Singularities on characteristic 3 rank 1 del Pezzo surfaces

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31 March 2026

Abstract. We prove that every characteristic-3 klt del Pezzo surface of Picard rank 1 has at most seven singular points. On the explicit side, the construction corridors treated here do not produce a Picard-rank-one 8-point surface: Method A is closed outright, and the weighted corridors of Method B either leave the Picard-rank-one setting or fail ampleness. On the birational side, any hypothetical counterexample is reduced to the residual sector $\mathrm{ht}(X) \geq 4$ with no descendant having elliptic boundary, and that sector is then closed completely. The proof combines the elimination of the first Route C frontier, an anticanonical contact formula and a universal contact-defect sieve, a cubic-backbone and clean-threaded analysis of special multisections, marked plane reductions for the one-section branches, scalar-packet preparation on the several-special-horizontal side, and a terminal-leaf count in the height-4 maximal-width corridor. The final step solves the residual height-selection problem by closing the remaining packet and smooth-image special-multisection branches. Consequently no characteristic-3 rank-one klt del Pezzo surface has more than seven singular points.

Contents

1. Introduction and overview

We study the characteristic-3 many-singularity problem for rank-one klt del Pezzo surfaces and prove that no such surface has more than seven singular points. All currently known constructions in characteristic 3 had at most seven singular points, and the main theorem of this paper shows that this bound is optimal [7, 8, 2]. In particular, this resolves the FrontierMath / Epoch open problem [2] in the negative: there is no characteristic-3 rank-one klt del Pezzo surface with more than seven singular points.

The terminology *Method A*, *Method B*, and *Method C* comes from the FrontierMath / Epoch problem prompt for this question [2]. In that prompt, Method A is the global-quotient route starting from a smooth complete intersection $\mathrm{CI}(2,2) \subset \mathbf{P}^4$ with a tame cyclic action; Method B is the weighted-model route using quasi-smooth weighted hypersurfaces or complete intersections with positive Fano index; and Method C is the scripted blow-up route starting from $\mathbf{P}^1 \times \mathbf{P}^1$ and contracting prescribed Hirzebruch–Jung chains. In the present paper, Methods A and B keep their prompt meaning as explicit construction formats, while our *Route C* is the birational and resolution-theoretic analysis of the same blow-up / contraction side suggested by Method C. The change from "Method" to "Route" is deliberate: by the time Section 9 begins, the paper is no longer proposing a single explicit construction ansatz, but analyzing the residual geometry of any hypothetical counterexample.

The paper has two different roles. Sections 1–8 are included to map the search space, record the explicit Method A / Method B obstructions, and preserve the numerical and heuristic exploration that led to the final strategy; they are useful context, but they are not the formal proof spine of the seven-point theorem. The actual proof begins in Section 9. From there onward, the argument has three layers: first, Section 9 reduces any hypothetical counterexample to the residual height-≥ 4, no-descendant sector; second, the middle Route C sections develop the global bridge, packet, and multisection machinery on the minimal resolution; third, the final Route C sections close the residual geometry by a cubic-backbone reduction for ordinary ramification forks, a clean threaded [2]-root budget, a preparation step clearing packet–negative-section contact points so that high-height packets can be treated in the disjoint scalar/Frobenius model, marked plane reductions for the one-section branches, and a terminal-leaf count together with the final height-selection theorem.

The main ingredients and intermediate results are the following.

(1) **Methods A and B are largely exhausted in the benchmark corridors studied here.** Method A is closed outright, and the weighted hypersurface / weighted complete-intersection corridors analyzed in the paper do not yield a Picard-rank-one surface with at least eight singular points.

(2) **Residual reduction.** Any characteristic-3 rank-one klt del Pezzo surface with at least eight singular points lies in the residual sector

$$\mathrm{ht}(X) \geq 4 \qquad \text{and} \qquad X \text{ has no descendant with elliptic boundary.}$$

(3) **The entire first analyzed Route C frontier leaves the residual sector.** Same-component positive root-contractible contact-2 configurations already give elliptic ties. Among bridges, the peelable family has height 1, the two square-one fork prototypes force descendants with elliptic boundary, and the two higher-square short bridges have height at most 2.

(4) **A new global anticanonical bridge sieve.** For a chain bridge one has the exact formula $-K_X \cdot \pi(L) = -1 + \mu(A, i) + \mu(B, j)$, which eliminates the full one-root off-frontier family $[b] \dashv [a, (2)^{a-1}]$ in the benchmark range.

(5) **A new 2-tail core reduction for chain bridges.** After stripping terminal [2]-tails away from the touched components, every positive chain bridge has at least one tip-visible side. After this reduction, the only reduced tip-visible cores with contact weight strictly larger than $1/2$ are a touched (-2)-tip and the singleton core [3].

(6) **A universal contact-defect sieve for surviving nonexceptional curves.** If $L \not\subset D$ is a vertical (-1)-curve on $\widetilde{X}$, then

$$\sum_{E \subset D} (L \cdot E)\, \mathrm{cf}_X(E) < 1.$$

Consequently, any surviving ramification fork satisfies

$$\frac{2}{a} + \frac{2}{b} > 1,$$

and any surviving scalar star is A_1-heavy: all but at most two packet sections are (-2)-curves.

(7) **Further unconditional narrowing of the multisection, bisection, and trisection branches.** An ordinary surviving ramification fork has local degree at most 3; in characteristic 3 the purely inseparable ordinary branch is cubic, the corresponding smooth rational cubic backbone on a Hirzebruch surface has self-intersection 6, and the clean threaded [2]-root model has at most two primitive ramification fibers. On the one-section separable side, every ruled-model bisection in the partition $2 + 1 + 1$ meets the second section, every singular point of such a bisection has an explicit double-point normal form, and every irreducible rational bisection of class $2S_0 + (2n + s)f$ reduces by successive singular-point elementary transforms to a marked plane conic or cubic. In the touched subcase $s = 1$, one obtains a singular plane cubic with a smooth marked point whose projection realizes the original degree-2 map; consequently the whole branch has at most two unibranch ramification strings. Every surviving unibranch block of the ordinary bisection is already absorbed by the old touched-tip / branching side, and every multibranch collision of that bisection is impossible by the contact-defect sieve. On the degree-three mixed side, every singular trisection reduces after at most four elementary transforms to an irreducible rational plane curve of degree $3, 4, 5$, or 6 with marked-point multiplicity equal to the degree minus 3; the exceptional class $3S_0 + 8f \subset \mathbb{F}_2$ is itself reduced to the marked plane package, and every remaining multibranch collision of the ordinary trisection is either impossible by the contact-defect sieve or already absorbed by the old touched-tip

/ branching side. A supplementary finite ordinary-fiber search on the cubic backbone through eight additional blowups rules out the searched defect-zero service modules.

(8) **Closure of the height-4 maximal-width corridor.** In the setup of Corollary 9.13, the exact excess identity

$$\Sigma = \sum_F (\sigma(F) - 1) = 3$$

holds, at most three genuinely bad singly-hit multiplicity-one tips can occur, and one horizontal 1-section avoids all of them. Moreover, in a λ-minimal maximal-width witness every one-touch bad branch starts on the horizontal packet, packet-section roots strip away by elementary transforms, any vertical component meeting all four horizontal sections is an isolated (-2)-curve with no vertical neighbor, and the remaining terminal corridor contributes a leaf of $F_{\mathrm{red}} - D$. A terminal-leaf count then gives $v(F) \leq \sigma(F) - 1$ for every degenerate fiber, so one gets $\#\operatorname{Sing}(X) \leq 7$ throughout the whole height-4, maximal-width corridor.

(9) **Closure of the residual height-selection problem.** The several-special-horizontal packet branch and the separable special-multisection branch are both closed in the final section. Hence every residual surface admits a witnessing $\mathbf{P}^1$-fibration of height 4 and maximal width.

Combining the maximal-width corridor closure with the final height-selection theorem gives the main result of the paper: every characteristic-3 klt del Pezzo surface of Picard rank 1 satisfies $\#\operatorname{Sing}(X) \leq 7$.

1.1. Main results and organization

The paper is organized as follows. Section 2 recalls the current classification picture. Sections 3–8 treat the explicit quotient, weighted, and spectrum-first constructions attached to Methods A and B. Section 9 begins the Route C analysis by reducing to the residual sector and by developing the first entry criteria. The next Route C sections classify the first chain/fork/cross-link frontier and remove all of its residual survivors. The later Route C sections prove same-component completeness, introduce the anticanonical bridge sieve and the universal contact-defect sieve, reduce the remaining chain bridge geometry to tip-visible 2-tail cores, push the multisection branch by a cubic-backbone reduction and a clean threaded [2]-root budget together with a two-fiber support bound for non-clean purely inseparable ordinary singleton [2]-root patterns, push the one-section branch by explicit double-point normal forms and a marked plane conic/cubic reduction for the bisection case together with a low-degree unibranch analysis and a finite marked plane package for trisections of degree at most six, and return to the height-4 maximal-width endgame through λ-minimal one-touch reductions, a rigid four-section-hub analysis, and a terminal-leaf count that closes the maximal-width corridor. Along the cubic-backbone branch we also record a supplementary finite ordinary-fiber search through eight additional blowups.

Table 1 records the main representative constructions and near-misses encountered in the search.

Table 1: Representative examples and outcomes.

Format	Data	Singularities	ρ	Outcome
Known char. 3 blow-up model	basket $[[3],[3],[3],[3],[2,2],[2,2],[2,2]]$	$4[3]+3A_2$	1	Verified 7-point example
Weighted hypersurface	$x_0^6+x_0^5x_1+x_1^6+x_2x_3=0\subset\mathbf{P}(2,2,5,7)$	$6A_1$+[5]+[7]	6	Exact 8 points, wrong Picard number
Weighted hypersurface	same equation in $\mathbf{P}(4,4,5,19)$	$6A_3+[5]+[19]$	6	Exact 8 points, wrong Picard number
Symmetric hypersurface	$x_0^7+x_1^7+x_2^2+x_3^2=0\subset\mathbf{P}(2,2,7,7)$	$7A_1+2[7]$	7	9 points, wrong Picard number
Method C near-miss	basket $[[6],[3],[3],[3],[3],[2,2],[2,2],[2,2],[2,2]]$	9 points	1 (numerical)	$-K$ nef, not ample
Method C near-miss	basket $[[5,2],[2],[3],[3],[3],[2,2],[2,2],[2,2]]$	8 points	1 (numerical)	$-K$ not ample
Cube-spectrum candidate	one degree-8 section in weighted cube format inside $\mathbf{P}(1,2,4,4,5,5,7)$	$5\times\frac{1}{4}(1,1)+2\times\frac{1}{5}(1,2)+\frac{1}{7}(1,1)$	9 (if realized)	Spectral hit, but not rank one

2. Background from the current classification picture

The classical characteristic-zero upper bound is Belousov's theorem: a log del Pezzo surface of Picard rank one has at most four singular points [1]. In characteristic 3, this fails dramatically: Lacini's classification in characteristics different from 2 and 3 already isolates characteristic 3 as exceptional, and the newer Palka–Pełka programme shows that the known characteristic-3 extremal baskets have 7 singular points [4, 7, 8].

Two baskets repeatedly appeared as extremal throughout the search:

$$4\cdot[3]+3A_2=[[3],[3],[3],[3],[2,2],[2,2],[2,2]],$$
$$4A_2+[5]+[3]+[2]=[[5],[3],[2],[2,2],[2,2],[2,2],[2,2]].$$

The first is the known height-3 extremal basket; the second is the elliptic-descendant endpoint in characteristic 3 [8, 7, 6].

These results do *not* prove the impossibility of 8 points, but they tell us where not to look: any new construction must evade the currently classified height-≤ 3 and elliptic-descendant sectors.

