A TOPOLOGICAL TWIN OF THE NULL-CORRELATION BUNDLE ON $\mathbf{CP}^5$ WITH NO MOTIVIC LIFT

PRZEMEK CHOJECKI

Abstract. Complex realization

$$[\mathbf{P}^5, BGL_4]_{\mathbb{A}^1} \longrightarrow \mathrm{Vect}_4^{\mathrm{top}}(\mathbf{CP}^5)$$

is not surjective. More precisely, a null-correlation bundle N on $\mathbf{P}^5_{\mathbf{C}}$ has a topological twin $N^\dagger$ with the same total Chern class $1+h^2+h^4$ and the same stable complex K-theory class, but with no motivic lift. Indeed, Opie's enumeration gives exactly two rank-four topological bundles with this Chern class, differing by the nonzero top-cell action of $\pi_{10}BU(4) \cong \mathbf{Z}/2$, whereas we prove that among motivic rank-four classes whose realizations have total Chern class $1+h^2+h^4$ there is exactly one, namely $[N]$. Thus $N^\dagger$ has neither an algebraic nor a holomorphic representative. This gives a negative answer to Question 1 and a counterexample to Conjecture 5 of Asok–Fasel–Hopkins, already for $(X, r) = (\mathbf{P}^5, 4)$ [3, Question 1 and Conjecture 5].

The main structural result is finite-rank oriented cancellation:

$$[\mathbf{P}^5, BSL_4]_{\mathbb{A}^1} \longrightarrow [\mathbf{P}^5, BSL]_{\mathbb{A}^1}$$

is injective. We prove it by combining a precise relative Moore–Postnikov comparison with a projective-source James–Thomas theorem. On $\mathbf{P}^4$ the relevant stabilization fiber is the cokernel of a derivative with values in

$$H^4(\mathbf{P}^4, \mathbf{K}_5^{\mathrm{MW}}) \cong H^4(\mathbf{P}^4, \mathbf{K}_5^{\mathrm{M}}) \cong \mathbf{C}^\times.$$

The Suslin class acts by $4! = 24$, and divisibility of $\mathbf{C}^\times$ makes the derivative surjective. Passage across the last cell of $\mathbf{P}^5$ is controlled by $\pi_{10,5}^{\mathbb{A}^1}(BSL_4)(\mathbf{C})$, which vanishes by combining the unstable sphere calculation of Asok–Bachmann–Hopkins with the complex-realization theorem of Asok–Fasel. An appendix verifies the exact specialization of Opie's topological enumeration. As an application, if $\eta \in [\mathbf{P}^5, BGL_2]_{\mathbb{A}^1}$ is any motivic lift of the Rees rank-two bundle ξ_3 on $\mathbf{CP}^5$, then

$$\eta \oplus [\mathcal{O}^{\oplus 2}] = [\mathcal{O}^{\oplus 4}] \qquad \text{in } [\mathbf{P}^5, BGL_4]_{\mathbb{A}^1}.$$

1. Introduction

Let X be a smooth complex projective variety. Complex realization sends an algebraic vector bundle to its underlying complex topological bundle and defines

$$\mathrm{Vect}_r(X) \longrightarrow \mathrm{Vect}_r^{\mathrm{top}}(X(\mathbf{C})). \tag{1}$$

By GAGA, replacing "algebraic" by "holomorphic" does not change the image when X is projective [20]. The algebraization problem asks which topological bundles lie in the image of (1).

Asok–Fasel–Hopkins ask whether the motivic realization map is surjective for every smooth complex cellular variety and every rank; their Conjecture 5 predicts, more strongly, that every topological vector bundle on such a variety has a unique motivic lift [3, Question 1 and Conjecture 5]. Theorem A gives a negative answer to their question and a counterexample to their conjecture, already for $(X, r) = (\mathbf{P}^5, 4)$.

Projective space is a stringent test. The cycle-class maps

$$CH^i(\mathbf{P}^n) \xrightarrow{\sim} H^{2i}(\mathbf{CP}^n; \mathbf{Z})$$

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are isomorphisms, so every integral topological Chern class on $\mathbf{CP}^n$ is algebraic. Chern classes therefore cannot, by themselves, detect failure of algebraization. There is a finer factorization

$$\mathrm{Vect}_r(\mathbf{P}^n) \longrightarrow [\mathbf{P}^n, BGL_r]_{\mathbb{A}^1} \xrightarrow{\mathrm{Re}_{\mathbf{C}}} \mathrm{Vect}_r^{\mathrm{top}}(\mathbf{CP}^n). \tag{2}$$

We call an element of the middle set a *motivic rank-r bundle class*. If $p\colon X' \to \mathbf{P}^n$ is a Jouanolou device, then p is an $\mathbb{A}^1$-equivalence and X' is affine; affine representability identifies the pullback of a motivic class with an algebraic rank-r vector bundle on X' [13, 4]. We do not assume that finite-rank $\mathbb{A}^1$-homotopy classes on the projective scheme itself classify algebraic bundles up to isomorphism. Only the forward construction is used: an actual algebraic bundle, and an orientation when required, gives a classifying map.

The second arrow of (2) is thus a motivic realization problem, strictly prior to descent from a Jouanolou device. The distinction is central here: our counterexample fails already at this second arrow.

Let V be a six-dimensional complex vector space and let $\omega \in \Lambda^2 V^*$ be nondegenerate. On $\mathbf{P}(V) = \mathbf{P}^5$ the associated linear monad

$$0 \longrightarrow \mathcal{O}(-1) \longrightarrow V \otimes \mathcal{O} \longrightarrow \mathcal{O}(1) \longrightarrow 0 \tag{3}$$

has a rank-four middle cohomology bundle $N = N_\omega$, the null-correlation bundle. Writing $h = c_1(\mathcal{O}(1))$, one has

$$c(N) = 1 + h^2 + h^4.$$

Opie's enumeration theorem, specialized in Appendix A, gives exactly two rank-four complex topological bundles on $\mathbf{CP}^5$ with this total Chern class. We write $N^\dagger$ for the class other than the topological realization N^{top}. The two classes differ by the nonzero top-cell action of $\pi_{10}BU(4) \cong \mathbf{Z}/2$ and have the same stable K-theory class.

The main result is stronger than nonalgebraizability.

Theorem A. *Let N be a null-correlation bundle on $\mathbf{P}^5_{\mathbf{C}}$, and let $\eta \in [\mathbf{P}^5, BGL_4]_{\mathbb{A}^1}$. If the complex realization of η has total Chern class $1 + h^2 + h^4$, then*

$$\eta = [N] \qquad \text{in } [\mathbf{P}^5, BGL_4]_{\mathbb{A}^1}.$$

Consequently, among the two topological rank-four bundles with this Chern class, N^{top} has a unique motivic lift and $N^\dagger$ has none. In particular, $N^\dagger$ is not algebraizable.

Theorem B. *The stabilization map*

$$[\mathbf{P}^5, BSL_4]_{\mathbb{A}^1} \longrightarrow [\mathbf{P}^5, BSL]_{\mathbb{A}^1}$$

is injective.

Theorem B is the structural cancellation theorem underlying Theorem A; it is independent of the final topological enumeration and may be useful in other finite-rank lifting problems on projective space.

Corollary 1.1. *Neither map*

$$[\mathbf{P}^5, BGL_4]_{\mathbb{A}^1} \longrightarrow \mathrm{Vect}_4^{\mathrm{top}}(\mathbf{CP}^5), \qquad \mathrm{Vect}_4(\mathbf{P}^5) \longrightarrow \mathrm{Vect}_4^{\mathrm{top}}(\mathbf{CP}^5)$$

is surjective. Thus not every complex topological vector bundle on $\mathbf{CP}^5$ admits a holomorphic or algebraic structure.

We outline the proof. Stable algebraic and topological K-theory agree on complex projective space:

$$K_0(\mathbf{P}^5) \xrightarrow{\sim} K^0(\mathbf{CP}^5).$$

Hence a motivic lift of either topological class has the same stable BGL-class as N. The split determinant sequence implies that

$$[\mathbf{P}^5, BSL]_{\mathbb{A}^1} \longrightarrow [\mathbf{P}^5, BGL]_{\mathbb{A}^1}$$

is injective. Theorem B then reduces the motivic lifting problem to stable K-theory and supplies the finite-rank uniqueness needed in Theorem A.

The proof of Theorem B has two independent parts. First we prove critical-rank cancellation on $\mathbf{P}^4$. For the stabilization map $BSL_4 \to BSL$, the first nonzero homotopy sheaf of the fiber is $\mathbf{K}_5^{\mathrm{MW}}$ in degree four. We record the complete relative Moore–Postnikov hypotheses, including convergence and the possible fundamental-group twisting, and then prove a projective James–Thomas theorem directly from derived mapping spaces. It identifies a nonempty stabilization fiber over $\xi \in [\mathbf{P}^4, BSL]_{\mathbb{A}^1}$ with

$$\operatorname{coker}\big([\mathbf{P}^4, SL]_{\mathbb{A}^1} \xrightarrow{\Delta(e_5,\xi)} H^4(\mathbf{P}^4, \mathbf{K}_5^{\mathrm{MW}})\big).$$

We derive the identifications

$$[\mathbf{P}^4, \mathbb{A}^5 \setminus 0]_{\mathbb{A}^1} \cong H^4(\mathbf{P}^4, \mathbf{K}_5^{\mathrm{MW}}) \cong H^4(\mathbf{P}^4, \mathbf{K}_5^{\mathrm{M}}) \cong \mathbf{C}^\times$$

from the first Postnikov map, the Rost–Schmid complex, and the projective bundle formula. The universal Suslin class contributes

$$\frac{4!}{2}\,\langle 1, -1\rangle = 12\langle 1, -1\rangle$$

in Milnor–Witt theory; reduction to Milnor K-theory sends $\langle 1, -1\rangle$ to 2, and therefore gives multiplication by $4! = 24$. Since $\mathbf{C}^\times$ is 24-divisible, the derivative is surjective and stabilization is injective on $\mathbf{P}^4$.

