

# A standalone repair of the late Route C plane-cubic spine

Replacing every cuspidal use of Proposition 12.55 in `master.tex`

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## Abstract

In the current `master.tex`, the proposition labeled `prop:routeC-one-section-plane-cubic-descendant` (Proposition 12.55 in the present numbering) is used only in three later closure theorems: `thm:routeC-one-section-bisection-closed`, `thm:routeC-one-section-cubic-pure-triple-closed`, and `thm:routeC-root-two-closed`. Its proof is correct for *nodal* plane cubics but false for *cuspidal* cubics.

This note replaces all three later uses. First, the degree-three pure-triple branch is repaired by forcing the final plane cubic to be *nodal*. Second, the degree-two one-section branch is closed without any plane-cubic descendant theorem at all: outside the several-special-horizontal corridor it contributes no separate residual case by the earlier bisection endgame already proved in `master.tex`. Third, the purely inseparable singleton [2]-root branch is also closed without any plane cubic: once the branch is stated with the geometry it actually has, it is already absorbed by the old ramification-frontier corridor.

Consequently the late Section 12 proof spine can be rewritten so that it no longer uses Proposition 12.55 in any cuspidal situation. This note is deliberately scoped only to that late plane-cubic defect; it does not address any independent earlier gaps elsewhere in the manuscript.

## 1 The nodal replacement for Proposition 12.55

**Proposition 1.1** (Nodal plane cubics still give elliptic ties). *Let  $H$  be a smooth rational curve on a smooth surface  $\tilde{X}$ . Assume that there exists a birational map*

$$\eta: \tilde{X} \dashrightarrow \mathbb{P}^2$$

*whose restriction to  $H$  is birational onto an irreducible nodal plane cubic  $C \subset \mathbb{P}^2$ . Then, on a smooth surface dominating  $\tilde{X}$ , there exists an elliptic tie rooted at the strict transform of  $H$ . In particular, every downstream use of Proposition 12.55 remains valid in the nodal subcase.*

*Proof.* Resolve the indeterminacy of  $\eta$  and the node of  $C$  simultaneously to obtain a smooth surface  $Y$ , a birational morphism  $\sigma: Y \rightarrow \tilde{X}$ , and a morphism

$$f := \eta \circ \sigma: Y \rightarrow \mathbb{P}^2$$

such that the reduced total transform

$$E := (f^{-1}(C))_{\text{red}}$$

is simple normal crossings. Let  $R \subset E$  be the strict transform of  $H$ , and let  $Z \subset E - R$  be the reduced exceptional divisor above the node of  $C$ .

Because  $C$  has exactly two local branches at its unique singular point, the embedded resolution graph over that point is a tree of smooth rational curves whose two ends are met by the strict

transform  $R$  of the normalization. Equivalently,  $Z$  is a connected tree of smooth rational curves and  $R$  meets  $Z$  in exactly two points. Therefore

$$p_a(R + Z) = p_a(R) + p_a(Z) + R \cdot Z - 1 = 0 + 0 + 2 - 1 = 1.$$

So  $T := R + Z$  is a connected reduced snc divisor with rational components and arithmetic genus 1.

Every component of  $Z$  is exceptional for the birational contraction to the nodal cubic, whereas  $R$  is not. Factoring that contraction into elementary blowdowns between smooth surfaces shows that one may contract the components of  $Z$  successively while never contracting  $R$ . Hence  $T$  is an elliptic tie rooted at  $R$ .  $\square$

**Corollary 1.2** (A smooth  $3S + f$  endpoint on  $\mathbb{F}_0$  can be made nodal). *Let  $Q \subset \mathbb{F}_0$  be a smooth rational curve of class  $3S + f$ , and let*

$$\pi: Q \rightarrow \mathbb{P}^1$$

*be the restriction of the  $f$ -ruling. Assume that  $\pi$  is separable of degree 3. Then there exists a birational map*

$$\rho: \mathbb{F}_0 \dashrightarrow \mathbb{P}^2$$

*such that  $\rho|_Q$  is birational onto an irreducible nodal plane cubic.*

*Proof.* Because  $\pi$  is separable of degree 3, only finitely many fibers are ramified. Choose a smooth point  $q \in Q$  on an unramified fiber  $F$  of the  $f$ -ruling. Then

$$F \cap Q = q + r + s$$

with  $q$ ,  $r$ , and  $s$  distinct. Let  $T \in |S|$  be the curve from the other ruling through  $q$ ; since  $Q \cdot T = 1$ , it meets  $Q$  only at  $q$ .

Blow up  $q$ , obtaining  $\sigma: Z \rightarrow \mathbb{F}_0$  with exceptional curve  $E$ , and let  $\tilde{Q}$ ,  $\tilde{F}$ , and  $\tilde{T}$  be the strict transforms. The curves  $\tilde{F}$  and  $\tilde{T}$  are disjoint  $(-1)$ -curves, so one may contract them successively to a smooth rational surface  $W$  of Picard rank 1, hence  $W \cong \mathbb{P}^2$ . Let  $\tau: Z \rightarrow W \cong \mathbb{P}^2$  be that contraction and set  $\rho := \tau \circ \sigma^{-1}$ .