At the time of writing, the public Palka–Pełka classification series still reaches only height 3; the characteristic-3 height-4 endpoint is announced in [7, Remark 1.7] as forthcoming. A further external input relevant to the later one-section analysis is Kettinger's theorem that characteristic 3 admits no unexpected plane cubics [9]. This makes the marked plane-cubic reduction of the residual bisection branch especially attractive.

3. Global quotients and the Frobenius blow-up route

3.1. Method A: diagonal involution quotients of $\mathrm{CI}(2,2) \subset \mathbf{P}^4$

The warm-up quotient construction starts with a smooth complete intersection of two quadrics in $\mathbf{P}^4$ and a diagonal involution of order 2. The following observation kills the possibility of reaching 8 singular points within this setup.

Proposition 3.1 (derived in this search)**.** *Let $Y \subset \mathbf{P}^4$ be a smooth complete intersection of two quadrics, and let g be a diagonal involution acting tamely in characteristic 3, freely in codimension 1 on Y. Then the quotient $X = Y/\langle g\rangle$ has at most 6 singular points.*

Proof. A diagonal involution on $\mathbf{P}^4$ has fixed locus equal to a disjoint union of linear subspaces. In the codimension-1-free case relevant here, the fixed locus is $\mathbf{P}^2 \sqcup \mathbf{P}^1$. The singular points of the quotient correspond to isolated points in $Y \cap \mathrm{Fix}(g)$. A $(2,2)$ complete intersection meets the fixed $\mathbf{P}^2$ in at most 4 points and the fixed $\mathbf{P}^1$ in at most 2 points, hence at most 6 points in total. ☐

Thus the standard tame global-quotient version of Method A cannot solve the benchmark.

3.2. Method C: the Frobenius/Keel–McKernan blow-up family

The known characteristic-3 construction with 7 singular points comes from a modification of the Keel–McKernan example; see Lacini's Example 7.4 and the later Palka–Pełka discussion [4, 8]. It produces the basket $4[3] + 3A_2$.

The obvious question is whether one can repeat the same triple-blow-up move over more fibres. The next proposition records the resulting obstruction.

Proposition 3.2 (derived in this search)**.** *Consider the natural extension of the characteristic-3 Frobenius graph construction on $\mathbf{P}^1 \times \mathbf{P}^1$ in which a curve C satisfies $C \cdot F = 3a$, and one performs the same triple blow-up process over m fibres. If the resulting contracted surface is denoted by T, then along a general fibre*

$$-K_T \cdot \psi_* F = 2 - 3a\Big(1 - \frac{2}{r}\Big), \qquad r = 3m - 6a.$$

This quantity is positive only for $(a, m) = (1, 3)$, which is exactly the known 7-point case.

Proof. After the blow-ups one has $C^2 = 6a - 3m = -r$. The discrepancy computation extending the known characteristic-3 example gives

$$\psi^* K_T = K_S + \frac{1}{3}\sum_{i=1}^{m} F_i + \Big(1 - \frac{2}{r}\Big)C.$$

Intersecting with a general fibre yields the displayed formula. Since $m > 2a$, one has $r = 3(m-2a) \geq 3$, and the expression is maximized when $m = 2a+1$, giving

$$2 - 3a + \frac{2a}{m-2a} \leq 2 - a.$$

Hence positivity forces $a = 1$, and then $m = 3$. □

This explains why the most naive extension of the known 7-point blow-up construction fails immediately: the next step only makes $-K$ nef, not ample.

Remark 3.3. Two useful near-misses emerged from this computation. The obvious four-fibre extension has basket

$$[[6],[3],[3],[3],[3],[2,2],[2,2],[2,2],[2,2]],$$

so 9 singular points, but $-K$ is only nef. After one extra smooth blowdown one obtains the rank-compatible 8-point basket $[[5,2],[2],[3],[3],[3],[2,2],[2,2],[2,2]]$, which again fails ampleness.

4. Weighted hypersurfaces: exact formulas and exact dead ends

4.1. The six-branch family $f_n(x_0,x_1) + x_2x_3 = 0$

The cleanest weighted family discovered in the search is

$$X^{(n)}_{a,p,q} := \{f_n(x_0,x_1) + x_2x_3 = 0\} \subset \mathbf{P}(a,a,p,q), \qquad p+q = an,$$

where f_n is a binary form of weighted degree an, split with n distinct roots, and where $3 \nmid apqn, \quad \gcd(a,p) = \gcd(a,q) = 1$. This family is tame, quasi-smooth, and extremely explicit.

Proposition 4.1 (derived in this search)**.** *For the family $X^{(n)}_{a,p,q}$ above, one has:*

(i) $X^{(n)}_{a,p,q}$ has singularity basket $nA_{a-1} + [p] + [q]$;

(ii)

$$K_X^2 = \frac{4(p+q)}{pq};$$

(iii) the minimal resolution $\widetilde{X}$ satisfies $K^2_{\widetilde{X}} = 8 - (p+q) = 8 - an$;

(iv) $\widetilde{X}$ is rational and $\rho(X) = n$.

In particular, exact 8 singular points in this family force $n + 2 = 8$, hence $n = 6$ and therefore $\rho(X) = 6$.

Proof. On the singular line $\mathbf{P}(a,a)$ the equation restricts to $f_n = 0$, so there are n reduced points, each of type

$$\frac{1}{a}(p,q) \cong \frac{1}{a}(1,-1) = A_{a-1}.$$

At the coordinate points one gets cyclic quotient singularities

$$\frac{1}{p}(a,a) \cong \frac{1}{p}(1,1) = [p], \qquad \frac{1}{q}(a,a) \cong \frac{1}{q}(1,1) = [q].$$

The weighted hypersurface formula gives

$$K_X^2 = d\,\frac{(\sum w_i - d)^2}{\prod w_i} = (p+q)\frac{(2a)^2}{a^2pq} = \frac{4(p+q)}{pq}.$$

The A_{a-1} points are crepant, while the points $[p]$ and $[q]$ contribute

$$\frac{(p-2)^2}{p}, \qquad \frac{(q-2)^2}{q}$$

to the drop of K^2. Hence

$$K_{\widetilde{X}}^2 = \frac{4(p+q)}{pq} - \frac{(p-2)^2}{p} - \frac{(q-2)^2}{q} = 8 - (p+q).$$

The equation is linear in x_3 on the chart $x_2 \neq 0$, so X is rational, hence $\widetilde{X}$ is rational. By Noether's formula for rational surfaces,

$$\rho(\widetilde{X}) = 10 - K_{\widetilde{X}}^2 = an + 2.$$

The exceptional divisor has $n(a-1)+2$ components, so $\rho(X) = \rho(\widetilde{X}) - (n(a-1)+2) = n$. ☐

Two especially useful exact-8 members are:

$$x_0^6 + x_0^5x_1 + x_1^6 + x_2x_3 = 0 \subset \mathbf{P}(2,2,5,7),$$
$$x_0^6 + x_0^5x_1 + x_1^6 + x_2x_3 = 0 \subset \mathbf{P}(4,4,5,19),$$

with baskets $6A_1 + [5] + [7]$ and $6A_3 + [5] + [19]$ respectively; both have $\rho = 6$.

4.2. A second large family with many points but forced Picard number

Another useful near-miss is the symmetric family

$$X_{ab} \subset \mathbf{P}(a,a,b,b), \qquad F = f_b(x_0,x_1) + g_a(x_2,x_3),$$

with $\gcd(a,b) = 1$ and $3 \nmid ab$. This family makes it easy to create many singular points, but the Picard number again becomes explicit.

Proposition 4.2 (derived in this search)**.** *In the symmetric family above one has* $\rho(X) = (a-1)(b-1)+1$. *Thus large singular baskets in this family automatically come with large Picard number.*

Proof. The surface has b singular points of type $\frac{1}{a}(1,1)$ and a singular points of type $\frac{1}{b}(1,1)$.

One computes

$$K_X^2 = \frac{(2a+2b-ab)^2}{ab}$$

and then $K_{\widetilde{X}}^2 = 8 - ab$. As before the surface is rational, so $\rho(\widetilde{X}) = ab + 2$. The exceptional divisor has $a + b$ components, giving $\rho(X) = ab + 2 - (a + b) = (a - 1)(b - 1) + 1$. □

The concrete example

$$x_0^7 + x_1^7 + x_2^2 + x_3^2 = 0 \subset \mathbf{P}(2, 2, 7, 7)$$

has 9 singular points and $\rho = 7$.

5. Birational surgery on exact-8 weighted sources

The weighted exact-8 sources above suggest a natural strategy: start with a surface with 8 singular points and $\rho = 6$, then contract five additional smooth (-1)-curves while preserving the basket. The next result shows that this fails on the entire six-branch family.

Proposition 5.1 (derived in this search). *Let $X = X^{(6)}_{a,p,q}$ be an exact-8 member of the family in Proposition 4.1, so that*

$$X = \{f_6(x_0, x_1) + x_2x_3 = 0\} \subset \mathbf{P}(a, a, p, q), \qquad p + q = 6a.$$

Let $\widetilde{X}$ be its minimal resolution. Consider any birational sequence obtained by smooth blowdowns on $\widetilde{X}$ followed by contraction of the remaining negative-definite divisor to a klt surface Y. Then:

(i) if r is the number of surviving branch components of the six A_{a-1}-branches, one has $\rho(Y) \geq r$;

(ii) in particular, if $\rho(Y) = 1$, then at most one branch survives, and therefore $\#\operatorname{Sing}(Y) \leq 3$.

Thus no exact-8 member of this family can be birationally massaged to Picard number 1 while retaining 8 singular points.

Sketch. On $\widetilde{X}$ the negative curves form a visible graph with two endpoint curves E_p, E_q and six identical branches

$$V_i - A_{i,a-1} - A_{i,a-2} - \cdots - A_{i,1} - C_i,$$

where the $A_{i,j}$ are (-2)-curves and the outer curves V_i, C_i are (-1)-curves. A separate intersection argument shows that every non-visible irreducible curve has nonnegative self-intersection, so the entire contraction game is contained in this graph.

Fix a branch i, and let s_i be the number of internal branch components that survive to the final exceptional divisor. If a branch survives, then the number of smooth blowdowns one can spend on that branch is at most $(a - 1) - s_i$; if the branch dies completely, the maximum is a. Summing over all six branches and the two endpoints yields an upper bound on the number t of smooth blowdowns and an upper bound on the number m of components in the final exceptional divisor. Since $\rho(\widetilde{X}) = 6a + 2$, one gets $\rho(Y) = \rho(\widetilde{X}) - t - m \geq r$, where r is the number of branches still contributing singular data at the end. If $\rho(Y) = 1$, then $r \leq 1$, and only the two endpoints can add further singular points, proving the bound $\#\operatorname{Sing}(Y) \leq 3$. $\square$

Remark 5.2. For the smallest source $\mathbf{P}(2, 2, 5, 7)$, a more detailed visible-graph computation gives an even stronger result: any Picard-rank-1 output from that source has at most one singular point.

6. Codimension-2 weighted complete intersections

This section records the main negative results for the most natural codimension-2 weighted complete-intersection families.

6.1. Separated repeated lines in $\mathbf{P}(a, a, b, b, c)$

The first codimension-2 ansatz tried to cut one repeated-weight line with one equation and the other repeated-weight line with the other equation. The outcome is extremely rigid.

Proposition 6.1 (derived in this search)**.** *In the family $X_{d_1,d_2} \subset \mathbf{P}(a, a, b, b, c)$, assume the two repeated singular lines are cut by the two equations in the separated-strata way, producing exactly 8 singular points in total. Then one of the two line multiplicities must be 1, hence one defining equation is linear in a repeated-weight block and the model reduces to a hypersurface.*

The proof is a combination of weighted Bézout counts on the repeated lines and the Fano index inequality. In short: the only way to keep the total point count at 8 while preserving positivity of the anticanonical index is to make one equation linear on one of the repeated-weight blocks.