Second, the pointed motivic cofiber sequence

$$\mathbb{A}^5 \setminus 0 \longrightarrow \mathbf{P}^4 \longrightarrow \mathbf{P}^5$$

attaches one cell with quotient $S^{10,5}$. We prove a mapping-space form of the one-cell action, with basepoints and action conventions made explicit. Its difference group is

$$[S^{10,5}, BSL_4]_{\mathbb{A}^1} = \pi_{10,5}^{\mathbb{A}^1}(BSL_4)(\mathbf{C}).$$

This group vanishes. Indeed, Asok–Bachmann–Hopkins compute the adjacent unstable motivic sphere group as $K_2^M(\mathbf{C})/24 = 0$, while Asok–Fasel identify the remaining target under complex realization with $\pi_9(U(5)) \cong \mathbf{Z}$; naturality makes the image from SL_4 factor through $\pi_9(SU(4)) \cong \mathbf{Z}/2$, hence vanish. The one-cell argument upgrades cancellation from $\mathbf{P}^4$ to $\mathbf{P}^5$.

The topological enumeration is used only in the final step. To make that dependency transparent, Appendix A restates Opie's result in the exact rank and dimension needed here, verifies the Schwarzenberger hypothesis using the actual null-correlation bundle, and explains the top-cell $\mathbf{Z}/2$-action via the Postnikov tower of $BU(4)$.

The Rees bundles provide a useful complementary comparison. Rees constructed nontrivial rank-two bundles ξ_p on $\mathbf{CP}^{2p-1}$ with vanishing Chern classes, and Asok–Fasel–Hopkins proved that every ξ_p has a motivic lift [18, 3]. Thus any obstruction to an algebraic bundle on projective space itself cannot occur at the realization arrow in (2). In Section 10 we show that for $p = 3$ every motivic lift on $\mathbf{P}^5$ becomes trivial after adding two trivial summands in rank four. This sharpens the contrast: $N^\dagger$ has no motivic lift, whereas a Rees class is motivic and the remaining issue is projective descent.

2. The null-correlation bundle and its topological twin

At a point $[x] \in \mathbf{P}(V)$, the two maps in (3) are $1 \mapsto x$ and $v \mapsto \omega(x, v)$. Nondegeneracy of ω gives exactness at the two ends, and the middle cohomology N_ω is locally free of rank four.

Lemma 2.1. *For every nondegenerate alternating form ω on V,*

$$[N_\omega] = 6[\mathcal{O}] - [\mathcal{O}(-1)] - [\mathcal{O}(1)] \quad \textit{in } K_0(\mathbf{P}^5),$$

$$c(N_\omega) = 1 + h^2 + h^4, \qquad \operatorname{ch}(N_\omega) = 4 - h^2 - \frac{1}{12}h^4.$$

Proof. The equality in K_0 follows from the monad. Equivalently, N_ω fits into

$$0 \longrightarrow \mathcal{O}(-1) \longrightarrow \Omega^1_{\mathbf{P}^5}(1) \longrightarrow N_\omega \longrightarrow 0.$$

The Euler sequence gives

$$c(N_\omega) = \frac{1}{(1-h)(1+h)} = \frac{1}{1-h^2} = 1 + h^2 + h^4$$

in $\mathbf{Z}[h]/(h^6)$. The Chern character is $6 - e^{-h} - e^{h}$. □

Proposition 2.2. *There are exactly two isomorphism classes of rank-four complex topological bundles W on $\mathbf{CP}^5$ with*

$$c(W) = 1 + h^2 + h^4.$$

One is N^{top}. The other, denoted $N^\dagger$, is obtained from its classifying map by the nonzero top-cell action of

$$\pi_{10} BU(4) = \pi_9 U(4) \cong \mathbf{Z}/2.$$

The two bundles have the same stable complex K-theory class.

Proof. This is the specialization proved in Appendix A. Stable equality also follows directly from the fact that the two bundles have the same Chern character and $K^0(\mathbf{CP}^5)$ is torsion-free. □

Lemma 2.3. *All null-correlation bundles on $\mathbf{P}^5$ have isomorphic underlying topological bundles.*

Proof. Any two nondegenerate alternating forms on V are carried to one another by an element of $GL(V)$. The corresponding bundles differ by pullback along a projective linear automorphism of $\mathbf{P}(V)$. Since $PGL(V)$ is connected, this automorphism is homotopic to the identity on $\mathbf{CP}^5$. □

Lemma 2.4 (Algebraic and topological K-theory)**.** *For every $n \geq 0$, complex realization induces an isomorphism*

$$K_0(\mathbf{P}^n_{\mathbf{C}}) \xrightarrow{\sim} K^0(\mathbf{CP}^n).$$

It carries $[\mathcal{O}(-1)]$ to the class of the tautological topological line bundle.

Proof. Put $u = 1 - [\mathcal{O}(-1)]$. The algebraic and topological projective bundle theorems give identical presentations

$$K_0(\mathbf{P}^n_{\mathbf{C}}) \cong \mathbf{Z}[u]/(u^{n+1}), \qquad K^0(\mathbf{CP}^n) \cong \mathbf{Z}[u]/(u^{n+1}).$$

Realization respects the tautological line bundle. □

Remark 2.5. An actual rank-four algebraic bundle on $\mathbf{P}^5$ determines an element of $[\mathbf{P}^5, BGL_4]_{\mathbb{A}^1}$, and a determinant trivialization determines a lift to BSL_4. No converse representability statement on the projective source is used. Consequently, the main theorem determines the motivic class and the underlying topological bundle of any algebraic representative, but it does not classify algebraic bundles up to isomorphism.

3. Motivic conventions and stable orientations

We work over $\mathbf{C}$ in the pointed Morel–Voevodsky motivic model category and its $\mathbb{A}^1$-homotopy category [16, 15]. Our sphere convention is

$$S^{p,q} = S_s^{p-q} \wedge \mathbf{G}_m^{\wedge q}, \qquad \pi_{p,q}^{\mathbb{A}^1}(Y)(\mathbf{C}) = [S^{p,q}, Y]_{\mathbb{A}^1,*}.$$

For a smooth scheme X and a pointed target Y, $[X, Y]_{\mathbb{A}^1}$ denotes the unpointed mapping set, equivalently $[X_+, Y]_{\mathbb{A}^1,*}$. When a rational basepoint is chosen we write $[X, Y]_{\mathbb{A}^1,*}$ for pointed classes. All cohomology is Nisnevich cohomology. For a smooth complex variety,

$$\mathrm{cd}_{\mathrm{Nis}}(X) \leq \dim X. \tag{4}$$

We use the standard fiber sequences

$$\mathbb{A}^{n+1} \setminus 0 \longrightarrow BSL_n \longrightarrow BSL_{n+1}, \qquad SL_n \longrightarrow SL_{n+1} \longrightarrow \mathbb{A}^{n+1} \setminus 0, \tag{5}$$

and put

$$BSL = \operatorname*{colim}_n BSL_n, \qquad SL = \operatorname*{colim}_n SL_n \simeq \Omega BSL.$$

The stable spaces BGL and BSL are grouplike homotopy-commutative H-spaces under direct sum. Complex realization sends BGL_n and BSL_n to $BU(n)$ and $BSU(n)$. Stable K-theory is represented by $\mathbf{Z} \times BGL$.

The following elementary point will remove all ambiguity from choices of determinant trivializations.

Lemma 3.1 (Stable orientations)**.** *For every smooth complex variety X, the forgetful map*

$$[X, BSL]_{\mathbb{A}^1} \longrightarrow [X, BGL]_{\mathbb{A}^1}$$

is injective. Thus two oriented stable classes with the same underlying stable BGL-class are equal, independently of the chosen determinant trivializations.

Proof. Apply $[X, -]_{\mathbb{A}^1}$ to the fiber sequence $BSL \to BGL \xrightarrow{\det} B\mathbf{G}_m$. The relevant portion of the exact sequence is

$$[X, GL]_{\mathbb{A}^1} \xrightarrow{\det} [X, \mathbf{G}_m]_{\mathbb{A}^1} \longrightarrow [X, BSL]_{\mathbb{A}^1} \longrightarrow [X, BGL]_{\mathbb{A}^1}.$$

The determinant has a section $u \mapsto \operatorname{diag}(u, 1, 1, \ldots)$, so the first arrow is surjective. The kernel of the last arrow is therefore trivial. $\square$

4. Relative Moore–Postnikov towers for finite-dimensional sources

We first record the precise form of relative obstruction theory used later. The point of including the full statement is that a relative stage need not be principal when the fundamental group acts nontrivially on the homotopy sheaf of the fiber.