As in the class computation already used in `master.tex`, the image  $\rho(Q)$  is a plane cubic. Now  $\tilde{Q}$  meets  $\tilde{F}$  exactly in the two distinct points  $r$  and  $s$ . So the first contraction creates an ordinary node on the image of  $\tilde{Q}$ . The later contraction of the disjoint curve  $\tilde{T}$  does not alter that singularity. Therefore  $\rho(Q)$  is an irreducible nodal plane cubic.  $\square$

## 2 The degree-three pure-triple branch

**Theorem 2.1** (Replacement for `thm:routeC-one-section-cubic-pure-triple-closed`). *Assume the setup of `cor:routeC-one-section-cubic-pure-triple` in `master.tex`. Suppose that after the possible unique double step one is in the pure-triple branch, so the current image of  $H$  has class  $3S_0 + (3m + 1)f$  on some  $\mathbb{F}_m$  and every later singular-point elementary transform is centered at a triple point. Then  $X$  has a descendant with elliptic boundary. In particular, this branch cannot occur in the residual characteristic-3 sector.*

*Proof.* Continue the lifted singular-point transform sequence until the image of  $H$  becomes smooth. By `prop:routeC-one-section-cubic-singular-eltrans-lifts`, every step lifts from  $\tilde{X}$ . Because every remaining step is triple, the parameter 1 is preserved throughout the sequence, and after exactly  $m$  steps one arrives at a birational morphism

$$\phi: \tilde{X} \rightarrow \mathbb{F}_0$$

whose image  $Q := \phi(H)$  is smooth of class  $Q \equiv 3S + f$ . This is exactly the final smooth model recorded in `cor:routeC-one-section-cubic-reduction-count`.

Since we are on the separable special-multisection side, the induced map

$$Q \cong H \rightarrow \mathbb{P}^1$$

has degree 3 and is separable. So Corollary 1.2 applies and yields a birational map

$$\rho: \mathbb{F}_0 \dashrightarrow \mathbb{P}^2$$

whose restriction to  $Q$  is birational onto an irreducible nodal plane cubic. Set

$$\eta := \rho \circ \phi: \tilde{X} \dashrightarrow \mathbb{P}^2.$$

Then  $\eta|_H$  is birational onto that nodal cubic. By Proposition 1.1,  $X$  has a descendant with elliptic boundary.  $\square$

**Corollary 2.2** (The degree-three part of the one-section singular-image side remains closed). *In `cor:routeC-special-multisection-one-section-singular-closed`, item (ii) remains valid after replacing the citation to `thm:routeC-one-section-cubic-pure-triple-closed` by Theorem 2.1.*

### 3 The degree-two one-section branch

**Proposition 3.1** (Replacement for `thm:routeC-one-section-bisection-closed`). *Assume the setup of `cor:routeC-one-section-height4-partitions-singular` in `master.tex`, and suppose that  $d = 2$ . Assume moreover that one is outside the several-special-horizontal corridor. Then the degree-two one-section singular-image branch contributes no separate residual case. Equivalently, outside the several-special-horizontal corridor the partition  $2 + 1 + 1$  does not create any new late Route C branch that still needs closure.*

*Proof.* This is exactly the content of the bisection endgame already proved earlier in `master.tex`. By `cor:routeC-bisection-no-new-local`, every singularity of the unique ordinary bisection is either impossible or already absorbed by the old branching / touched-[2] geometry. Equivalently, the degree-two mixed branch contributes no genuinely new local geometry. Then `cor:routeC-lowdegree-mixed-no-new-local` packages this globally: outside the several-special-horizontal corridor, the low-degree mixed branch creates no separate local corridor at all. The degree-two case is precisely the  $2 + 1 + 1$  branch.  $\square$

**Corollary 3.2** (The touched marked-cubic endpoint needs no node/cusp analysis). *In the touched degree-two subcase  $d = 2$ ,  $s = 1$  of the one-section branch, the marked singular plane cubic produced by `cor:routeC-one-section-bisection-cubic` is only a global compression. Its singularity may be nodal or cuspidal; that distinction is irrelevant for Route C. The branch is already absorbed by Proposition 3.1.*

**Corollary 3.3** (The degree-two part of the one-section singular-image side remains closed). *In `cor:routeC-special-multisection-one-section-singular-closed`, item (i) remains valid after replacing the citation to `thm:routeC-one-section-bisection-closed` by Proposition 3.1.*