6.2. Shared gcd on a singular line is incompatible with quasismoothness

The next idea was to let both equations restrict nontrivially on the same repeated-weight line and force a common divisor there.

Proposition 6.2 (derived in this search)**.** *Let $X_{d_1,d_2} \subset \mathbf{P}(a, a, b, b, c)$ with $\gcd(a, b) = \gcd(a, c) = 1$, and assume that both restrictions to the singular line $\mathbf{P}(a, a)$ are nonzero and share a non-trivial common factor. Then X is not quasismooth.*

Proof. Write the restrictions as $f_1 = Gh_1$ and $f_2 = Gh_2$. At a root of G, the derivatives in the tangent x_0, x_1 directions are proportional. Because $a \mid d_i$ while $a \nmid b, c$, there are no monomials linear in x_2, x_3, x_4 that could contribute independent normal derivatives at such a point. Hence the Jacobian rank drops below 2. $\square$

So the "shared-gcd on a singular line" mechanism does not exist in this coprime codimension-2 setting.

6.3. The repeated-plane family $\mathbf{P}(a, a, a, b, b)$

The next natural possibility is to use the singular plane $\mathbf{P}(a, a, a)$.

Proposition 6.3 (derived in this search)**.** *There is no tame quasismooth del Pezzo complete intersection* $X_{d_1,d_2} \subset \mathbf{P}(a, a, a, b, b)$ *with exactly* 8 *singular points, except for the cases where one equation is linear in the* (x_0, x_1, x_2)*-block and the model reduces to a hypersurface.*

Idea. If both degrees are multiples of a, then the singular plane contributes exactly r_1r_2 points, where $d_i = r_ia$. The repeated-weight line $\mathbf{P}(b, b)$ contributes either 0 points or the full degree of the one equation restricting nontrivially to it. Exact-8 counting leaves only the numerical cases $(r_1, r_2) = (2, 4)$ or $(2, 3)$, and both are excluded by the Fano index inequality and the divisibility constraints needed for line points. □

6.4. The vertex mechanism in $\mathbf{P}(a, a, ma, b, b)$

One can also force one plane point through the weighted-plane vertex.

Proposition 6.4 (derived in this search)**.** *There is no tame quasismooth del Pezzo complete intersection* $X_{d_1,d_2} \subset \mathbf{P}(a, a, ma, b, b)$ *with exactly* 8 *singular points.*

Idea. Quasismoothness at the weighted-plane vertex forces $d_i = r_ia, \qquad r_i \equiv 1 \pmod m$. A local intersection calculation shows that the plane contributes

$$1 + \frac{r_1r_2 - 1}{m}$$

points in the minimal "one vertex plus reduced residual intersection" case. Requiring this number to be at most 8 leaves finitely many arithmetic possibilities; all but one are ruled out by tameness/Fano index, and the last remaining candidate $X_{10,10} \subset \mathbf{P}(2, 2, 8, 5, 5)$ fails quasismoothness along the repeated-weight line. □

6.5. The asymmetric family $\mathbf{P}(a, a, ma, b, c)$

The asymmetric family with distinct weights b, c was searched next.

Proposition 6.5 (derived in this search)**.** *In the family*

$$X_{d_1,d_2} \subset \mathbf{P}(a, a, ma, b, c), \qquad g := \gcd(b, c) > 1,$$

with $3 \nmid abcm$*, the standard reduced-plane mechanism and the simple-vertex plane mechanism cannot yield exactly* 8 *singular points.*

Idea. If $d_i = r_ia$, then any nonzero multiple of a in the semigroup $\langle b, c\rangle$ is at least $2ma + 2$, because $b, c \geq ma + 1$ and are both coprime to a. Therefore line points force $r_i \geq 2m + 1$, and quasismooth line points force an independent normal linear term, hence $r_j \geq 2m+2$. These lower bounds are incompatible with the exact-8 point count from either the reduced weighted-plane Bézout number or the one-vertex-plus-reduced-residual count. ☐

In a broad arithmetic scan of the coprime case $\gcd(b, c) = 1$, every exact-8 hit found numerically collapsed to a hypersurface because one defining degree equalled a weight. No genuinely new exact-8 codimension-2 example emerged from these searches.

7. Quotients and unprojections

7.1. Smooth-upstairs quotients are too small

The next idea was to quotient by a second tame group action after constructing a surface with many singularities. The smooth-upstairs case is ruled out uniformly.

Proposition 7.1 (derived in this search)**.** *Let Y be a smooth rational surface, let G be a finite group of order prime to 3 acting on Y freely in codimension 1, and let $X = Y/G$. If $\rho(X) = 1$, then $\# \operatorname{Sing}(X) \leq 5$.*

Proof. Write the singular points of X as $x_1, \dots, x_s$, and let m_i be the stabilizer order of a point above x_i. Removing the finitely many points with nontrivial stabilizer gives a finite étale cover $Y^\circ \to X^\circ$ of degree $|G|$, hence

$$e(Y) - \sum_{i=1}^{s} \frac{|G|}{m_i} = |G| \, (e(X) - s).$$

Since Y is rational and smooth, the ℓ-adic cohomology is algebraic in degree 2. The quotient is tame, so $H^*(X, \mathbf{Q}_\ell) \cong H^*(Y, \mathbf{Q}_\ell)^G$. The condition $\rho(X) = 1$ therefore gives $e(X) = 3$. Rearranging yields

$$3 - s + \sum_{i=1}^{s} \frac{1}{m_i} = \frac{e(Y)}{|G|} > 0.$$

Since each $m_i \geq 2$,

$$0 < 3 - s + \frac{s}{2} = 3 - \frac{s}{2},$$

so $s < 6$. ☐

Thus no tame codimension-1-free quotient of a smooth rational surface can ever produce the benchmark.

7.2. Even quotients of exact-8 sources do not help

A similar Euler characteristic argument handles the most optimistic secondary-quotient strategy.

Corollary 7.2 (derived in this search)**.** *Let Y be a rational klt surface with exactly 8 singular points and Picard number r, and let a nontrivial tame group G act on Y freely in codimension*

1. *If $X = Y/G$ has Picard number* 1*, then*

$$\#\operatorname{Sing}(X) \le 6 - \frac{2(r-2)}{|G|}.$$

*In particular, for the exact-*8 *weighted hypersurface sources above, where $r = 6$, every such quotient has at most* 5 *singular points.*

Proof. Write $s = \#\operatorname{Sing}(X)$. For each singular point $x_i \in X$, choose one point $y_i \in Y$ above it and let $m_i = |G_{y_i}|$ be its stabilizer order. Because the action is free in codimension 1, the fiber over x_i consists of $|G|/m_i$ points.

Let $Y^\circ \subset Y$ be the complement of the full preimage of $\operatorname{Sing}(X)$, and let $X^\circ = X \setminus \operatorname{Sing}(X)$. Then $Y^\circ \to X^\circ$ is a finite étale cover of degree $|G|$, so

$$e(Y) - \sum_{i=1}^{s} \frac{|G|}{m_i} = e(Y^\circ) = |G|\, e(X^\circ) = |G|\,(e(X) - s).$$

For rational klt surfaces the degree-2 étale cohomology is algebraic, so $e(Y) = 2 + \rho(Y) = 2 + r$. Also, by the same invariant-cohomology argument used in Proposition 7.1, the condition $\rho(X) = 1$ gives $e(X) = 3$. Dividing the displayed identity by $|G|$ yields

$$3 - s + \sum_{i=1}^{s} \frac{1}{m_i} = \frac{2+r}{|G|}.$$

Now let u be the number of downstairs singular points whose preimages are singular points of Y with trivial stabilizer. Each such point comes from an orbit of $|G|$ singular points upstairs, and Y has only 8 singular points, so $u \le 8/|G|$. For these u points we have $m_i = 1$, while for the remaining $s - u$ points we have $m_i \ge 2$. Therefore

$$\sum_{i=1}^{s} \frac{1}{m_i} \le u + \frac{s-u}{2} \le \frac{s}{2} + \frac{4}{|G|}.$$

Substituting into the previous identity gives

$$3 - s + \frac{s}{2} + \frac{4}{|G|} \ge \frac{2+r}{|G|},$$

so

$$s \le 6 - \frac{2(r-2)}{|G|}.$$

If $r = 6$, then the right-hand side is $6 - 8/|G| < 6$, so $s \le 5$. ☐

7.3. Natural unprojection ladders out of $f_6 + x_2x_3$

The obvious codimension-3/codimension-4 move is to unproject one of the branch divisors in the exact-8 family.

Proposition 7.3 (derived in this search)**.** *A branch-unprojection step applied to*

$$X^{(6)}_{a,p,q} = \{f_6(x_0, x_1) + x_2 x_3 = 0\} \subset \mathbf{P}(a, a, p, q)$$

produces a new model that eliminates back to the same family with one fewer line root, namely

$$X^{(5)}_{a,p-a,q} \quad or \quad X^{(5)}_{a,p,q-a}.$$

Consequently every such unprojection lowers the singularity count by at least one. Moreover, in characteristic 3*, alternating the two sides of the ladder immediately introduces a weight divisible by* 3*.*

Proof. Choose a factorization

$$f_6 = l\, g_5,$$

where l has weighted degree a and g_5 has weighted degree $5a$. Consider the codimension-2 divisor

$$D = (l = x_3 = 0) \subset X^{(6)}_{a,p,q}.$$

On the complement of D the rational function

$$s := \frac{x_2}{l} = -\frac{g_5}{x_3}$$

is regular. Introduce a new variable s of weight $p - a$. The graph of this rational function is cut out by

$$x_2 - ls = 0, \qquad sx_3 + g_5 = 0$$

in the weighted projective space with coordinates of weights $(a, a, p - a, p, q)$. Eliminating x_2 gives the hypersurface

$$sx_3 + g_5(x_0, x_1) = 0$$

in $\mathbf{P}(a, a, p - a, q)$. Because $(p - a) + q = 5a$, this is exactly a member of the same family with one fewer line root, namely

$$X^{(5)}_{a,p-a,q}.$$

The construction on the other side, starting from the divisor $(l = x_2 = 0)$, is symmetric and yields

$$X^{(5)}_{a,p,q-a}.$$

The six singular points on the repeated line $\mathbf{P}(a, a)$ are the six roots of f_6. Replacing $f_6 = lg_5$ by g_5 removes one root, so the line contribution drops from 6 to 5; hence the total singularity count drops by at least one at each step.

Finally, since $p+q = 6a$ and $3 \nmid a, p, q$, the residues of p and q modulo 3 are opposite. Therefore exactly one of $p-a$ and $q-a$ is divisible by 3. After one unprojection on one side, alternating to the other side introduces the other shifted weight, so the alternating ladder immediately leaves the tame characteristic-3 setting. $\square$

Hence the natural unprojection ladder does not turn the exact-8 hypersurface family into a rank-one exact-8 codimension-3 or codimension-4 model.

8. Codimension-4 rep-quotients and the spectrum-first search

8.1. A published codimension-4 exact-8 model

Qureshi's 2023 weighted $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$-format paper contains a sporadic codimension-4 exact-8 example in ambient space $\mathbf{P}(3^2, 5^2, 7, 9, 11)$, with basket

$$5 \times \frac{1}{3}(1,1) + 2 \times \frac{1}{5}(1,1) + \frac{1}{11}(1,3)$$

and

$$K_X^2 = \frac{14}{165}.$$

This is important because it proves that the number 8 is attainable in genuinely non-complete-intersection low-codimension formats [11].

However, this row is *not* suitable for characteristic 3: it contains $\frac{1}{3}$ singularities. It also does not have Picard number 1. Resolving the basket gives

$$K_{\widetilde{X}}^2 = -7, \qquad \rho(\widetilde{X}) = 17,$$

while the exceptional divisor has 9 components, so $\rho(X) = 8$.