Proposition 4.1 (Relative Moore–Postnikov system)**.** *Let k be a perfect field and let*

$$F \longrightarrow E \xrightarrow{q} B \tag{6}$$

be an $\mathbb{A}^1$-homotopy fiber sequence of pointed simplicial presheaves on Sm_k. Assume that F, E, and B are $\mathbb{A}^1$-connected and $\mathbb{A}^1$-fibrant, and put $G = \pi_1^{\mathbb{A}^1}(E)$. Then there are $\mathbb{A}^1$-fibrant spaces E_n, maps

$$E \xrightarrow{i_n} E_n \xrightarrow{p_n} B, \qquad E_n \xrightarrow{q_n} E_{n-1},$$

and the following properties.

(i) *$E_0 \to B$ is an $\mathbb{A}^1$-equivalence, $p_n i_n = q$, $q_n i_n = i_{n-1}$, and $p_{n-1} q_n = p_n$.*

(ii) *For $n \geq 2$, the homotopy fiber of q_n is $K(\pi_n^{\mathbb{A}^1} F, n)$, and i_n induces an isomorphism*

$$\pi_j^{\mathbb{A}^1}(E) \xrightarrow{\sim} \pi_j^{\mathbb{A}^1}(E_n) \qquad (j \leq n).$$

Moreover, there is an exact sequence

$$0 \longrightarrow \pi_{n+1}^{\mathbb{A}^1}(E_n) \xrightarrow{(p_n)_*} \pi_{n+1}^{\mathbb{A}^1}(B) \xrightarrow{\partial} \pi_n^{\mathbb{A}^1}(F), \tag{7}$$

where ∂ is the boundary map of (6).

(iii) *The sheaf $\pi_n^{\mathbb{A}^1}(F)$ carries the monodromy action induced by G. For $n \geq 2$, the stage q_n is classified in the slice over BG by a homotopy pullback square*

$$\begin{tikzcd} E_n \arrow[r] \arrow[d, "q_n"'] & BG \arrow[d] \\ E_{n-1} \arrow[r, "k_{n+1}"] & K_G(\pi_n^{\mathbb{A}^1} F, n+1), \end{tikzcd}$$

where $K_G(-, n+1)$ is the twisted Eilenberg–Mac Lane object associated with the G-module $\pi_n^{\mathbb{A}^1} F$. If the G-action is trivial—in particular, if E is $\mathbb{A}^1$-simply connected—then the stage is a principal $K(\pi_n^{\mathbb{A}^1} F, n)$-fibration and k_{n+1} is an ordinary cohomology class.

(iv) *The tower converges:*

$$E \xrightarrow{\sim} \operatorname{holim}_n E_n \tag{8}$$

is an $\mathbb{A}^1$-equivalence.

(v) *For $n \geq 2$, let X be smooth and let $a\colon X \to E_{n-1}$ be a map. Pullback of the G-module along $X \to E_{n-1} \to BG$ gives a Nisnevich local coefficient sheaf $(\pi_n^{\mathbb{A}^1} F)_a$ on X. The obstruction to lifting a through q_n lies in*

$$H^{n+1}_{\mathrm{Nis}}(X, (\pi_n^{\mathbb{A}^1} F)_a).$$

It vanishes exactly when a lift exists; when it vanishes, the set of homotopy classes of lifts is a torsor under $H^n_{\mathrm{Nis}}(X, (\pi_n^{\mathbb{A}^1} F)_a)$. In the principal case these are the usual untwisted cohomology groups.

Proof. Parts (i)–(iv), including the construction in the slice over BG, are the relative Moore–Postnikov theorem in the motivic model category [9, Proposition 3.4 and Corollary 3.5]. We spell out the obstruction statement. After pulling the homotopy-cartesian square in (iii) back along a, a lift of a is a null-homotopy of the pulled-back k-invariant. Its homotopy class is the obstruction in H^{n+1} with the indicated local coefficients. If one null-homotopy is chosen, all others are obtained by concatenation with a loop in the twisted Eilenberg–Mac Lane mapping space. Its connected components form H^n; hence the lift set is a torsor. When the action is trivial, $K_G(M, n+1) = BG \times K(M, n+1)$ over BG, so the same construction is the ordinary principal Eilenberg–Mac Lane fibration. □

Corollary 4.2 (Finite-dimensional comparison). *In the situation of Proposition 4.1, let $d \geq 1$ and suppose*

$$\pi_i^{\mathbb{A}^1}(F) = 0 \qquad (0 \leq i \leq d).$$

If X is smooth and $\mathrm{cd}_{\mathrm{Nis}}(X) \leq d$, *then*

$$q_*\colon [X, E]_{\mathbb{A}^1} \xrightarrow{\sim} [X, B]_{\mathbb{A}^1}$$

is a bijection. The conclusion remains valid with twisted stages; no principality assumption is required.

Proof. The first possibly nontrivial stage has index $n \geq d+1$. For every map from X to E_{n-1}, the obstruction and ambiguity groups of Proposition 4.1(v) occur in degrees $n+1$ and n, both greater than d. They vanish by the cohomological-dimension hypothesis, for ordinary or twisted Nisnevich coefficient sheaves. Inductively, every map to $B \simeq E_0$ lifts uniquely on connected components through every finite stage.

For completeness, apply $\mathrm{RMap}(X, -)$ to the tower. Over a chosen lift, a stage has mapping-space fiber the section space of the pulled-back $K(M, n)$-local system, and its homotopy groups are

$$\pi_j \cong H^{n-j}_{\mathrm{Nis}}(X, M_a).$$

In the principal case this is simply $\mathrm{RMap}(X, K(M, n))$. They vanish for $j < n - d$, so the connectivities of the mapping-space fibers tend to infinity. Together with the convergence (8), this identifies $\pi_0 \,\mathrm{RMap}(X, E)$ with the inverse limit of the singleton lift sets. Hence existence and uniqueness persist at the homotopy limit. □

Remark 4.3. In all applications below the total space of the relevant relative tower is $\mathbb{A}^1$-simply connected. Thus the coefficient systems are untwisted and every stage is principal. Proposition 4.1 records the twisting only to make clear why this simplification is legitimate rather than implicit.

5. A projective James–Thomas theorem

The next theorem is a mapping-space statement and has no affineness hypothesis. It is the formal mechanism behind the James–Thomas enumeration of unstable lifts [12].

Theorem 5.1 (James–Thomas formula for a smooth source). *Let B be a pointed $\mathbb{A}^1$-local grouplike H-space, let M be a strictly $\mathbb{A}^1$-invariant sheaf of abelian groups, and let*

$$\theta\colon B \longrightarrow K(M, n+1)$$

be a pointed map. Put $E = \mathrm{hofib}(\theta)$ and let $q\colon E \to B$. Let X be a smooth scheme and let $\xi \in [X, B]_{\mathbb{A}^1}$ satisfy $\theta_*(\xi) = 0$. After choosing one lift of ξ to E, there is a natural bijection

$$q_*^{-1}(\xi) \cong \mathrm{coker}\Big([X, \Omega B]_{\mathbb{A}^1} \xrightarrow{\Delta(\theta,\xi)} H^n(X, M)\Big). \tag{9}$$

Here $H^n(X, M)$ acts on the lift set by concatenating the chosen null-homotopy of $\theta\xi$ with a loop, and $\Delta(\theta, \xi)$ is the stabilizer homomorphism induced by loops in the component of ξ.

Proof. Apply $\mathrm{RMap}(X, -)$ to the homotopy fiber sequence $E \to B \xrightarrow{\theta} K(M, n+1)$. At the component ξ and the zero component of the Eilenberg–Mac Lane mapping space, the homotopy exact sequence contains

$$\pi_1 \mathrm{RMap}(X, B, \xi) \longrightarrow H^n(X, M) \longrightarrow \pi_0 \mathrm{RMap}(X, E) \longrightarrow [X, B]_{\mathbb{A}^1}. \tag{10}$$

The middle arrow changes a chosen null-homotopy of $\theta\xi$ by a loop in $\mathrm{RMap}(X, K(M, n+1), 0)$. Exactness says that this action is transitive on the fiber over ξ and that its stabilizer is the image of the first arrow.

Translation by ξ in the grouplike H-space B identifies the component $\mathrm{RMap}(X, B, \xi)$ with the base component. Looping and adjunction then give

$$\pi_1 \mathrm{RMap}(X, B, \xi) \cong [X, \Omega B]_{\mathbb{A}^1}.$$

Under this identification the first arrow in (10) is, by definition, $\Delta(\theta, \xi)$. Since $H^n(X, M)$ is abelian, its orbit set is the cokernel in (9). □

The derivative has a concrete characteristic-class formula. Let $m\colon B \times B \to B$ be the H-space multiplication. Translation by ξ gives a map $\Omega B \to \Omega K(M, n+1) = K(M, n)$; evaluating on X is $\Delta(\theta, \xi)$. For stable oriented bundles, reduction of the Euler class gives the following formula.