**Remark 3.4** (How the smooth-image proof should be edited). *In the proof of `prop:routeC-special-multisection-smooth-image-lowdegree-corridor-impossible`, Theorem `routeC-one-section-bisection-closed` is used twice, exactly in the two places where `lem:routeC-special-multisection-one-sided-threaded-height` produces a witnessing ruling of height 4 with horizontal degree partition  $2 + 1 + 1$ . Under the standing hypothesis of that proof one is already outside the several-special-horizontal corridor. So in both places the correct replacement is Proposition 3.1, not any plane-cubic descendant theorem.*

## 4 The purely inseparable singleton [2]-root branch

**Lemma 4.1** (A singleton [2]. -root is special) *Let  $H \subset D_{\text{hor}}$  be a horizontal component in the sense of `def:routeC-global-special-horizontal`. If the connected component of  $D$  containing  $H$  is a chain whose reduced touched core relative to  $H$  is a touched tip of weight 2, then  $H$  is special. In particular, a branch described geometrically as a singleton [2]-root branch is a special branch, not an ordinary one.*

*Proof.* This is immediate from the definition of “ordinary” and “special” in `def:routeC-global-special-horizontal`: a horizontal component is ordinary exactly when its reduced touched core is neither a touched tip of weight 2 nor the singleton chain [3].  $\square$

**Proposition 4.2** (Replacement for `thm:routeC-root-two-closed`). *Assume the purely inseparable singleton [2]-root branch, stated by its actual geometry: the horizontal multisection  $H \rightarrow \mathbb{P}^1$  is purely inseparable of degree greater than 1, and the horizontal side is of touched-[2] type. Then this branch is already absorbed by the old ramification-frontier corridor. In particular, it contributes no separate residual case to the final height-selection theorem.*

*Proof.* By Lemma 4.1, the horizontal multisection  $H$  is special. Now apply `cor:routeC-final-lemma-ramification`. It produces a finite bad set  $B_\nu \subset S$  on the ruled-model image  $S$  such that every point  $q \in S \setminus B_\nu$  of local degree at least 2 yields a surviving primitive ramification fork on  $\tilde{X}$ . Because the map  $S \rightarrow \mathbb{P}^1$  is purely inseparable of degree greater than 1, every point of  $S$  has local degree at least 2. So one may choose

$$q \in S \setminus B_\nu.$$

For that point, the primitive ramification fork survives.

Now invoke `cor:routeC-special-multisection-ramification-frontier-old`. Its statement requires only that  $H$  be special and that  $q \in S \setminus B_\nu$  have local degree at least 2; those hypotheses are satisfied here. Therefore the surviving primitive ramification fork over  $q$  is already absorbed by the old Route C geometry. So this branch cannot survive as an independent final case. No plane-cubic descendant theorem is needed.  $\square$

**Corollary 4.3** (How the final split should be repaired). *In `prop:routeC-height-selection-final-split`, the separate case currently labeled the purely inseparable ordinary singleton [2]-root branch should be deleted. The corrected final split has only two residual cases:*

- (i) *the several-special-horizontal branch;*
- (ii) *the separable special-multisection branch.*

*The singleton [2] purely inseparable branch is already absorbed earlier by Proposition 4.2.*

## 5 Repaired late closure

**Theorem 5.1** (The Proposition 12.55-dependent late closure can be repaired). *Replace the three later uses of `prop:routeC-one-section-plane-cubic-descendant` in `master.tex` as follows:*

- (1) *replace `thm:routeC-one-section-cubic-pure-triple-closed` by Theorem 2.1;*
- (2) *replace `thm:routeC-one-section-bisection-closed` by Proposition 3.1;*
- (3) *replace `thm:routeC-root-two-closed` by Proposition 4.2.*

*Then the late Section 12 proof no longer uses Proposition 12.55 in any cuspidal situation. More precisely, the proof of `thm:routeC-height-selection-complete` can be rewritten exactly as follows:*

- (i) *the one-section singular-image part of the separable special-multisection branch is closed by Corollaries 3.3 and 2.2;*
- (ii) *the smooth-image part is still closed by the existing smooth-image closure outside the several-special-horizontal corridor;*
- (iii) *the purely inseparable singleton [2] branch is removed from the final split by Corollary 4.3;*
- (iv) *the several-special-horizontal packet side is still closed by the existing packet-side closure theorem.*

*Therefore the late height-selection theorem and the final seven-point corollary can be rewritten without any remaining cuspidal use of Proposition 12.55.*

*Proof.* Only the three cited later theorems currently use Proposition 12.55. After the replacements above, the degree-three branch is closed by Theorem 2.1, the degree-two branch contributes no separate residual case by Proposition 3.1, and the purely inseparable singleton [2] branch is already old by Proposition 4.2. So the proof of the late height-selection theorem needs no other change than the corresponding citation edits. The concluding seven-point corollary in Section 12 then follows from the same height-selection reduction as before.  $\square$

**Remark 5.2** (Scope). *This note repairs the late Proposition 12.55 / cuspidal-cubic defect only. It should be read as a replacement package for that specific late dependency spine. It does not claim to repair any independent earlier defects elsewhere in the manuscript.*