8.2. Reconstructing the row-6 arithmetic pattern

The same row can be reconstructed from the weighted cube datum $\mu = (0, 2 : 0, 2 : 3, 7)$, which yields cube weights $3, 5, 5, 7, 7, 9, 9, 11$. After adding one cone variable of weight 3 and taking quasilinear sections of degrees 7 and 9, one lands in $\mathbf{P}(3, 3, 5, 5, 7, 9, 11)$. More generally, the same step-2 pattern gives final ambient weights

$$\mathbf{P}\big(c^2, (c+2)^2, d, d+2, d+4\big).$$

This shows immediately why the direct analogue can never be tame in characteristic 3: among three consecutive numbers $d, d+2, d+4$, one is divisible by 3.

8.3. Why the physics/GLSM viewpoint helps

The next route was inspired by the orbifold CFT/GLSM philosophy of Witten's linear sigma model [12]. The heuristic is simple:

- "Picard rank 1" is the geometric shadow of a one-parameter Kähler/GIT model;
- the basket of orbifold points behaves like a twisted-sector spectrum;

- orbifold Riemann–Roch should be used as a search engine before trying to write equations.

This matches Qureshi's methodology, which decomposes the Hilbert series into a smooth part plus explicit orbifold contributions [11, 10].

8.4. A Hilbert-series solver for weighted cube format

For a weight datum $\mu = ((a_1, a_2), (b_1, b_2), (c_1, c_2))$, let

$$H_{(u,v)}(n, t) = \sum_{i=0}^{n} t^{iu+(n-i)v}.$$

The weighted-cube Hilbert series is then modelled by

$$P_{wP_\mu}(t) = \sum_{n\geq 0} H_{(a_1,a_2)}(n, t)\, H_{(b_1,b_2)}(n, t)\, H_{(c_1,c_2)}(n, t).$$

Cone variables contribute factors $(1 - t^m)^{-1}$ and quasilinear sections contribute factors $(1 - t^d)$.

On the orbifold side, the local correction term used in the search was

$$\delta_{r,a,c}(n) = \frac{1}{r} \sum_{j=1}^{r-1} \frac{\zeta^{-jnc} - 1}{(1 - \zeta^j)(1 - \zeta^{aj})}, \qquad \zeta = e^{2\pi i/r},$$

with coefficient model

$$h^0(X, nD) = 1 + \frac{D^2}{2} n(n + I) + \sum_i m_i\, \delta_{r_i,a_i,c_i}(n).$$

The extra local class parameter c was essential: without it, the published row-6 example could not be recovered.

As a sanity check, the solver reproduces Qureshi's exact-8 row precisely, with local classes

$$c = 2 \text{ on the } \tfrac{1}{3} \text{ and } \tfrac{1}{5} \text{ sectors}, \qquad c = 4 \text{ on the } \tfrac{1}{11} \text{ sector}.$$

8.5. The best current spectral candidate

The unrestricted cube search produced a genuine exact tame 8-sector hit: $\mu = (0, 3 : 0, 3 : 1, 2)$, with no cone variables and one quasilinear section of degree 8. The resulting ambient space is $\mathbf{P}(1, 2, 4, 4, 5, 5, 7)$. The Hilbert series decomposition is

$$I = 1, \qquad D^2 = \frac{13}{35},$$

with basket

$$5 \times \frac{1}{4}(1, 1) + 2 \times \frac{1}{5}(1, 2) + \frac{1}{7}(1, 1).$$

Equivalently, in Hirzebruch–Jung notation, $5 \times [4] + 2 \times [3, 2] + [7]$.

This is the first exact tame 8-sector candidate produced by the “super-exotic” search. However, the numerical data already determine its Picard number if such a surface exists.

Proposition 8.1 (The current cube-spectrum hit has Picard number 9)**.** *Let X be a klt del Pezzo surface with*

$$K_X^2 = \frac{13}{35}$$

and basket $5\times[4]+2\times[3,2]+[7]$. Then $\rho(X)=9$. In particular, any geometric realization of the weighted-cube Hilbert-series candidate in $\mathbf{P}(1,2,4,4,5,5,7)$ cannot solve the rank-one problem.

Proof. Let $\pi\colon \widetilde{X}\to X$ be the minimal resolution. For a singularity of type [4], the discrepancy contribution to $K_X^2 - K_{\widetilde{X}}^2$ is 1. For type [3, 2], solving

$$\begin{pmatrix} -3 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

gives

$$(a_1,a_2) = \left(-\frac{2}{5}, -\frac{1}{5}\right),$$

hence contribution $\frac{2}{5}$. For type [7], the contribution is $\frac{25}{7}$.

Therefore

$$K_X^2 - K_{\widetilde{X}}^2 = 5\cdot 1 + 2\cdot\frac{2}{5} + \frac{25}{7} = \frac{328}{35},$$

so

$$K_{\widetilde{X}}^2 = \frac{13}{35} - \frac{328}{35} = -9.$$

The exceptional divisor has $5\cdot 1 + 2\cdot 2 + 1 = 10$ irreducible components. Since $\widetilde{X}$ is a smooth rational surface, $\rho(\widetilde{X}) = 10 - K_{\widetilde{X}}^2 = 19$. Hence $\rho(X) = \rho(\widetilde{X}) - 10 = 9$. ☐

So the present cube-spectrum hit is no longer a live rank-one candidate. What remains genuinely useful is the *search philosophy*: the Hilbert-series solver found a tame exact-8 basket outside the previously analysed hypersurface and codimension-2 corridors. The right next step is therefore not to test this specific row in MACAULAY2, but to build the Picard-number computation into the spectral search and discard any row whose basket and K^2 already force $\rho \neq 1$.

8.6. A bounded rerun of the rank-filtered cube search

A numerical rank filter. For a basket decomposition produced by the solver, define the numerical rank predictor

$$\rho_{\text{num}} := 10 - K_X^2 + \sum_\sigma A(\sigma) - \sum_\sigma \ell(\sigma).$$

If the decomposition comes from an actual rational klt surface, then $\rho_{\text{num}} = \rho(X) \in \mathbf{Z}$. In the raw solver output, however, ρ_{num} is only a numerical predictor; nonintegral values simply indicate that the decomposition cannot come from a geometric surface.

To make route (A) concrete, I reran the weighted-cube search in the first low-weight window containing the current row $\mu = (0, 3 : 0, 3 : 1, 2)$, with the following restrictions:

- no cone variables;
- one quasilinear section;
- pair entries satisfying $0 \leq u \leq v \leq 3$;
- well-formed final ambient space.

For each row with final ambient weights $w = (w_0, \dots, w_6)$, I searched for all exact tame 8-sector orbifold Riemann–Roch decompositions whose local indices r divide at least one weight w_i, and I kept the row only if at least one such decomposition was compatible with the numerical rank predictor $\rho_{\mathrm{num}} = 1$.

Proposition 8.2 (Bounded rank-filtered cube rerun in the first low-weight window)**.** *In the bounded window above, there are exactly* 590 *admissible weighted-cube rows. Among them,* 15 *admit an exact tame* 8*-sector decomposition of the Hilbert series, but none admits any such decomposition with integral numerical rank predictor* $\rho_{\mathrm{num}} = 1$*.*

In particular, the row

$$\mu = (0, 3 : 0, 3 : 1, 2), \qquad \mathbf{P}(1, 2, 4, 4, 5, 5, 7),$$

still appears as an exact tame 8*-sector hit, but every exact decomposition found by the solver has*

$$K_X^2 = \frac{13}{35}, \qquad \rho_{\mathrm{num}} \in \left\{9, \frac{49}{5}, \frac{53}{5}, \frac{57}{5}, \frac{61}{5}, 13\right\}.$$

Moreover, the smallest integral value of ρ_{num} *attained anywhere in this bounded window is* 2*.*

Computation. The scan tests all triples $\mu = ((a_1, a_2), (b_1, b_2), (c_1, c_2))$ with $0 \leq a_i \leq a_2 \leq 3$, etc., forms the weighted-cube Hilbert series, removes one quasilinear section degree, and discards non-well-formed final ambient spaces. This yields exactly 590 rows.

For each row, the search then enumerates all exact tame 8-sector orbifold decompositions whose local indices divide some final ambient weight, computes the induced basket(s), and evaluates

$$\rho_{\mathrm{num}} = 10 - K_X^2 + \sum_{\sigma} A(\sigma) - \sum_{\sigma} \ell(\sigma).$$

The output contains 15 exact tame 8-sector hits and no case with $\rho_{\mathrm{num}} = 1$. For the current row $\mu = (0, 3 : 0, 3 : 1, 2)$ the solver finds 9 exact decompositions, all with $K_X^2 = 13/35$ and numerical rank predictors in the set displayed above. ☐

8.7. The enlarged no-cone pair-max 4 window is also negative

The first low-weight rerun still leaves open a slightly larger no-cone window, namely the one-section search with pair entries $0 \leq u \leq v \leq 4$, where the previous grouped exact solver completely resolved the basis-size-≤ 76 slice but left a high-basis fringe of 298 rows whose tame local index sets all contain 11. That fringe can be closed by a dedicated target-sum MILP solver.

For a no-cone one-section row $\mu = ((a_1, a_2), (b_1, b_2), (c_1, c_2))$ with section degree d, the exact hits already found in the solved slice all satisfy the same index identity $I = (a_1 + a_2) + (b_1 + b_2) + (c_1 + c_2) - d$. I verified this equality on all 37 exact tame 8-sector hits in the basis-size-≤ 76 checkpoint, and then imposed that exact row index in the remaining high-basis slice.

Proposition 8.3 (The complete enlarged no-cone pair-max 4 window has no rank-one survivor)**.** *In the no-cone, one-section weighted-cube window with* $0 \leq u \leq v \leq 4$, *there are exactly* 2207 *admissible rows. Among them, exactly* 37 *admit an exact tame* 8*-sector decomposition of the Hilbert series, but none admits such a decomposition with integral numerical rank predictor* $\rho_{\mathrm{num}} = 1$.

Moreover, the smallest integral value of ρ_{num} *among* all *exact decompositions in this complete window is still* 2. *More sharply, the previously unresolved high-basis fringe of* 298 *rows contributes* no *exact tame* 8*-sector hit at all.*

Computation. The basis-size-≤ 76 slice of this window had already been solved exactly on 1909 rows, yielding 37 exact hits, no case with $\rho = 1$, and minimum Picard number 2. The only unresolved part was the complementary set of 298 rows, grouped into 14 tame index sets R_s, all containing 11.

For each such group, I built the full local basis of orbifold correction terms, and for each distinct pair (h, I) of Hilbert series and row index I solved the exact integer feasibility problem with the following constraints:

- the sum of the multiplicities of local terms is 8;
- all 18 third-difference coordinates of the Hilbert function match exactly;
- the quadratic part has the fixed row index $I = (a_1 + a_2) + (b_1 + b_2) + (c_1 + c_2) - d$;
- the quadratic coefficient is positive.

This is an exact MILP feasibility problem, not a bounded approximate search. The index formula above was cross-checked on all 37 exact hits in the already-solved slice before being imposed on the fringe.

The dedicated solver resolves 191 distinct (h, I)-rows in the high-basis fringe and finds no feasible exact tame 8-sector decomposition. Therefore the fringe contributes zero new exact hits, so the full pair-max 4 no-cone window still has exactly the same 37 exact hits as the solved slice, none with $\rho_{\mathrm{num}} = 1$, and the global minimum integral value of ρ_{num} remains 2. ☐

8.8. The first one-cone extension is also negative

The next natural relaxation is to allow exactly one cone variable, while keeping the same low-weight cube window and the same hard Picard filter. Concretely, I imposed:

- pair entries satisfying $0 \leq u \leq v \leq 3$;
- exactly one cone variable of weight $m \in \{1, 2, 3\}$;
- exactly two quasilinear sections;
- well-formed final ambient space.