Proposition 5.2 (Chern-class derivative)**.** *Let*

$$e_5\colon BSL \longrightarrow K(\mathbf{K}_5^{\mathrm{MW}}, 5)$$

be normalized so that its reduction under $\tau\colon \mathbf{K}_5^{\mathrm{MW}} \to \mathbf{K}_5^{\mathrm{M}}$ is $\pm c_5$. For every smooth X, every $\xi \in [X, BSL]_{\mathbb{A}^1}$ with $e_5(\xi) = 0$, and every $\beta \in [X, SL]_{\mathbb{A}^1}$,

$$\tau\big(\Delta(e_5, \xi)(\beta)\big) = \pm\Delta(c_5, \xi)(\beta), \tag{11}$$

where

$$\Delta(c_5, \xi)(\beta) = (\Omega c_5)(\beta) + \sum_{r=1}^{4} (\Omega c_r)(\beta) c_{5-r}(\xi). \tag{12}$$

Proof. Naturality of the mapping-space boundary under $K(\mathbf{K}_5^{\mathrm{MW}}, 5) \to K(\mathbf{K}_5^{\mathrm{M}}, 5)$ gives (11). The Whitney formula

$$m^* c_5 = \sum_{r=0}^{5} c_r \times c_{5-r}$$

and looping in the first variable give (12). This is the formal calculation of [9, Proposition 6.7]; the proof uses only the H-space law, derived mapping spaces, and the Whitney identity. □

5.1. **The rank-four projective formula.** Set

$$BSL = \operatorname*{colim}_n BSL_n, \qquad SL = \operatorname*{colim}_n SL_n \simeq \Omega BSL,$$

and let

$$\varphi_4\colon BSL_4 \longrightarrow BSL$$

be stabilization. Write $F_4 = \mathrm{hofib}(\varphi_4)$. The unstable stability calculation gives

$$\pi_j^{\mathbb{A}^1}(F_4) = 0 \quad (j < 4), \qquad \pi_4^{\mathbb{A}^1}(F_4) \cong \mathbf{K}_5^{\mathrm{MW}}; \tag{13}$$

see [9, Sections 4 and 6]. Let

$$BSL_4 \xrightarrow{i_4} \mathcal{E}_4 \xrightarrow{p_4} BSL \tag{14}$$

be the first nontrivial relative Moore–Postnikov stage. Thus p_4 has fiber $K(\mathbf{K}_5^{\mathrm{MW}}, 4)$ and is classified by

$$e_5 \in H^5(BSL, \mathbf{K}_5^{\mathrm{MW}}).$$

Choose the identification in (13) so that reduction $\tau\colon \mathbf{K}_5^{\mathrm{MW}} \to \mathbf{K}_5^{\mathrm{M}}$ sends e_5 to $\pm c_5$. This is the universal Euler class calculation in [9, Section 6]. The sign and the choice of a unit in $GW(\mathbf{C})$ will be immaterial.

The following degree-five surjectivity is essential at the critical cohomological dimension.

Lemma 5.3 (Comparison with the first stage)**.** *Let $\mathcal{H}_4 = \mathrm{hofib}(i_4)$. Then*

$$\pi_j^{\mathbb{A}^1}(\mathcal{H}_4) = 0 \qquad (j \leq 4).$$

Consequently, if X is smooth, $\mathbb{A}^1$-connected, and $\mathrm{cd}_{\mathrm{Nis}}(X) \leq 4$, then

$$[X, BSL_4]_{\mathbb{A}^1} \xrightarrow{\sim} [X, \mathcal{E}_4]_{\mathbb{A}^1}.$$

Proof. The relative Moore–Postnikov construction gives isomorphisms

$$\pi_j^{\mathbb{A}^1}(BSL_4) \xrightarrow{\sim} \pi_j^{\mathbb{A}^1}(\mathcal{E}_4) \qquad (j \leq 4).$$

To deduce vanishing of the homotopy fiber in degree four, one must also prove surjectivity in degree five. The edge exact sequence (7), applied to the first stage, gives

$$0 \longrightarrow \pi_5^{\mathbb{A}^1}(\mathcal{E}_4) \xrightarrow{(p_4)_*} \pi_5^{\mathbb{A}^1}(BSL) \xrightarrow{\partial} \pi_4^{\mathbb{A}^1}(F_4). \tag{15}$$

The long exact sequence of $F_4 \to BSL_4 \xrightarrow{\varphi_4} BSL$ gives

$$\mathrm{im}\big(\pi_5^{\mathbb{A}^1}(BSL_4) \to \pi_5^{\mathbb{A}^1}(BSL)\big) = \ker(\partial).$$

Equation (15) identifies $\pi_5^{\mathbb{A}^1}(\mathcal{E}_4)$ with the same kernel. Since $p_4 i_4 = \varphi_4$, the map

$$\pi_5^{\mathbb{A}^1}(BSL_4) \longrightarrow \pi_5^{\mathbb{A}^1}(\mathcal{E}_4)$$

is surjective. The long exact sequence of $\mathcal{H}_4 \to BSL_4 \to \mathcal{E}_4$ now yields $\pi_j^{\mathbb{A}^1}(\mathcal{H}_4) = 0$ for all $j \leq 4$.

Both BSL_4 and $\mathcal{E}_4$ are $\mathbb{A}^1$-simply connected, so the remaining relative stages are principal. The final assertion follows from Corollary 4.2. □

We now give the projective-source James–Thomas formula. The proof is included to separate the argument from the affine hypotheses under which the corresponding bundle-enumeration theorem in [9] is stated.

Theorem 5.4 (Projective James–Thomas formula)**.** *Let X be a smooth complex variety with $\mathrm{cd}_{\mathrm{Nis}}(X) \leq 4$, and let $\xi \in [X, BSL]_{\mathbb{A}^1}$ admit a lift to BSL_4. After choosing one rank-four lift, the stabilization fiber is noncanonically identified with*

$$\varphi_{4,*}^{-1}(\xi) \cong \mathrm{coker}\Big([X, SL]_{\mathbb{A}^1} \xrightarrow{\Delta(e_5,\xi)} H^4(X, \mathbf{K}_5^{\mathrm{MW}})\Big). \tag{16}$$

Under reduction of coefficients,

$$\tau\big(\Delta(e_5, \xi)(\beta)\big) = \pm\Delta(c_5, \xi)(\beta), \tag{17}$$

where

$$\Delta(c_5, \xi)(\beta) = (\Omega c_5)(\beta) + \sum_{r=1}^{4} (\Omega c_r)(\beta) c_{5-r}(\xi). \tag{18}$$

Proof. The first stage is the homotopy fiber of the classifying map

$$e_5\colon BSL \longrightarrow K(\mathbf{K}_5^{\mathrm{MW}}, 5).$$

Applying $\mathrm{RMap}(X, -)$ gives a homotopy fiber sequence

$$\mathrm{RMap}(X, \mathcal{E}_4) \longrightarrow \mathrm{RMap}(X, BSL) \longrightarrow \mathrm{RMap}(X, K(\mathbf{K}_5^{\mathrm{MW}}, 5)).$$

Because ξ lifts, $e_5(\xi) = 0$. The homotopy exact sequence at the component ξ contains

$$\pi_1 \operatorname{RMap}(X, BSL, \xi) \longrightarrow H^4(X, \mathbf{K}_5^{\mathrm{MW}}) \longrightarrow \pi_0 \operatorname{RMap}(X, \mathcal{E}_4) \longrightarrow [X, BSL]_{\mathbb{A}^1}. \tag{19}$$

Stable direct sum makes BSL a grouplike, homotopy-commutative H-space [9, Corollary 5.6 and Section 6]. Translation by ξ identifies its mapping space component with the base component, and adjunction gives

$$\pi_1 \operatorname{RMap}(X, BSL, \xi) \cong \pi_1 \operatorname{RMap}(X, BSL, 0) \cong [X, \Omega BSL]_{\mathbb{A}^1} = [X, SL]_{\mathbb{A}^1}.$$

Choose one point of the lifting space over ξ. Exactness of (19) identifies the components of that lifting space with the cokernel of its first arrow. Lemma 5.3 identifies these components with the actual stabilization fiber for BSL_4, proving (16). Notice that no affine representability statement enters this mapping-space calculation.

It remains to identify the boundary. Let $m\colon BSL \times BSL \to BSL$ be stable direct sum. Looping in the first variable the difference between $e_5 m$ and the value at ξ gives the homomorphism $\Delta(e_5, \xi)$. Naturality for $\tau\colon \mathbf{K}_5^{\mathrm{MW}} \to \mathbf{K}_5^{\mathrm{M}}$ and $\tau(e_5) = \pm c_5$ gives (17). Finally, the Whitney formula

$$m^* c_5 = \sum_{r=0}^{5} c_r \times c_{5-r}$$

and looping in the first factor yield (18). This is the formal calculation of [9, Proposition 6.7]; the argument above shows directly that it applies to the present projective source. □

6. The coefficient group and the Suslin factor

We next compute the target in Theorem 5.4 and the universal class that acts on it.

Proposition 6.1 (The complete coefficient calculation)**.** *The first Postnikov map of $\mathbb{A}^5 \setminus 0$ and reduction of coefficients induce canonical bijections and isomorphisms*

$$[\mathbf{P}^4, \mathbb{A}^5 \setminus 0]_{\mathbb{A}^1} \xrightarrow{\sim} H^4(\mathbf{P}^4, \mathbf{K}_5^{\mathrm{MW}}) \xrightarrow{\sim} H^4(\mathbf{P}^4, \mathbf{K}_5^{\mathrm{M}}) \xrightarrow{\sim} \mathbf{C}^\times. \tag{20}$$

Under the last identification, multiplication by an integer m in cohomology is the m-th power map on $\mathbf{C}^\times$. In particular, this group is 24-divisible.

Proof. We give the three steps separately.

