[1, N]$ containing no $k + 1$ pairwise coprime integers and proved that the natural construction, the integers divisible by one of the first k primes, is asymptotically best and eventually exactly best [12]. The case $k = 1$ is precisely the pairwise non-coprime problem.

The Beatty version is harder because divisibility by a fixed prime does not partition the index set into a simple residue class; it is a Sturmian-type set for each prime. Intersections of several such divisibility sets may allow exotic pairwise-intersecting constructions not visible in the interval case.

12 Cross-problem observations and suggested first attacks

Finite type versus arbitrary irrational

Many Beatty prime theorems assume that α has finite Diophantine type. The proposed problems were stated for all irrational $\alpha > 1$ in Erdős style. A sensible first program is therefore:

(i) prove the conjecture for badly approximable α;

(ii) extend to finite-type α;

(iii) identify whether Liouville α create counterexamples or merely technical barriers.

Problem 5 illustrates this division: badly approximable α already give a square-root-range theorem by elementary discrepancy.

Random models

For $\mathcal{B}_\alpha \cap [1, X]$, a crude model is a random subset of $[1, X]$ of density $1/\alpha$, but this model ignores strong rotation structure. It predicts:

- Problem 1: many prime-indexed values have a prime factor $> X^{1-\varepsilon}$;
- Problem 2: Beatty primes should dominate the maximum pairwise coprime family;
- Problem 3: for large α, the primitive density may exceed $1/(2\alpha)$;
- Problem 7: reciprocal-prime subset sums should have near-random covering radius.

The main risk is that Beatty sequences are low-complexity deterministic sequences, not independent random sets.

Problems with proofs already reducing the question

Three of the ten problems have partial answers strong enough to sharpen the open formulation:

(1) Problem 5: for badly approximable α, uniformity up to $N^{1/2-\varepsilon}$ follows from discrepancy.

(2) Problem 9: the qualitative infinitude of prime divisors follows from the S-unit theorem; the remaining question is quantitative.

(3) Problem 8: irrationality is elementary, and sparse S gives explicit lower bounds for the irrationality exponent.

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