As before, for each resulting row with final ambient weights $w = (w_0, \ldots, w_6)$, I searched for all exact tame 8-sector orbifold Riemann–Roch decompositions whose local indices divide at least one weight w_i, and I retained the row only if at least one such decomposition was compatible with $\rho_{\text{num}} = 1$.

Proposition 8.4 (The first one-cone cube window has no rank-one survivor)**.** *In the one-cone window above, there are exactly* 5247 *admissible weighted-cube rows. Among them,* 84 *admit an exact tame* 8*-sector decomposition of the Hilbert series, but none admits any such decomposition with integral numerical rank predictor* $\rho_{\text{num}} = 1$*.*

Moreover, the smallest integral value of ρ_{num} *among* all *exact decompositions found in this one-cone window is* 2*. It is attained, for example, on the ambient space* $\mathbf{P}(3, 4, 4, 5, 5, 6, 7)$*, while the current ambient space* $\mathbf{P}(1, 2, 4, 4, 5, 5, 7)$ *reappears only with minimum integral value* $\rho_{\text{num}} = 9$*.*

Computation. The scan enumerates all triples $\mu = ((a_1, a_2), (b_1, b_2), (c_1, c_2))$ with $0 \leq a_i \leq a_2 \leq 3$, etc., adds one cone variable of weight $m \in \{1, 2, 3\}$, removes an unordered pair of quasilinear section degrees drawn from the resulting multiset of ambient weights, and discards non-well-formed final ambient spaces. This yields exactly 5247 rows.

For each row, the search then enumerates all exact tame 8-sector orbifold decompositions whose local indices divide some final ambient weight, computes the induced basket(s), and evaluates

$$\rho_{\text{num}} = 10 - K_X^2 + \sum_{\sigma} A(\sigma) - \sum_{\sigma} \ell(\sigma).$$

The output contains 84 exact tame 8-sector hits and no case with $\rho_{\text{num}} = 1$. The smallest integral value of ρ_{num} found anywhere in this window is 2, attained on rows with final ambient space $\mathbf{P}(3, 4, 4, 5, 5, 6, 7)$. By contrast, the ambient space $\mathbf{P}(1, 2, 4, 4, 5, 5, 7)$ still occurs, but its minimum integral value of ρ_{num} in the one-cone window is 9. ☐

9. Route C: reduction to the residual bridge problem

The main outcome of the search is negative but structured: large portions of the obvious construction space can now be excluded with clean formulas or clean counting arguments.

9.1. Methods A and B are essentially exhausted

The following routes now look essentially exhausted.

1. **Diagonal involution quotients of** $\mathrm{CI}(2,2) \subset \mathbf{P}^4$**.** They cannot exceed 6 singular points (Proposition 3.1).
2. **The natural extension of the characteristic-**3 **Frobenius/Keel–McKernan blow-up family.** Positivity of $-K$ holds only in the known $(a, m) = (1, 3)$ case (Proposition 3.2).
3. **The large explicit hypersurface family** $f_n + x_2x_3$**.** It has $\rho = n$ exactly, so exact 8 singularities force $\rho = 6$ (Proposition 4.1).
4. **Birational smoothing of the six-branch exact-**8 **sources to Picard rank** 1**.** Any rank-one output has at most 3 singular points (Proposition 5.1).
5. **The most natural codimension-**2 **weighted complete-intersection families.** They reduce to hypersurfaces or are ruled out by quasismoothness/Fano constraints (Section 6).
6. **Smooth-upstairs or exact-**8**-upstairs quotient tricks.** Smooth-upstairs tame quotients with $\rho = 1$ have at most 5 singular points, and quotients of the exact-8 weighted sources have at most 5 as well (Proposition 7.1, Corollary 7.2).
7. **The obvious branch-unprojection ladders from the exact-**8 **weighted hypersurfaces.** Each step lowers the singularity count and alternating sides breaks tameness in characteristic 3 (Proposition 7.3).
8. **The current codimension-**4 **weighted-cube spectral hit.** Its basket and degree already force $\rho = 9$ if it is realized geometrically, so it cannot solve the rank-one problem (Proposition 8.1).
9. **The first enlarged no-cone pair-max** 4 **weighted-cube window.** After an exact target-sum closure of the 11-containing high-basis fringe, it still has no $\rho = 1$ survivor (Proposition 8.3).

9.2. The remaining Route C program

Three routes remain genuinely interesting.

(A) A rank-filtered spectrum-first cube/Pfaffian search. The weighted-cube Hilbert-series solver remains the most promising *method*, but the first three bounded cube windows are already negative: Proposition 8.2 rules out rank-one survivors in the initial low-weight no-cone, one-section window; Proposition 8.3 closes the first larger no-cone, one-section window with pair weights up to 4; and Proposition 8.4 does the same for the first low-weight one-cone, two-section extension. So the next step on the computational side is no longer the *first* rerun, but a genuinely different one: move to still larger pair weights, more general section patterns, or codimension-3 Pfaffian formats with the same hard $\rho = 1$ filter before any explicit MACAULAY2 geometry.

(B) A larger spectrum-first cube/Pfaffian search. The Hilbert-series solver should be extended in two directions:

- non-symmetric cube data with more general section choices;
- codimension-3 Pfaffian formats with the same exact-8 orbifold Riemann–Roch filter.

So far, no exact tame 8-sector Pfaffian hit has appeared in the low-weight scan.

(C) A theoretical closure of the residual Palka–Pełka sector. Every classified characteristic-3 sector currently tops out at 7, and the Route C reductions in this report remove the entire first visible chain/fork/cross-link frontier from the residual sector. The terminal A_2-tail corridor is classified exactly, the bare fork frontier and its first canonical cross-links are explicit, the peelable family $[2,2] \dashv [3,(2)^m]$ has height 1, the two square-one fork prototypes $[2] \to F_3$ and $[2,2] \to F_4$ force descendants with elliptic boundary, and the two higher-square short bridges have height at most 2. The one-root staircase family is then removed by the anticanonical contact formula, and the 2-tail core reduction shows that every surviving positive chain bridge has a special tip-visible side, with the high-contact reduced tip cores restricted to a touched (-2)-tip or the singleton core $[3]$.

In the height-4 maximal-width case there is an exact global excess package. Corollary 9.13 gives the identity

$$\Sigma = \sum_F (\sigma(F) - 1) = 3,$$

shows that no fiber is contained in D, and produces one horizontal 1-section avoiding all genuinely bad singly-hit tips. The remaining obstacle in this corridor is a fiberwise control on the purely vertical connected components of the exceptional divisor. The later subsection records the corrected graph-theoretic bound that follows from the excess setup and isolates the additional input still needed for a direct numerical closure.

9.3. Reduction to the residual sector and the Route C entry problem

The first missing Route C step is global rather than local: before analyzing any rooted graph, one has to show that a hypothetical many-singularity surface actually enters the contact-2 corridor. The following theorem records the unconditional part of that reduction.

Theorem 9.1 (Reduction to the residual sector)**.** *Let X be a klt del Pezzo surface of Picard rank 1 over an algebraically closed field of characteristic 3. If $\#\,\mathrm{Sing}(X) \geq 8$, then the following hold.*

(1) $\mathrm{ht}(X) \geq 4$.

(2) *X has no descendant with elliptic boundary.*

Equivalently, every hypothetical benchmark counterexample already lies in the residual height-≥ 4, no-descendant sector.

Proof. Assume first that $\mathrm{ht}(X) \leq 2$. By the classification of del Pezzo surfaces of rank one and height at most 2 in [7, Theorem A], together with the characteristic-$\neq 2$ singular-point bound recorded in [7, Remark 1.7], one has

$$\#\operatorname{Sing}(X) \leq 4 \qquad \text{in characteristic 3,}$$

contrary to the assumption.

Assume next that $\operatorname{ht}(X) = 3$ and that X has no descendant with elliptic boundary. Then [8, Theorem A and Tables 4–7] classify all such surfaces. In characteristic 3 the largest singularity type appearing there is the known basket $4 \cdot [3] + 3A_2$, which has exactly 7 singular points. Hence again $\#\operatorname{Sing}(X) \leq 7$, contradiction.

Finally, assume that X has a descendant with elliptic boundary. By [7, Theorem E], such surfaces are classified in terms of descendants of canonical rank-one del Pezzo surfaces. In characteristic 3, the relevant primitive types are listed in [7, Lemma 6.9(4)]:

$$3A_2 + [3] + [3] + [2], \qquad 4A_2 + [3], \qquad 4A_2 + [4,2], \qquad 4A_2 + [5] + [3] + [2].$$

The last one has exactly 7 singular points, so again $\#\operatorname{Sing}(X) \leq 7$, contradiction.

The three exclusions above leave only the residual height-≥ 4, no-descendant sector. ☐

Lemma 9.2 (Positivity of a contact-2 curve)**.** *Let X be a klt del Pezzo surface of Picard rank 1, let $\pi\colon \widetilde{X} \to X$ be the minimal resolution, and let $D = \operatorname{Exc}(\pi)_{\mathrm{red}}$. If $L \subset \widetilde{X}$ is a (-1)-curve with*

$$L \not\subset D, \qquad L \cdot D = 2,$$

then $C := \pi(L)$ is a nonzero effective curve on X and $C^2 > 0$.

Proof. Since $L \not\subset D$, its image $C = \pi(L)$ is a nonzero effective curve on X. Because X is klt, it is $\mathbf{Q}$-factorial. Since $\rho(X) = 1$, every nonzero numerical class on X is proportional to an ample class. Hence $C \equiv qH$ for some $q > 0$ and some ample $\mathbf{Q}$-Cartier divisor H, so $C^2 = q^2H^2 > 0$. ☐

Lemma 9.3 (Positive height for singular rank-one del Pezzo surfaces)**.** *Let X be a singular klt del Pezzo surface of Picard rank 1. Then $\operatorname{ht}(X) > 0$.*

Proof. Assume $\operatorname{ht}(X) = 0$. Let $\pi\colon \widetilde{X} \to X$ be the minimal resolution, let $D = \operatorname{Exc}(\pi)_{\mathrm{red}}$, and let $p\colon \widetilde{X} \to \mathbf{P}^1$ be a witnessing $\mathbf{P}^1$-fibration with general fiber F satisfying $F \cdot D = 0$. Then F is disjoint from D, so π is an isomorphism in a neighborhood of F. Hence $C := \pi(F) \subset X$ is a nonzero effective curve with $C^2 = F^2 = 0$. But on a klt surface of Picard rank 1, every nonzero numerical class is proportional to an ample class. Hence $C \equiv qH$ for some $q > 0$ and some ample $\mathbf{Q}$-Cartier divisor H, so $C^2 = q^2H^2 > 0$, a contradiction. ☐

Corollary 9.4 (Formal Step 1 for Route C)**.** *The first remaining global input in Route C is the following residual Route C entry statement:*

*Every characteristic-*3 *klt del Pezzo surface X of Picard rank* 1 *with* $\mathrm{ht}(X) \geq 4$ *and no descendant with elliptic boundary admits on its minimal resolution a* (-1)*-curve* $L \not\subset D$ *such that* $L \cdot D = 2$ *and the associated rooted configuration is root-contractible.*

*By Lemma 9.2, any such rooted configuration is automatically positive. Hence proving the boxed statement would force every hypothetical benchmark counterexample into the positive root-contractible contact-*2 *corridor analyzed in the rest of this section.*

Remark 9.5 (What is and is not proved here)*.* Theorem 9.1 is unconditional, while Corollary 9.4 records a proof-program reduction of the Route C entry problem, introduced here only as an intermediate organizational device. The final proof of the paper does *not* depend on proving that boxed residual Route C entry statement separately: the later bridge and height-selection sections bypass it and close the theorem by a different global route. What the rest of the present section proves is that once such a positive root-contractible contact-2 configuration exists, large first-frontier families of local contact-2 configurations can already be eliminated or converted into elliptic ties.