Step 1: the first Postnikov map. Morel's connectivity theorem [15, Chapter 6] gives

$$\pi_j^{\mathbb{A}^1}(\mathbb{A}^5 \setminus 0) = 0 \quad (j < 4), \qquad \pi_4^{\mathbb{A}^1}(\mathbb{A}^5 \setminus 0) \cong \mathbf{K}_5^{\mathrm{MW}}.$$

Let

$$\omega\colon \mathbb{A}^5 \setminus 0 \longrightarrow K(\mathbf{K}_5^{\mathrm{MW}}, 4)$$

be the first Postnikov map. It is an isomorphism on homotopy sheaves through degree four. If R_ω is its homotopy fiber, the long exact sequence shows $\pi_j^{\mathbb{A}^1}(R_\omega) = 0$ for $j < 4$; in degree four one also uses $\pi_5^{\mathbb{A}^1}(K(\mathbf{K}_5^{\mathrm{MW}}, 4)) = 0$, so $\pi_4^{\mathbb{A}^1}(R_\omega) = 0$. Both source and target are $\mathbb{A}^1$-simply connected. Since $\mathrm{cd}_{\mathrm{Nis}}(\mathbf{P}^4) \le 4$, Corollary 4.2 gives

$$[\mathbf{P}^4, \mathbb{A}^5 \setminus 0]_{\mathbb{A}^1} \xrightarrow{\sim} [\mathbf{P}^4, K(\mathbf{K}_5^{\mathrm{MW}}, 4)]_{\mathbb{A}^1} = H^4(\mathbf{P}^4, \mathbf{K}_5^{\mathrm{MW}}).$$

Step 2: removal of the Witt-theoretic term. There is an exact sequence of strictly $\mathbb{A}^1$-invariant sheaves [15, Chapter 3]

$$0 \longrightarrow I^6 \longrightarrow \mathbf{K}_5^{\mathrm{MW}} \xrightarrow{\tau} \mathbf{K}_5^{\mathrm{M}} \longrightarrow 0. \tag{21}$$

The group $H^5(\mathbf{P}^4, I^6)$ vanishes because $\mathrm{cd}_{\mathrm{Nis}}(\mathbf{P}^4) \le 4$. To compute degree four, use the Rost–Schmid complex. Its codimension-four term for I^6 is

$$\bigoplus_{x \in (\mathbf{P}^4)^{(4)}} I^{6-4}(\kappa(x), \omega_{x/\mathbf{P}^4}) = \bigoplus_{x \in (\mathbf{P}^4)^{(4)}} I^2(\mathbf{C}, \omega_{x/\mathbf{P}^4}).$$

Every codimension-four point is closed with residue field $\mathbf{C}$, and $I(\mathbf{C}) = 0$. The displayed term is therefore zero, so $H^4(\mathbf{P}^4, I^6) = 0$. The long exact sequence of (21) now gives

$$H^4(\mathbf{P}^4, \mathbf{K}_5^{\mathrm{MW}}) \xrightarrow{\sim} H^4(\mathbf{P}^4, \mathbf{K}_5^{\mathrm{M}}).$$

This also explains explicitly why no residual I-cohomology survives in the critical degree.

Step 3: the projective bundle formula. The projective bundle formula for Chow groups with coefficients gives

$$H^4(\mathbf{P}^4, \mathbf{K}_5^{\mathrm{M}}) \cong H^0(\operatorname{Spec}\mathbf{C}, \mathbf{K}_1^{\mathrm{M}}) \cdot h^4 \cong K_1^{\mathrm{M}}(\mathbf{C}) = \mathbf{C}^\times$$

[19]. The additive multiple $m\alpha$ in Milnor K-theory corresponds to the m-th power of the representing unit. Since every complex number has a 24-th root, the group is 24-divisible. □

We now make the factorial calculation explicit. Let

$$Q_9 = \operatorname{Spec}\mathbf{C}[x_0, \dots, x_4, y_0, \dots, y_4]/(\textstyle\sum x_i y_i - 1).$$

Projection to the x-coordinates is a Zariski-locally trivial affine-space bundle $Q_9 \to \mathbb{A}^5 \setminus 0$, hence an $\mathbb{A}^1$-equivalence. Suslin's matrix construction [21] gives

$$\beta_5 \colon Q_9 \longrightarrow SL_5.$$

Transporting along the projection and stabilizing defines

$$\overline{\beta}_5 \colon \mathbb{A}^5 \setminus 0 \longrightarrow SL_5 \longrightarrow SL. \tag{22}$$

Let

$$\iota \in H^4(\mathbb{A}^5 \setminus 0, \mathbf{K}_5^{\mathrm{MW}})$$

be the first Postnikov class, and let $\mathfrak{h} = \langle 1, -1\rangle \in K_0^{\mathrm{MW}}(\mathbf{C})$ be the hyperbolic element.

Proposition 6.2 (The Suslin factor 4!)**.** *For the class* (22)*,*

$$\overline{\beta}_5^*(\Omega c_r) = 0 \qquad (1 \le r < 5), \tag{23}$$

and

$$\overline{\beta}_5^*(\Omega c_5) = \pm 24\,\tau(\iota) \quad \textit{in } H^4(\mathbb{A}^5 \setminus 0, \mathbf{K}_5^{\mathrm{M}}). \tag{24}$$

More precisely, before reduction to Milnor K-theory, the first-row map associated with the Suslin matrix acts on the Postnikov class by

$$\frac{4!}{2}\,\mathfrak{h} \cdot \iota = 12\mathfrak{h} \cdot \iota. \tag{25}$$

Since $\tau(\mathfrak{h}) = 2$, reduction of (25) *is multiplication by* $12 \cdot 2 = 24 = 4!$.

Proof. Let $\mathrm{pr}_1 \colon SL_5 \to \mathbb{A}^5 \setminus 0$ be projection to the first row and put $\psi_5 = \mathrm{pr}_1\,\beta_5$. The universal Suslin degree calculation, in the notation of [9, Proposition 6.11 and Remark 6.12], is

$$\psi_5^*(\iota) = \frac{4!}{2}\,\mathfrak{h} \cdot \iota.$$

This is (25). In Milnor–Witt K-theory $\mathfrak{h} = \eta[-1] + 2$, and the reduction τ kills η; hence $\tau(\mathfrak{h}) = 2$. Therefore

$$\tau\big(\psi_5^*(\iota)\big) = 24\,\tau(\iota).$$

The loop of the universal Euler k-invariant is, up to a unit in $GW(\mathbf{C})$, the first Postnikov map composed with pr_1. Reducing the Euler class to c_5 turns that unit into a sign. Thus the preceding identity is exactly (24); this is the calculation written in [9, equations (6.4)–(6.6)]. Finally, (23) is [9, Proposition 6.9]. Combining the two statements makes clear that the coefficient is 24, not merely a nonzero multiple of it. □

Theorem 6.3 (Critical-rank cancellation on $\mathbf{P}^4$)**.** *The stabilization map*

$$[\mathbf{P}^4, BSL_4]_{\mathbb{A}^1} \longrightarrow [\mathbf{P}^4, BSL]_{\mathbb{A}^1}$$

is injective.

Proof. Fix a stable class ξ admitting a rank-four lift. By Theorem 5.4, its stabilization fiber is the cokernel of $\Delta(e_5, \xi)$. Proposition 6.1 and (17) reduce surjectivity to the Milnor derivative

$$\Delta(c_5, \xi)\colon [\mathbf{P}^4, SL]_{\mathbb{A}^1} \longrightarrow H^4(\mathbf{P}^4, \mathbf{K}_5^{\mathrm{M}}).$$

Let γ lie in the target. By 24-divisibility choose δ with $24\delta = \gamma$, and lift δ uniquely to $\widetilde{\delta} \in H^4(\mathbf{P}^4, \mathbf{K}_5^{\mathrm{MW}})$. The first isomorphism in (20) supplies $u\colon \mathbf{P}^4 \to \mathbb{A}^5 \setminus 0$ with $u^*(\iota) = \widetilde{\delta}$. Set

$$\beta = \overline{\beta}_5 \circ u \in [\mathbf{P}^4, SL]_{\mathbb{A}^1}.$$

The lower terms in (18) vanish by (23), and (24) gives

$$\Delta(c_5, \xi)(\beta) = \pm 24\delta = \pm\gamma.$$

Thus the Milnor derivative is surjective. Since $H^4(\mathbf{P}^4, \mathbf{K}_5^{\mathrm{MW}}) \to H^4(\mathbf{P}^4, \mathbf{K}_5^{\mathrm{M}})$ is an isomorphism, $\Delta(e_5, \xi)$ is also surjective. Its cokernel vanishes, so every nonempty stabilization fiber is a singleton. □

7. The motivic top cell

The passage from $\mathbf{P}^4$ to $\mathbf{P}^5$ is governed by the next unstable homotopy group of BSL_4.

Theorem 7.1. *One has*

$$\pi_{10,5}^{\mathbb{A}^1}(BSL_4)(\mathbf{C}) = 0.$$

Proof. Looping identifies the group with $\pi_{9,5}^{\mathbb{A}^1}(SL_4)(\mathbf{C})$. The standard fiber sequence

$$SL_4 \longrightarrow SL_5 \longrightarrow \mathbb{A}^5 \setminus 0 \simeq S^{9,5}$$

gives an exact sequence

$$\pi_{10,5}^{\mathbb{A}^1}(S^{9,5})(\mathbf{C}) \longrightarrow \pi_{9,5}^{\mathbb{A}^1}(SL_4)(\mathbf{C}) \longrightarrow \pi_{9,5}^{\mathbb{A}^1}(SL_5)(\mathbf{C}). \tag{26}$$

Asok–Bachmann–Hopkins prove, in characteristic zero and for $d \geq 4$, that

$$\mathbf{K}_{d+2-j}^{\mathrm{M}}/24 \xrightarrow{\sim} \pi_{d+j,j}^{\mathbb{A}^1}(S^{2d-1,d}) \qquad (j \geq d-3)$$

[1, Theorem 7.2.1]. Taking $d = j = 5$ and evaluating at $\mathbf{C}$ gives

$$\pi_{10,5}^{\mathbb{A}^1}(S^{9,5})(\mathbf{C}) \cong K_2^{\mathrm{M}}(\mathbf{C})/24 = 0. \tag{27}$$

Indeed, $K_2^{\mathrm{M}}(\mathbf{C})$ is generated by symbols and, if $\alpha^{24} = a$, then $\{a, b\} = 24\{\alpha, b\}$. Thus the second arrow in (26) is injective.

We show that this injective map has zero image. Since $\mathbf{G}_m$ is simplicially 0-truncated,

$$\pi_{10,5}^{\mathbb{A}^1}(\mathbf{G}_m) = \pi_{9,5}^{\mathbb{A}^1}(\mathbf{G}_m) = 0.$$

The determinant fiber sequence $SL_5 \to GL_5 \to \mathbf{G}_m$ therefore gives

$$\pi_{9,5}^{\mathbb{A}^1}(SL_5)(\mathbf{C}) \xrightarrow{\sim} \pi_{9,5}^{\mathbb{A}^1}(GL_5)(\mathbf{C}). \tag{28}$$

Asok–Fasel index the same group as $\pi_{4,5}^{\mathbb{A}^1}(GL_5)(\mathbf{C})$, with the first index recording simplicial suspensions. Their complex realization theorem gives an isomorphism

$$\pi_{9,5}^{\mathbb{A}^1}(GL_5)(\mathbf{C}) \xrightarrow{\sim} \pi_9(U(5)) \cong \mathbf{Z} \tag{29}$$

[2, Theorem 5.5]. The last identification is also the stable-range unitary calculation of Bott [7].

Let $x \in \pi_{9,5}^{\mathbb{A}^1}(SL_4)(\mathbf{C})$. By naturality, the realization of its image in SL_5 factors through

$$\pi_9(SU(4)) \longrightarrow \pi_9(SU(5)).$$

The source is $\mathbf{Z}/2$ [14], whereas the target is $\mathbf{Z}$ by Bott periodicity. Hence the homomorphism is zero. Since (29) is injective and (28) is an isomorphism, the motivic image of x in SL_5 is zero. The injection obtained from (26) and (27) forces $x = 0$. □

8. The single projective cell and cancellation on $\mathbf{P}^5$

We make the relative extension argument, including its basepoint and action conventions, explicit. The statement is model-categorical and will then be applied in the pointed motivic model category.

Proposition 8.1 (Extension space for one attached cell)**.** *Let $\mathcal{M}$ be a pointed simplicial model category. Suppose that A, K, and X are cofibrant, Y is fibrant, and X is represented by a homotopy pushout*

$$\begin{array}{ccc} K & \xrightarrow{u} & A \\ \downarrow & & \downarrow \\ CK & \longrightarrow & X, \end{array} \tag{30}$$

where CK is a pointed cone on K. Let

$$\rho\colon \operatorname{RMap}_*(X,Y) \longrightarrow \operatorname{RMap}_*(A,Y)$$

be restriction.

Fix $a\colon A \to Y$. The homotopy fiber $\mathcal{L}_a$ of ρ over a is the space of nullhomotopies of $a \circ u$ in $\operatorname{RMap}_*(K,Y)$. If $\mathcal{L}_a$ is nonempty and H_0 is one chosen nullhomotopy, concatenation with loops gives a free and transitive action

$$\pi_1 \operatorname{RMap}_*(K,Y,a\circ u) \curvearrowright \pi_0(\mathcal{L}_a). \tag{31}$$

After transporting the basepoint along H_0, this acting group is

$$\pi_1 \operatorname{RMap}_*(K,Y,*) = [\Sigma K, Y]_*.$$

Thus the relative homotopy classes of extensions of a form a torsor under $[\Sigma K, Y]_*$.

If $[\Sigma K, Y]_* = 0$, every nonempty homotopy fiber of ρ is connected and restriction

$$[X,Y]_* \longrightarrow [A,Y]_*$$

is injective.

Proof. Applying $\operatorname{RMap}_*(-,Y)$ to (30) turns the homotopy pushout into a homotopy pullback

$$\operatorname{RMap}_*(X,Y) \simeq \operatorname{RMap}_*(A,Y) \times^h_{\operatorname{RMap}_*(K,Y)} \operatorname{RMap}_*(CK,Y).$$

Since CK is contractible, $\operatorname{RMap}_*(CK,Y)$ is contractible and maps to the constant map in $\operatorname{RMap}_*(K,Y)$. The homotopy fiber over a is therefore

$$\mathcal{L}_a \simeq \operatorname{Path}_{\operatorname{RMap}_*(K,Y)}(a\circ u, *),$$

the space of paths from $a \circ u$ to the constant map.

Choose $H_0 \in \mathcal{L}_a$. A loop λ at $a \circ u$ acts on a path H by left concatenation $\lambda * H$; using the opposite convention would give the naturally isomorphic right torsor. Given two paths H, H', the composite $H' * H^{-1}$ is a loop at $a \circ u$, so the action is free and transitive on components. Transport along H_0 identifies loops based at $a \circ u$ with loops based at the constant map, and the suspension–loop adjunction gives

$$\pi_1 \operatorname{RMap}_*(K,Y,*) = [\Sigma K, Y]_*.$$

If this group vanishes, each nonempty homotopy fiber is connected. For a fibration of mapping spaces, connected homotopy fibers imply injectivity on π_0: a path between the restrictions of two maps lifts to put the maps in one homotopy fiber, where they are connected. Replacing the restriction map by a fibration does not change its derived homotopy fibers, so the conclusion holds for ρ. □

Remark 8.2 (Pointed versus unpointed classes). For an $\mathbb{A}^1$-connected source T and an $\mathbb{A}^1$-connected target Y, the forgetful map from pointed to unpointed classes is the orbit map for the standard action of $\pi_1^{\mathbb{A}^1}(Y)$; see [15, Section 6]. If Y is $\mathbb{A}^1$-simply connected, this action is trivial and pointed and unpointed classes agree. This is the only basepoint issue in the application below.

Dugger–Isaksen construct, after choosing the standard rational basepoints, a pointed homotopy cofiber sequence

$$\mathbb{A}^5 \setminus 0 \longrightarrow \mathbf{P}^4 \longrightarrow \mathbf{P}^5 \longrightarrow \Sigma(\mathbb{A}^5 \setminus 0) \tag{32}$$

in the motivic homotopy category [8, Proposition 2.13]. Since $\mathbb{A}^5 \setminus 0 \simeq S^{9,5}$, the quotient is $S^{10,5}$.

Corollary 8.3 (One-cell uniqueness for $\mathbf{P}^5$)**.** *Restriction induces an injection*

$$[\mathbf{P}^5, BSL_4]_{\mathbb{A}^1} \longrightarrow [\mathbf{P}^4, BSL_4]_{\mathbb{A}^1}.$$

More precisely, after choosing a homotopy between two restrictions on $\mathbf{P}^4$, the relative difference between the two extensions is the action of

$$[S^{10,5}, BSL_4]_{\mathbb{A}^1} = \pi^{\mathbb{A}^1}_{10,5}(BSL_4)(\mathbf{C}),$$

with no fundamental-group twisting.

Proof. Apply Proposition 8.1 to $K = \mathbb{A}^5 \setminus 0$, $A = \mathbf{P}^4$, $X = \mathbf{P}^5$, and $Y = BSL_4$, using cofibrant and fibrant replacements. The acting group is

$$[\Sigma(\mathbb{A}^5 \setminus 0), BSL_4]_{\mathbb{A}^1} = [S^{10,5}, BSL_4]_{\mathbb{A}^1},$$

which vanishes by Theorem 7.1. The projective spaces are $\mathbb{A}^1$-connected and BSL_4 is $\mathbb{A}^1$-simply connected, so Remark 8.2 removes both the basepoint ambiguity and the fundamental-group twisting. □

Theorem 8.4 (Oriented cancellation on $\mathbf{P}^5$)**.** *The stabilization map*

$$[\mathbf{P}^5, BSL_4]_{\mathbb{A}^1} \longrightarrow [\mathbf{P}^5, BSL]_{\mathbb{A}^1}$$

is injective.

Proof. Let $f, g\colon \mathbf{P}^5 \to BSL_4$ have the same stable class. Their restrictions to $\mathbf{P}^4$ have the same stable class. By Theorem 6.3,

$$f|_{\mathbf{P}^4} = g|_{\mathbf{P}^4} \quad \text{in } [\mathbf{P}^4, BSL_4]_{\mathbb{A}^1}.$$

Corollary 8.3 then gives $f = g$ on $\mathbf{P}^5$. □

9. The nonmotivic topological twin

Proof of Theorem A. Let $\eta \in [\mathbf{P}^5, BGL_4]_{\mathbb{A}^1}$, and let W be its complex realization. Assume

$$c(W) = 1 + h^2 + h^4.$$

The determinant of η is a class in

$$[\mathbf{P}^5, B\mathbf{G}_m]_{\mathbb{A}^1} = \mathrm{Pic}(\mathbf{P}^5) = CH^1(\mathbf{P}^5).$$

Its cycle class is $c_1(W) = 0$. Since $CH^1(\mathbf{P}^5) \xrightarrow{\sim} H^2(\mathbf{CP}^5; \mathbf{Z})$, the determinant is trivial. Choose an oriented lift

$$f_\eta\colon \mathbf{P}^5 \longrightarrow BSL_4.$$

Choose also a determinant trivialization of N and an oriented classifying map $f_N\colon \mathbf{P}^5 \to BSL_4$.