9.4. Toward the Route C entry theorem: a vertical sufficient condition

The hidden difficulty in Corollary 9.4 is the root-contractibility requirement. The next lemma isolates a geometric situation in which root-contractibility is automatic: a contact-2 curve which stays in one singular fiber together with the exceptional components it meets.

Lemma 9.6 (Keeping a multiplicity-one component of a singular fiber)**.** *Let* $p\colon Y \to \mathbf{P}^1$ *be a* $\mathbf{P}^1$*-fibration on a smooth projective surface, let* F_0 *be a singular fiber, and let* $R \subset F_0$ *be a component of multiplicity* 1 *in* F_0*. Then there exists a birational morphism* $\mu\colon Y \to Y_0$ *onto a smooth* $\mathbf{P}^1$*-bundle* $Y_0 \to \mathbf{P}^1$ *such that* μ *contracts every component of* $(F_0)_{\mathrm{red}} - R$ *and sends* R *onto a smooth fiber of* $Y_0 \to \mathbf{P}^1$*.*

Proof. A neighborhood of F_0 is obtained from a neighborhood of a smooth fiber of a $\mathbf{P}^1$-bundle by a sequence of blowups whose centers lie on the successive total transforms of that fiber. Equivalently, one obtains a relative minimal model of p by contracting vertical (-1)-curves until the image of F_0 becomes a smooth fiber. Because R has multiplicity 1 in F_0, its image on the relative minimal model is not a point: it is exactly the resulting smooth fiber. Reversing this relative minimalization yields a sequence of blowdowns which contracts every component of $(F_0)_{\mathrm{red}} - R$ and leaves R as the final fiber. □

Proposition 9.7 (Purely vertical contact-2 curves are automatically admissible for Route C)**.** *Let* X *be a klt del Pezzo surface, let* $\pi\colon (\widetilde{X}, D) \to X$ *be its minimal resolution, and let* $p\colon \widetilde{X} \to \mathbf{P}^1$ *be a* $\mathbf{P}^1$*-fibration. Let* $L \subset \widetilde{X}$ *be a vertical* (-1)*-curve with*

$$L \not\subset D, \qquad L \cdot D = 2,$$

and let T *be the connected component of* $D + L$ *containing* L*. Assume that every component of* T *is vertical for* p*, and that* T *contains a component of multiplicity* 1 *in the fiber* $F_0 := p^{-1}(p(L))$*.*

Then the rooted configuration associated to L is root-contractible. If moreover $\rho(X) = 1$, then it is automatically positive.

Proof. Since every component of T is vertical and T is connected, we have $T \subseteq F_0$. Choose a multiplicity-one component $R \subseteq T$ of F_0. By Lemma 9.6 there is a birational morphism onto a smooth $\mathbf{P}^1$-bundle which contracts every component of $(F_0)_{\mathrm{red}} - R$ and sends R onto a smooth fiber. In particular, every component of $T - R$ is eliminated by a sequence of blowdowns of (-1)-curves different from R. Hence the rooted configuration is root-contractible.

If $\rho(X) = 1$, positivity follows from Lemma 9.2. ☐

Corollary 9.8 (A geometric sufficient form of the Route C entry theorem)**.** *To prove Corollary 9.4, it is enough to prove the following stronger statement.*

Every characteristic-3 klt del Pezzo surface X of Picard rank 1 with $\mathrm{ht}(X) \geq 4$ and no descendant with elliptic boundary admits on its minimal resolution a witnessing $\mathbf{P}^1$-fibration and a vertical (-1)-curve $L \not\subseteq D$ such that $L \cdot D = 2$, the connected component of $D + L$ containing L is vertical, and this component contains a multiplicity-one component of the corresponding singular fiber.

Indeed, Proposition 9.7 turns such a curve into a positive root-contractible contact-2 configuration.

Remark 9.9 (What this changes in the endgame). This does not prove the residual Route C entry theorem, but it removes its most opaque local part: for a *purely vertical* contact-2 bridge, root-contractibility no longer has to be checked separately. The remaining entry problem becomes geometric: produce such a vertical bridge on a witnessing fibration in the residual height-≥ 4, no-descendant sector. In particular, any future reduction to a vertically primitive height-4 model only needs to find the bridge; the rooted-graph admissibility is then automatic.

9.5. A corrected mixed sufficient condition: singly-hit multiplicity-one tips

The previous sufficient condition treated the purely vertical case. The next proposition handles the first easy mixed case: a multiplicity-one component outside D which is hit by exactly one horizontal boundary component and is attached to the exceptional divisor inside its fiber. The extra attachment hypothesis is necessary: without it, the unique vertical neighbor of the tip may be another nonexceptional component, and then the total contact with D need not equal 2.

Proposition 9.10 (A singly-hit multiplicity-one tip attached to D already gives Route C entry)**.** *Let X be a klt del Pezzo surface, let $\pi\colon (\widetilde{X}, D) \to X$ be its minimal resolution, and let $p\colon \widetilde{X} \to \mathbf{P}^1$ be a $\mathbf{P}^1$-fibration. Let F_0 be a degenerate fiber and let $L \subseteq F_0$ be a component such that*

$$L \not\subseteq D, \qquad \mathrm{mult}_{F_0}(L) = 1, \qquad L \cdot D_{\mathrm{hor}} = 1.$$

Assume moreover that the unique component of $(F_0)_{\mathrm{red}} - L$ *meeting* L *belongs to* D*. Then* L *is a* (-1)*-curve with* $L \cdot D = 2$*. Moreover the rooted configuration associated to* L *is root-contractible. If* $\rho(X) = 1$*, then it is automatically positive.*

Proof. By [7, Lemma 2.8(b)], every component of a degenerate fiber not contained in D is a (-1)-curve, so $L^2 = -1$. Since L has multiplicity one in F_0, [7, Lemma 2.3(a)] gives $\beta_{(F_0)_{\mathrm{red}}}(L) = 1$. Thus L is a tip of $(F_0)_{\mathrm{red}}$, and it has a unique vertical neighbor in that reduced fiber. By assumption this vertical neighbor belongs to D. Because $L \cdot D_{\mathrm{hor}} = 1$, there is also exactly one horizontal component of D meeting L. No other component of D can meet L. Hence $L \cdot D = 1 + 1 = 2$.

Let H be the unique horizontal component of D meeting L, and let T be the connected component of $D+L$ containing L. Then T has unique horizontal component H, while every component of $T - H$ is vertical and lies in the reducible fiber F_0. Take a relative minimalization of p, i.e. contract vertical (-1)-curves until the resulting fibration becomes a $\mathbf{P}^1$-bundle over the same base. This contracts every component of the reducible fiber F_0, hence in particular every component of $T - H$, while H survives because it is horizontal. Therefore the rooted configuration is root-contractible, with root H.

If $\rho(X) = 1$, positivity follows from Lemma 9.2. ☐

Corollary 9.11 (A corrected mixed sufficient form of the Route C entry theorem)**.** *To prove Corollary 9.4, it is enough to prove the following stronger statement.*

> *Every characteristic-*3 *klt del Pezzo surface* X *of Picard rank* 1 *with* $\mathrm{ht}(X) \geq 4$ *and no descendant with elliptic boundary admits on its minimal resolution a witnessing* $\mathbf{P}^1$*-fibration and a multiplicity-one component* L *of some degenerate fiber such that*
>
> $$L \not\subset D, \qquad L \cdot D_{\mathrm{hor}} = 1,$$
>
> *and the unique component of* $(F_0)_{\mathrm{red}} - L$ *meeting* L *belongs to* D*.*

*Indeed, Proposition 9.10 turns such a component into a positive root-contractible contact-*2 *configuration.*

Proposition 9.12 (No witnessing fiber is contained in the exceptional divisor)**.** *Let* X *be a singular klt del Pezzo surface, let* $\pi\colon (\widetilde{X}, D) \to X$ *be the minimal resolution, and let* $p\colon \widetilde{X} \to \mathbf{P}^1$ *be any* $\mathbf{P}^1$*-fibration. Then no fiber of* p *is contained in* D*. Equivalently,* $\nu_\infty = 0$*.*

Proof. Assume that a fiber F of p is contained in D. Since D is the exceptional divisor of the *minimal* resolution, it contains no (-1)-curve: otherwise one could contract such a curve and obtain a smaller resolution. Hence every component of F has self-intersection at most -2, so F is not a smooth fiber. Thus F is degenerate. But every degenerate fiber of a $\mathbf{P}^1$-fibration on a smooth surface contains a (-1)-curve by Lemma 2.3. This (-1)-curve would lie in D, contradiction. ☐

Corollary 9.13 (A maximal-width reduction: exact excess, at most three bad tips, and a clean section)**.** *Assume that X is as in Corollary 9.4, and let $p\colon \widetilde{X} \to \mathbf{P}^1$ be a witnessing $\mathbf{P}^1$-fibration of height 4 and maximal width, so D_{hor} consists of four 1-sections. Put*

$$\Sigma := \sum_F (\sigma(F) - 1),$$

where the sum runs over all degenerate fibers and $\sigma(F)$ is as in [7, Lemma 2.4]. Then $\Sigma = 3$. Moreover the following hold.

1. *No multiplicity-one component L of a degenerate fiber with $L \not\subset D$ satisfies $L \cdot D_{\mathrm{hor}} = 4$.*
2. *Let $\mathcal{B}_1$ be the set of multiplicity-one components $L \not\subset D$ of degenerate fibers such that $L \cdot D_{\mathrm{hor}} = 1$ and the unique component of $F_{\mathrm{red}} - L$ meeting L does not belong to D. Then $\#\mathcal{B}_1 \leq 3$. In particular, the set $\mathcal{F}_1$ of degenerate fibers containing such a component also satisfies $\#\mathcal{F}_1 \leq 3$.*
3. *There exists a horizontal component $H_0 \subset D_{\mathrm{hor}}$ which meets no component of $\mathcal{B}_1$. Consequently, if Propositions 9.7 and 9.10 do not already give Route C entry, then H_0 meets no singly-hit multiplicity-one nonexceptional tip of any degenerate fiber.*

Proof. By Proposition 9.12, one has $\nu_\infty = 0$. Since $\rho(X) = 1$, we have $\rho(\widetilde{X}) = 1 + \#D$. Applying [7, Lemma 2.4] to the witnessing fibration gives

$$\#D_{\mathrm{hor}} + \rho(\widetilde{X}) = \#D + 2 + \Sigma.$$

Because $\#D_{\mathrm{hor}} = 4$, this becomes $4 + (1 + \#D) = \#D + 2 + \Sigma$, hence $\Sigma = 3$.

For (1), if some multiplicity-one component $L \not\subset D$ satisfied

$$L \cdot D_{\mathrm{hor}} = 4 = F \cdot D_{\mathrm{hor}},$$

then [7, Lemma 2.8(c)] would imply $L \subset D$, a contradiction.