By Proposition 2.2, W is either N^{top} or $N^\dagger$, and in both cases it has the same stable topological K-theory class as N^{top}. In the commutative square

$$\begin{array}{ccc} [\mathbf{P}^5, BGL_4]_{\mathbb{A}^1} & \longrightarrow & [\mathbf{P}^5, BGL]_{\mathbb{A}^1} \cong \widetilde{K}_0(\mathbf{P}^5) \\ {\scriptstyle \mathrm{Re}_{\mathbf{C}}}\big\downarrow & & {\scriptstyle \mathrm{Re}_{\mathbf{C}}}\big\downarrow{\scriptstyle \cong} \\ [\mathbf{CP}^5, BU(4)] & \longrightarrow & [\mathbf{CP}^5, BU] \cong \widetilde{K}^0(\mathbf{CP}^5), \end{array} \tag{33}$$

the right vertical map is an isomorphism by Lemma 2.4. Here the stable classifying spaces represent reduced K-theory; equivalently, one may add the trivial rank-four class and work in the rank-four components of unreduced K-theory. Equality of the stable realized classes therefore implies equality of the stable motivic classes of η and N in $[\mathbf{P}^5, BGL]_{\mathbb{A}^1}$.

Lemma 3.1 now shows that f_η and f_N have the same image in $[\mathbf{P}^5, BSL]_{\mathbb{A}^1}$. Oriented cancellation on $\mathbf{P}^5$, Theorem 8.4, gives

$$[f_\eta] = [f_N] \qquad \text{in } [\mathbf{P}^5, BSL_4]_{\mathbb{A}^1}.$$

Forgetting the orientations yields the stronger equality

$$\eta = [N] \qquad \text{in } [\mathbf{P}^5, BGL_4]_{\mathbb{A}^1}.$$

This proves the first assertion and also shows that the realization fiber over N^{top} is a singleton. Since $N^\dagger$ is the distinct second topological class, its realization fiber is empty.

Finally, suppose an algebraic vector bundle E on $\mathbf{P}^5$ realized $N^\dagger$. Compatibility of algebraic and topological Chern classes, together with the cycle-class isomorphisms for projective space, would give

$$c(E) = 1 + h^2 + h^4 \quad \text{in } CH^*(\mathbf{P}^5).$$

Its motivic class $[E] \in [\mathbf{P}^5, BGL_4]_{\mathbb{A}^1}$ would realize $N^\dagger$, contrary to the empty-fiber statement. Hence $N^\dagger$ is not algebraizable. GAGA excludes a holomorphic representative as well. □

Proof of Corollary 1.1. The topological class $N^\dagger$ lies in the target of both maps but, by Theorem A, lies in the image of neither. □

Remark 9.1. The theorem determines the motivic realization fiber, not the algebraic isomorphism classes of all projective bundles with the same Chern vector. In particular, the argument does not claim that every algebraic rank-four bundle with $c = 1 + h^2 + h^4$ is algebraically isomorphic to a null-correlation bundle. It says that all such algebraic bundles determine the same motivic class and the same underlying topological bundle as N.

10. Application to the Rees bundles

The Rees bundles give a complementary application of Theorem B and clarify the factorization (2). For a prime p, let $\alpha_1^2 \in \pi_{4p-3}(S^3)$ be the usual p-primary composite. Collapsing the lower cells of $\mathbf{CP}^{2p-1}$ gives

$$\pi_{4p-3}(S^3) \cong \pi_{4p-2}(BSU(2)) \longrightarrow [\mathbf{CP}^{2p-1}, BSU(2)],$$

and Rees proved that the image ξ_p is a nontrivial rank-two topological bundle with $c_1(\xi_p) = c_2(\xi_p) = 0$ [18, 3]. These bundles are classical tests for algebraization: Barth's bounds place any hypothetical algebraic representative in the unstable range, while the Grauert–Schneider picture predicts splitting in sufficiently high dimension [6, Corollary 1, p. 127] and [10].

Asok–Fasel–Hopkins show that every ξ_p nevertheless has a motivic lift. They construct a nontrivial algebraic rank-two bundle on a Jouanolou device $q_p \colon X_p \to \mathbf{P}^{2p-1}$; because q_p is an $\mathbb{A}^1$-equivalence, its class determines

$$\widetilde{\xi}_p \in [\mathbf{P}^{2p-1}, BSL_2]_{\mathbb{A}^1}, \tag{34}$$

whose complex realization is ξ_p [3, Theorem 2.2.15]. Thus the Rees classes pass the motivic-realization stage. For $p = 2$, algebraic representatives on $\mathbf{CP}^3$ are known [11, 5]; in higher dimensions, motivic existence leaves a separate projective-descent problem.

The cancellation theorem above gives a precise structural consequence for every motivic Rees lift.

Proposition 10.1 (Rees classes lie in the unstable kernel). *Let p be a prime and let*

$$\eta \in [\mathbf{P}^{2p-1}, BGL_2]_{\mathbb{A}^1}$$

be a motivic class whose complex realization is ξ_p. Then $\det(\eta)$ *is trivial. Every oriented lift*

$$\bar{\eta} \in [\mathbf{P}^{2p-1}, BSL_2]_{\mathbb{A}^1}$$

is nonzero, but its stabilization is the basepoint in $[\mathbf{P}^{2p-1}, BSL]_{\mathbb{A}^1}$. In particular, the stabilization map

$$[\mathbf{P}^{2p-1}, BSL_2]_{\mathbb{A}^1} \longrightarrow [\mathbf{P}^{2p-1}, BSL]_{\mathbb{A}^1}$$

is not injective.

Proof. Compatibility of motivic and topological Chern classes gives

$$\mathrm{cl}(c_i(\eta)) = c_i(\xi_p) = 0, \qquad i = 1, 2.$$

The cycle-class maps

$$CH^i(\mathbf{P}^{2p-1}) \xrightarrow{\sim} H^{2i}(\mathbf{CP}^{2p-1}; \mathbf{Z})$$

are isomorphisms, so $c_1(\eta) = c_2(\eta) = 0$. In particular, the determinant is trivial and an oriented lift exists.

The stable Chern character of η is the rank-two class:

$$\mathrm{ch}([\eta]) = 2.$$

The Chern character is injective after tensoring $K_0(\mathbf{P}^{2p-1})$ with $\mathbb{Q}$, so $[\eta] - 2[\mathcal{O}]$ is torsion in $K_0(\mathbf{P}^{2p-1})$. The projective bundle formula identifies this group with a free abelian group, so it has no torsion and

$$[\eta] = 2[\mathcal{O}] \qquad \text{in } K_0(\mathbf{P}^{2p-1}).$$

Thus the underlying stable BGL-class is trivial. By Lemma 3.1, the corresponding stable BSL-class is also trivial, independently of the chosen determinant trivialization. On the other hand, $\bar{\eta}$ cannot be the basepoint in finite rank, because its complex realization is the nontrivial topological bundle ξ_p. ☐

The case $p = 3$ lives on the same projective space as the null-correlation twin and therefore interacts directly with Theorem 8.4.

Corollary 10.2 (Rank-four cancellation for the Rees class on $\mathbf{P}^5$)**.** *Let $\eta \in [\mathbf{P}^5, BGL_2]_{\mathbb{A}^1}$ be any motivic lift of the topological Rees bundle ξ_3. Then*

$$\eta \oplus [\mathcal{O}^{\oplus 2}] = [\mathcal{O}^{\oplus 4}] \qquad \textit{in } [\mathbf{P}^5, BGL_4]_{\mathbb{A}^1}. \tag{35}$$

Proof. Choose an orientation $\bar{\eta}$ of η and equip the two trivial summands with the standard orientation. Proposition 10.1 shows that the oriented rank-four class $\bar{\eta} \oplus \mathcal{O}^2$ has trivial stable image in $[\mathbf{P}^5, BSL]_{\mathbb{A}^1}$. Oriented cancellation on $\mathbf{P}^5$, Theorem 8.4, identifies it with the trivial oriented rank-four class. Forgetting the orientation gives (35). ☐

After pullback along a Jouanolou device $q\colon X \to \mathbf{P}^5$, affine representability [4] turns (35) into the actual algebraic stable trivialization

$$E_\eta \oplus \mathcal{O}_X^2 \cong \mathcal{O}_X^4$$

for the bundle E_η representing $q^*\eta$. This does not imply an algebraic splitting on $\mathbf{P}^5$: finite-rank $\mathbb{A}^1$-homotopy classes on the projective scheme are not being used as an isomorphism classification of algebraic bundles. The remaining issue is effectivity, equivalently descent along the Čech nerve of the Jouanolou torsor.

The two examples therefore isolate different stages of (2). The twin $N^\dagger$ has no motivic lift, whereas ξ_3 has motivic lifts and every such lift becomes trivial after adding two trivial summands already in rank four. Any further obstruction for ξ_3 must come from projective descent, not from Chern classes, stable K-theory, or the top-cell cancellation used here.

Appendix A. The topological rank-four calculation on $\mathbf{CP}^5$

This appendix makes the topological input in Proposition 2.2 explicit. We specialize Opie's Moore–Postnikov argument to $n = 4$ and $X = \mathbf{CP}^5$. The only imported unstable ingredients are the classical homotopy groups of $U(4)$ and Opie's identification and coaction formula for the final secondary k-invariant. All consequences for the Chern vector of the null-correlation bundle are then derived below.

Let $t \in H^2(\mathbf{CP}^5; \mathbf{Z})$ be the positive generator. We use $\bar{t}$ for its mod-2 reduction.

A.1. **The stable class.**

Lemma A.1. *All rank-four complex bundles W on $\mathbf{CP}^5$ satisfying*

$$c_1(W) = 0, \qquad c_2(W) = t^2, \qquad c_3(W) = 0, \qquad c_4(W) = t^4$$

have the same stable class in $K^0(\mathbf{CP}^5)$.

Proof. Newton's identities show that the Chern character is determined by these Chern classes; in fact

$$\mathrm{ch}(W) = 4 - t^2 - \frac{1}{12}t^4.$$

The group

$$K^0(\mathbf{CP}^5) \cong \mathbf{Z}[u]/(u^6)$$

is torsion-free, and the Chern character becomes injective after tensoring with $\mathbb{Q}$. Hence two classes with the same Chern character differ by torsion and therefore are equal. Lemma 2.1 shows that this stable class is realized by N^{top}. □

Denote the resulting stable classifying map by

$$V\colon \mathbf{CP}^5 \longrightarrow BU.$$

Because it is represented by a rank-four bundle, its fifth Chern class is zero.

A.2. **The final two-valued Postnikov ambiguity.** The unstable homotopy groups entering the Moore–Postnikov tower of $BU(4) \to BU$ give a final $\mathbf{Z}/2$ ambiguity in dimension ten. We first recall the upper bound.

Proposition A.2 (At most two classes)**.** *For any prescribed Chern classes $a_i t^i$, $1 \le i \le 4$, there are at most two rank-four complex bundles on $\mathbf{CP}^5$ realizing them.*

Proof. This is the $n = 4$ specialization of [17, Proposition 2.1]. We spell out the degree bookkeeping. Since $\mathbf{CP}^5$ is ten-dimensional, maps into $BU(4)$ depend only on $\tau_{\le 10}BU(4)$. The even Postnikov stages through degree eight are parametrized by c_1, c_2, c_3, c_4. The next obstruction is the Schwarzenberger congruence in

$$H^{10}(\mathbf{CP}^5; \mathbf{Z}/4!) \cong \mathbf{Z}/24.$$

Once it vanishes, the last obstruction lies in

$$H^{11}(\mathbf{CP}^5; \mathbf{Z}/2) = 0,$$

and the choices are acted on transitively by a quotient of

$$H^{10}(\mathbf{CP}^5; \mathbf{Z}/2) \cong \mathbf{Z}/2.$$

Thus there are at most two classes. For the Chern vector $(0, 1, 0, 1)$ the Schwarzenberger condition is automatically satisfied because the null-correlation bundle exists. □

To decide whether the last quotient is zero or $\mathbf{Z}/2$, consider

$$F = \mathrm{hofib}\big(c_5\colon BU \longrightarrow K(\mathbf{Z}, 10)\big). \tag{36}$$

The stable class V has a unique lift $\widetilde{V}\colon \mathbf{CP}^5 \to F$, because $c_5(V) = 0$ and $H^9(\mathbf{CP}^5; \mathbf{Z}) = 0$. For even rank $n = 4$, Opie constructs a space G and a fiber sequence

$$G \longrightarrow F \xrightarrow{U} K(\mathbf{Z}/2, 11) \tag{37}$$

together with a map $g\colon BU(4) \to G$ that induces isomorphisms on homotopy through degree nine and a surjection in degree ten. Opie's Lemma 3.6 identifies both degree-ten groups:

$$\pi_{10}BU(4) \cong \pi_{10}G \cong \mathbf{Z}/2.$$

The surjection in degree ten is therefore an isomorphism. Thus $\tau_{\le 10}g$ is an equivalence, and, because $\mathbf{CP}^5$ has dimension ten,

$$[\mathbf{CP}^5, BU(4)] \xrightarrow{\sim} [\mathbf{CP}^5, G]. \tag{38}$$

This is precisely [17, Lemma 3.6 and Corollary 3.7]. The class $U \in H^{11}(F; \mathbf{Z}/2)$ is the secondary k-invariant detected by $Sq^2\iota_9$ in the Serre spectral sequence of

$$K(\mathbf{Z}, 9) \longrightarrow F \longrightarrow BU;$$

see [17, Definitions 3.5–3.6 and Corollary 3.7].

Applying mapping spaces to (37) gives the exact James–Thomas description [12]: the set of rank-four lifts of V is a torsor under

$$\operatorname{coker}\Big(\pi_1 \operatorname{Map}(\mathbf{CP}^5, F, \widetilde{V}) \xrightarrow{\pi_1(U\circ -)} H^{10}(\mathbf{CP}^5; \mathbf{Z}/2)\Big). \tag{39}$$

This is [17, Lemma 3.9]; it is also the ordinary topological version of Theorem 5.1.

A.3. **The parity calculation.** The fibration $K(\mathbf{Z}, 9) \to F \to BU$ has a fiber action

$$m\colon K(\mathbf{Z}, 9) \times F \longrightarrow F.$$

Let $\iota_9' \in H^9(K(\mathbf{Z}, 9); \mathbf{Z}/2)$ be the fundamental class. Opie's coaction formula is

$$m^*U = U \otimes 1 + 1 \otimes Sq^2\iota_9' + c_1 \otimes \iota_9' \quad \text{in } H^{11}(K(\mathbf{Z}, 9) \times F; \mathbf{Z}/2), \tag{40}$$

up to the harmless interchange of tensor factors determined by the product convention [17, Proposition 3.15].

Let $\iota_1 \in H^1(S^1; \mathbf{Z})$ be a generator. The generator of

$$\pi_1 \operatorname{Map}(\mathbf{CP}^5, K(\mathbf{Z}, 9)) = H^8(\mathbf{CP}^5; \mathbf{Z}) \cong \mathbf{Z}$$

is represented on $S^1 \times \mathbf{CP}^5$ by $\iota_1 t^4$. If $c_1(V) = a_1 t$, then evaluation of (40) gives

$$(m^*U - U)(\iota_1 t^4, \widetilde{V}) = Sq^2(\bar{\iota}_1\bar{t}^4) + a_1\bar{\iota}_1\bar{t}^5 \tag{41}$$

$$= a_1\bar{\iota}_1\bar{t}^5 \quad \text{in } H^{11}(S^1 \times \mathbf{CP}^5; \mathbf{Z}/2). \tag{42}$$

Here

$$Sq^2(\bar{\iota}_1\bar{t}^4) = \bar{\iota}_1 Sq^2(\bar{t}^4) = 4\bar{\iota}_1\bar{t}^5 = 0.$$

The map

$$\pi_1 \operatorname{Map}(\mathbf{CP}^5, K(\mathbf{Z}, 9)) \longrightarrow \pi_1 \operatorname{Map}(\mathbf{CP}^5, F, \widetilde{V})$$

is surjective because $K^1(\mathbf{CP}^5) = 0$; see [17, Lemma 3.10]. Thus translation by the generator acts transitively on loops in the F-mapping space. Formula (41) shows that translation changes the value of $\pi_1(U \circ -)$ by a_1 modulo 2. Equivalently,

$$\operatorname{coker} \pi_1(U \circ -) \cong \begin{cases} \mathbf{Z}/2, & a_1 \equiv 0 \pmod 2, \\ 0, & a_1 \equiv 1 \pmod 2. \end{cases} \tag{43}$$

Opie's Corollary 3.14 identifies vanishing of this cokernel with the nonvanishing of the difference in (41). Hence, when a_1 is even, the cokernel is nonzero and therefore equals $\mathbf{Z}/2$; when a_1 is odd, the difference is the generator and the cokernel is zero. This is precisely [17, Corollaries 3.14–3.17].

Theorem A.3 (Specialized topological classification)**.** *Let V be a stable complex bundle on $\mathbf{CP}^5$ with $c_5(V) = 0$. If $c_1(V)$ is even, then V has exactly two rank-four representatives; if $c_1(V)$ is odd, it has exactly one.*

Proof. The torsor description (39), the parity computation (43), and the upper bound of Proposition A.2 give the result. This is the case $n = 4$ of [17, Theorem 3.18]. □

Applying Theorem A.3 to the stable class of Lemma A.1, whose first Chern class is zero, gives exactly two rank-four representatives with Chern vector $(0, 1, 0, 1)$.

A.4. **The top-cell action and stabilization.** Let $s\colon \mathbf{CP}^5 \to \mathbf{CP}^5 \vee S^{10}$ be the pinching map obtained by collapsing the boundary of a disk in the top cell. For $\sigma \in \pi_{10}BU(4)$ and $f\colon \mathbf{CP}^5 \to BU(4)$, define

$$\sigma \cdot f = (f \vee \sigma) \circ s. \tag{44}$$

Classical calculations give

$$\pi_{10}BU(4) = \pi_9 U(4) \cong \mathbf{Z}/2$$

[14, 17]. Opie's Postnikov analysis shows that the two classes with fixed Chern vector form one orbit for the action (44) [17, Proposition 2.2]. Since the orbit has two elements, the nonzero class acts nontrivially and exchanges them. Stabilization sends this element to

$$\pi_{10}BU \cong \mathbf{Z},$$

so it maps to zero. Hence the two rank-four bundles have the same stable K-theory class. Taking one representative to be N^{top} proves Proposition 2.2 in full.

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