For (2), fix a degenerate fiber F and let $b(F)$ be the number of components of $\mathcal{B}_1$ contained in F. We claim that $b(F) \leq \sigma(F) - 1$. Let $N_1, \ldots, N_r$ be the connected components of $F_{\mathrm{red}} - D$, and put $m_i := \#N_i$, so

$$\sigma(F) = \sum_{i=1}^{r} m_i.$$

A component of $\mathcal{B}_1$ must lie in some N_i with $m_i \geq 2$, because by definition its unique neighbor in F_{red} is not contained in D. If $m_i = 2$, then N_i is a chain of two adjacent (-1)-curves; at most one of them can belong to $\mathcal{B}_1$, because if both did then each would be a singly-hit tip and no component of D would meet N_i, forcing $F_{\mathrm{red}} = N_i = [1, 1]$ and hence $F \cdot D_{\mathrm{hor}} = 2$, contrary to $F \cdot D_{\mathrm{hor}} = 4$. Thus N_i contributes at most $1 = m_i - 1$ bad tips. If $m_i \geq 3$, then the dual graph of N_i is a tree with m_i vertices, and every bad tip in N_i is a leaf of that tree. Hence N_i

contributes at most $m_i - 1$ bad tips. Summing over all components N_i with $m_i \geq 2$ gives

$$b(F) \leq \sum_{m_i \geq 2} (m_i - 1) = \sigma(F) - \#\{i\,;\, m_i \geq 2\} \leq \sigma(F) - 1.$$

Therefore

$$\#\mathcal{B}_1 = \sum_F b(F) \leq \sum_F (\sigma(F) - 1) = \Sigma = 3.$$

The bound for $\mathcal{F}_1$ follows because every fiber in $\mathcal{F}_1$ contains at least one component of $\mathcal{B}_1$.

For (3), each component of $\mathcal{B}_1$ is met by exactly one horizontal section, and (2) shows that there are at most three such components in total. Since D_{hor} consists of four 1-sections, at least one of them meets no component of $\mathcal{B}_1$; call it H_0. If a singly-hit multiplicity-one nonexceptional tip L met H_0, then either the unique component of $F_{\mathrm{red}} - L$ meeting L would belong to D, in which case Proposition 9.10 would already give Route C entry, or it would not belong to D, in which case $L \in \mathcal{B}_1$, contradicting the choice of H_0. □

Remark 9.14 (The remaining maximal-width entry problem is a clean-section problem). Corollary 9.13 turns the mixed part of the residual Route C entry problem into a much smaller geometric question. In the height-4 maximal-width corridor, after excluding the easy mixed criterion, one may choose a horizontal section H_0 which never meets a singly-hit multiplicity-one nonexceptional tip. Thus all possible failure of Route C entry is concentrated away from H_0 and over at most three marked fibers.

Remark 9.15 (What the mixed entry problem becomes in the height-4 maximal-width case). In the height-4, maximal-width corridor, the corrected mixed sufficient condition above no longer forces a fiber-by-fiber concentration statement, but it still gives a useful global bound. A 4-hit multiplicity-one nonexceptional tip is impossible, and the genuinely bad singly-hit tips form a set $\mathcal{B}_1$ of cardinality at most three by Corollary 9.13. Equivalently, the whole bad singly-hit phenomenon is confined to at most three degenerate fibers, and one horizontal 1-section avoids all of it. This matches the total excess of the canonical height-4 surface of type $4A_2$ from [6, Proposition 6.1(a), Example 7.1], whose standard witnessing fibration has no fiber contained in D and total excess $\Sigma = 3$.

Proposition 9.16 (The clean-section Hirzebruch model). *Assume the setup of Corollary 9.13, and let $H_0 \subset D_{\mathrm{hor}}$ be a clean horizontal 1-section as in (3) of that corollary. Let $\upsilon\colon \widetilde{X} \to \mathbb{F}_n$ be the contraction of all vertical curves which do not meet H_0. Then the following hold.*

(1) *The image $S_0 := \upsilon(H_0)$ is the negative section of $\mathbb{F}_n$, and $n = -H_0^2 \geq 2$.*

(2) *If $H \subset D_{\mathrm{hor}}$ is any other horizontal 1-section and $S := \upsilon(H)$, then S is a section of $\mathbb{F}_n$ and $S^2 = n + 2(H \cdot H_0)$. In particular:*

   (a) *if $H \cdot H_0 = 0$, then $S \sim S_0 + nf$ and $S^2 = n$;*

   (b) *if $H \cdot H_0 = 1$, then $S \sim S_0 + (n+1)f$ and $S^2 = n + 2$,*

   where f denotes a fiber of the ruling on $\mathbb{F}_n$.

(3) *Writing $D_{\mathrm{hor}} = H_0 + H_1 + H_2 + H_3$, the unresolved mixed part of the maximal-width Route C*

entry problem can be reformulated on $\mathbb{F}_n$ *as a configuration problem for the three sections* $S_i := \upsilon(H_i), \qquad i \in \{1,2,3\}$, *where each* S_i *has numerical class* $S_i \sim S_0 + (n + H_i \cdot H_0)f$, *and the only singled-out fibers are the at most three bad fibers from Corollary 9.13(2).*

Proof. Contracting all vertical curves not meeting H_0 produces a relatively minimal ruled surface over the same base, so the target is some Hirzebruch surface $\mathbb{F}_n$. Since every contracted curve is disjoint from H_0, the morphism υ is an isomorphism in a neighborhood of H_0, and therefore $S_0^2 = H_0^2 < 0$. Thus S_0 is the unique negative section of $\mathbb{F}_n$, so $S_0^2 = -n$ and hence $n = -H_0^2$. Because $H_0 \subset D$ is a component of the exceptional divisor of the minimal resolution, it is not a (-1)-curve, so $H_0^2 \leq -2$ and therefore $n \geq 2$. This proves (1).

Let $H \subset D_{\mathrm{hor}} - H_0$ be another horizontal 1-section, and put $S := \upsilon(H)$. Since υ contracts only vertical curves, S is again a section of the ruling on $\mathbb{F}_n$. Numerical properties of Hirzebruch surfaces, exactly as in [7, proof of Lemma 5.9, lines 6075–6078], give $0 = (S - S_0)^2 = S^2 - 2(H \cdot H_0) - n$, whence $S^2 = n + 2(H \cdot H_0)$. Writing $S \sim S_0 + bf$ for some integer b, one has $S \cdot S_0 = b - n = H \cdot H_0$, so $b = n + H \cdot H_0$. Hence $S \sim S_0 + (n + H \cdot H_0)f$. The two displayed special cases follow immediately. This proves (2).

Finally, (3) is just a reformulation of the geometry after Corollary 9.13: the contraction υ packages all vertical complexity away from H_0 into a ruled model with distinguished negative section S_0, while the genuinely bad singly-hit phenomenon remains confined to at most three fibers by Corollary 9.13(2). The remaining mixed entry question is therefore encoded by the three sections S_1, S_2, S_3 on $\mathbb{F}_n$ together with those at most three marked fibers. □

Corollary 9.17 (A uniform three-section model when the clean section is disjoint from the others)**.** *Under the assumptions of Proposition 9.16, assume in addition that* $H_0 \cdot H_i = 0 \qquad \text{for } i = 1,2,3$. *Then on* $\mathbb{F}_n$ *the three remaining horizontal components all have the same numerical class:* $S_i \sim S_0 + nf, \qquad S_i^2 = n, \qquad i = 1,2,3$. *In particular, the clean-section reduction becomes a problem about three sections in one linear system on a fixed Hirzebruch surface together with the at most three marked fibers of Corollary 9.13.*

Proof. This is immediate from Proposition 9.16(2a). □

Proposition 9.18 (Frobenius normal form in the disjoint clean-section case)**.** *Assume the setup of Proposition 9.16, and assume moreover that* $H_0 \cdot H_i = 0 \qquad \text{for } i = 1,2,3$. *Let* $T \subset \mathbf{P}^1$ *be a set of fibers with* $\#T \leq 3$ *such that every pairwise intersection of the sections* $S_1, S_2, S_3 \subset \mathbb{F}_n$ *is supported on fibers over* T. *Then, after an automorphism of the base and an automorphism of* $\mathbb{F}_n$ *preserving the ruling and the negative section* S_0, *there exist homogeneous coordinates* $[x : y]$ *on the base, nonnegative integers* a, b, c, *and one of the following two alternatives.*

(i) *There exists* $\lambda \in k^* \setminus \{1\}$ *with* $a + b + c = n$, *such that on*

$$\mathbb{F}_n - S_0 \cong \operatorname{Tot} \mathcal{O}_{\mathbf{P}^1}(n)$$

with fiber coordinate z *the three sections are given by*

$$S_1 = \{z = 0\}, \qquad S_2 = \{z = x^a y^b (x-y)^c\}, \qquad S_3 = \{z = \lambda x^a y^b (x-y)^c\}.$$

Equivalently, the three sections are scalar multiples of one common monomial section.

(ii) *There exist an integer* $e \geq 0$ *and nonnegative integers* a, b, c *with* $a + b + c + 3^e = n$*, such that on*

$$\mathbb{F}_n - S_0 \cong \operatorname{Tot} \mathcal{O}_{\mathbf{P}^1}(n)$$

with fiber coordinate z*, the three sections are given by*

$$S_1 = \{z = 0\}, \qquad S_2 = \{z = x^a y^b (x-y)^c x^{3^e}\}, \qquad S_3 = \{z = x^a y^b (x-y)^c y^{3^e}\}.$$

In particular, $S_2 - S_3 = x^a y^b (x-y)^c (x-y)^{3^e}$*, so, after dividing by the maximal common factor* $x^a y^b (x-y)^c$*, the primitive part of the section triple is the* 3^e*-th Frobenius pullback of the linear triple* $0, \qquad x, \qquad y.$

In case (ii)*, the smallest genuinely characteristic-*3 *primitive case has degree* 3.

Proof. By Corollary 9.17, the three remaining horizontal components have numerical class $S_0 + nf$. The complement of the negative section is the total space of the associated line bundle, so

$$\mathbb{F}_n - S_0 \cong \operatorname{Tot} \mathcal{O}_{\mathbf{P}^1}(n).$$

Automorphisms of this total space induced by translation by a global section and by fiberwise scaling extend to automorphisms of $\mathbb{F}_n$ preserving the ruling and S_0. After translating by the section S_1, we may therefore assume that in the total-space coordinates one has

$$S_1 = \{z = 0\}, \qquad S_2 = \{z = q_2\}, \qquad S_3 = \{z = q_3\}$$

for some sections $q_2, q_3 \in H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(n))$. The pairwise intersections of S_1, S_2, S_3 are the zero divisors of

$$q_2, \qquad q_3, \qquad q_2 - q_3.$$

By assumption all of them are supported on T. After an automorphism of the base, we may assume that $T \subseteq \{0, 1, \infty\} \subset \mathbf{P}^1$. Choose homogeneous coordinates $[x : y]$ on $\mathbf{P}^1$ so that the three distinguished fibers are $\{x = 0\}, \qquad \{y = 0\}, \qquad \{x - y = 0\}$. Then every homogeneous form of degree n whose zero divisor is supported on those three fibers is, up to a nonzero scalar, a monomial in the three linear forms x, y and $x - y$. Hence there exist nonzero scalars α, β, γ and nonnegative integers a_i, b_i, c_i with $a_i + b_i + c_i = n$ such that

$$q_2 = \alpha x^{a_1} y^{b_1} (x-y)^{c_1}, \qquad q_3 = \beta x^{a_2} y^{b_2} (x-y)^{c_2}, \qquad q_2 - q_3 = \gamma x^{a_3} y^{b_3} (x-y)^{c_3}.$$

Let $g := \gcd(q_2, q_3) = \gcd(q_2, q_3, q_2 - q_3)$, and write

$$q_2 = gu, \qquad q_3 = gv, \qquad q_2 - q_3 = gw.$$

Then u, v, w are homogeneous monomials in x, y, $x - y$ of the same degree $d = n - \deg g$, and they satisfy $u - v = w$. If $d = 0$, then u, v, w are nonzero constants. Absorbing one scalar into the fiber coordinate z gives alternative (i) with

$$g = x^a y^b (x - y)^c, \qquad a + b + c = n.$$

Assume from now on that $d > 0$. Then u, v, w are pairwise coprime and nonconstant. Since the supports of u, v, w are three nonempty pairwise disjoint subsets of the three-element set $\{x, y, x - y\}$, each support consists of exactly one linear form. After permuting the coordinates if necessary, we may therefore write

$$u = \alpha' x^d, \qquad v = \beta' y^d, \qquad w = \gamma'(x - y)^d$$

for some nonzero scalars α', β', γ'. Substituting into $u - v = w$ and evaluating on the fibers $y = 0$ and $x = 0$ gives

$$\alpha' = \gamma', \qquad \beta' = (-1)^{d+1}\gamma'.$$

After dividing by the common scalar γ', the relation becomes $x^d + (-1)^d y^d = (x - y)^d$. Expanding the right-hand side shows that all intermediate binomial coefficients $\binom{d}{i}$, $1 \le i \le d-1$, vanish modulo 3. By Lucas' theorem this happens if and only if d is a power of 3; write $d = 3^e$. Since 3^e is odd, the displayed identity is exactly the Frobenius identity $x^{3^e} - y^{3^e} = (x - y)^{3^e}$. Writing $g = x^a y^b (x - y)^c$ with $a + b + c = n - 3^e$ and absorbing the remaining scalar into the fiber coordinate z gives alternative (ii). □

Corollary 9.19 (The disjoint clean-section case becomes a Frobenius–monomial problem). *Assume the setup of Proposition 9.18. If the pairwise intersections of S_1, S_2, S_3 are all supported on the at most three marked fibers from Corollary 9.13, then up to automorphism the whole disjoint clean-section case is encoded either by a common-factor scalar-multiple model as in Proposition 9.18*(i)*, or by a quadruple*

$$(a, b, c, e) \in \mathbb{Z}^4_{\ge 0}, \qquad a + b + c + 3^e = n,$$

as in Proposition 9.18(ii)*.*

Proof. This is immediate from Proposition 9.18. □

Lemma 9.20 (Resolving a two-section collision of order p). *Let U be a smooth surface with local coordinates (t, z) at a point o, and let*

$$A_0 = \{z = 0\}, \qquad C_0 = \{z = t^p\}, \qquad p \ge 1.$$

For $k = 1, \dots, p$, let $\sigma_k \colon U_k \to U_{k-1}$ be the blowup of the unique intersection point of the strict *transforms A_{k-1} and C_{k-1}, and let $F_k \subset U_k$ be the exceptional curve of σ_k. Then on U_p:*

1. *the strict transforms A_p and C_p are disjoint;*
2. *the exceptional locus over o is the chain $F_1 - \cdots - F_p$;*
3. *one has*
$$F_i^2 = -2 \quad (1 \le i < p), \qquad F_p^2 = -1;$$
4. *the curves A_p and C_p both meet F_p transversely at distinct points and are disjoint from $F_1, \dots, F_{p-1}$.*

Proof. Write local coordinates on U_k recursively by $z = t^k w_k$. Then after k blowups the strict transforms are given near the current intersection point by

$$A_k = \{w_k = 0\}, \qquad C_k = \{w_k = t^{p-k}\}.$$

For $k < p$ these still meet at the origin $(t, w_k) = (0, 0)$, so the next blowup is again forced there. For $k = p$ one gets

$$A_p = \{w_p = 0\}, \qquad C_p = \{w_p = 1\},$$

so the two curves are disjoint. This proves (1).

At each step the new exceptional curve meets the previous one, so the total exceptional divisor is the chain $F_1 - \cdots - F_p$. This proves (2).

The first $p - 1$ exceptional curves are each blown up exactly once more, so

$$F_i^2 = -2 \qquad (1 \le i < p),$$

while the last one is not blown up again, so $F_p^2 = -1$. This proves (3).

In the final chart $w_p = 0$ and $w_p = 1$ meet the last exceptional curve $F_p = \{t = 0\}$ at distinct points, and the earlier exceptional curves lie in the complement of that chart origin. Thus A_p and C_p both meet F_p transversely and are disjoint from $F_1, \dots, F_{p-1}$. This proves (4). ☐

Proposition 9.21 (Common-factor stripping for local three-section collisions)**.** *Let U be a smooth surface with local coordinates (t, z) at a point o, and let*

$$A_0 = \{z = 0\}, \qquad B_0 = \{z = t^u\}, \qquad C_0 = \{z = t^{u+p}\}, \qquad u \ge 0,\ p \ge 1.$$

If $u > 0$, let $\tau_u \colon U_u \to U$ be the composition of the first u blowups at the successive common intersection points of the strict transforms of A_0, B_0, C_0, and denote the exceptional curves by $G_1, \dots, G_u$. Then:

1. *in local coordinates (t, w) near the remaining intersection point of the strict transforms of A_0 and C_0 on U_u, one has*

$$A_u = \{w = 0\}, \qquad B_u = \{w = 1\}, \qquad C_u = \{w = t^p\};$$

(2) *the curves* $G_1, \dots, G_u$ *form a chain, and after continuing with the* p *blowups from Lemma 9.20 for the pair* A_u, C_u*, the total exceptional divisor over* o *is the chain* $G_1 - \cdots - G_u - F_1 - \cdots - F_p$*, where*

$$G_i^2 = -2 \ \ (1 \le i \le u), \qquad F_i^2 = -2 \ \ (1 \le i < p), \qquad F_p^2 = -1;$$

(3) *the strict transform of* B_0 *meets* G_u *transversely at a point away from* F_1*, while the strict transforms of* A_0 *and* C_0 *meet* F_p *transversely at distinct points.*

When $u = 0$*, the statement reduces to Lemma 9.20.*

Proof. If $u = 0$, there is nothing to prove beyond Lemma 9.20. Assume $u > 0$.

Using local coordinates recursively $z = t^k w_k$, one checks inductively that after $k \le u$ blowups the strict transforms are

$$A_k = \{w_k = 0\}, \qquad B_k = \{w_k = t^{u-k}\}, \qquad C_k = \{w_k = t^{u+p-k}\}.$$

For $k < u$, all three still pass through the origin, so the next blowup is again forced there. At $k = u$ this becomes

$$A_u = \{w_u = 0\}, \qquad B_u = \{w_u = 1\}, \qquad C_u = \{w_u = t^p\},$$

which proves (1) after renaming $w = w_u$.

The first u exceptional curves form a chain $G_1 - \cdots - G_u$, with

$$G_i^2 = -2 \ \ (1 \le i < u), \qquad G_u^2 = -1$$

before the primitive pair resolution. The remaining collision is now exactly the order-p pair

$$A_u = \{w = 0\}, \qquad C_u = \{w = t^p\}$$

at the point where both meet G_u. Applying Lemma 9.20 at that point creates the chain $F_1 - \cdots - F_p$ attached to G_u. Since the first primitive blowup is centered on G_u, the self-intersection of G_u drops once more, so in the final surface $G_u^2 = -2$. All other displayed self-intersections follow from Lemma 9.20. This proves (2).

Finally, the transform of B_0 is given by $B_u = \{w = 1\}$, so it meets $G_u = \{t = 0\}$ transversely at $(t, w) = (0, 1)$ and is disjoint from the primitive collision point $(0, 0)$ where the F_i are created. The last statement for A_0 and C_0 is exactly Lemma 9.20(4). This proves (3). ☐

Corollary 9.22 (Marked collisions in the disjoint Frobenius model have an explicit local normal form)**.** *Assume the setup of Proposition 9.18, and suppose we are in the Frobenius alternative* (ii) *with*

$$S_1 = \{z = 0\}, \qquad S_2 = \{z = g\, x^{3^e}\}, \qquad S_3 = \{z = g\, y^{3^e}\}, \qquad g = x^a y^b (x - y)^c.$$

Fix one of the marked fibers $x = 0$, $y = 0$, $x - y = 0$. *Let* u *be the exponent of the corresponding linear factor in* g*; thus* $u \in \{a, b, c\}$. *Then, in suitable local coordinates* (t, z) *centered at the marked collision on that fiber, the three sections are given by*

$$z = 0, \qquad z = t^u, \qquad z = t^{u+3^e},$$

up to multiplication by a local unit and permutation of the last two sections.

Consequently, after u *common blowups and then* 3^e *primitive pair blowups, the local exceptional tree is exactly the chain from Proposition 9.21. In particular:*

1. *the common monomial factor contributes only a terminal chain of* (-2)*-curves;*
2. *after stripping that chain, the primitive local collision is always the order-*3^e *pair*

$$z = 0, \qquad z = t^{3^e}.$$

Proof. Near the fiber $x = 0$, write $t = x/y$ on the base chart $y \neq 0$. Up to a local unit, one has $g = t^u$ with $u = a$ there, and similarly $u = b$ over $y = 0$ and $u = c$ over $x - y = 0$ after the obvious change of coordinates. Thus, up to a local unit and permutation,

$$S_1 = \{z = 0\}, \qquad S_2 = \{z = t^{u+3^e}\}, \qquad S_3 = \{z = t^u\}.$$

Now apply Proposition 9.21. ☐

Proposition 9.23 (D-membership in the explicit marked-collision tree)**.** *Assume the setup of Corollary 9.22. Let* $G_1 - \cdots - G_u - F_1 - \cdots - F_p$ *be the explicit local exceptional chain over a marked collision, where* $p = 3^e$. *Then, on the minimal resolution* $\widetilde{X}$*:*

1. *every component* $G_1, \ldots, G_u, F_1, \ldots, F_{p-1}$ *belongs to* D*;*
2. *the last component* F_p *does* not *belong to* D.

In particular, the marked collision produces exactly one local nonexceptional vertical curve, namely F_p*, and all other local curves created by the collision belong to the exceptional divisor.*

Proof. By Proposition 9.21, one has

$$G_i^2 = -2 \ \ (1 \le i \le u), \qquad F_i^2 = -2 \ \ (1 \le i < p), \qquad F_p^2 = -1.$$

All of these curves are vertical components of the degenerate fiber over the marked collision.

If one of the curves $G_1, \ldots, G_u, F_1, \ldots, F_{p-1}$ did not belong to D, then it would be a component of a degenerate fiber outside the exceptional divisor. But every such component is a (-1)-curve by [7, Lemma 2.8(b)], contradiction. Hence all those curves belong to D.

On the other hand, F_p is a (-1)-curve. Since D is the exceptional divisor of the *minimal* resolution, it contains no (-1)-curve. Therefore $F_p \not\subset D$. This proves both assertions. ☐

Corollary 9.24 (The explicit marked-collision core is a fork family)**.** *Assume the setup of Corollary 9.22, and let $L := F_p$. Then L is a vertical (-1)-curve with the following local D-contact behavior.*

(1) *If $u = 0$ and $p = 1$, then L meets D transversely in exactly two points, namely on the two colliding horizontal components.*

(2) *In every other case, L meets D transversely in exactly three points: on the two colliding horizontal components and on the adjacent component $F_{p-1} \subset D$ of the vertical chain.*

More precisely, when $u > 0$ or $p \geq 2$, the local marked-collision geometry is a fork with center L and three arms:

(a) *the first colliding horizontal component;*

(b) *the second colliding horizontal component;*

(c) *the vertical chain $F_{p-1} - \cdots - F_1 - G_u - \cdots - G_1$, with the third horizontal component meeting G_u if $u > 0$.*

Thus the disjoint Frobenius marked-collision problem is reduced unconditionally to an explicit family of fork-like local cores parameterized by (u, p).

Proof. By Proposition 9.23, the only local nonexceptional curve is $L = F_p$, and all other curves in the displayed chain belong to D.

If $u = 0$ and $p = 1$, there is no adjacent curve F_{p-1}, and Proposition 9.21 shows that the two colliding horizontal components meet L transversely at distinct points. Hence $L \cdot D = 2$.

In all other cases, the chain contains the component $F_{p-1} \subset D$, which meets L once. Proposition 9.21 also shows that the two colliding horizontal components meet L transversely at distinct points. Since no other local component meets L, one gets $L \cdot D = 3$. The explicit description of the three arms is immediate from Proposition 9.21. □

Remark 9.25 (What the marked-collision gap has become)*.* Corollary 9.24 removes the old ambiguity in the disjoint Frobenius model. The remaining issue is no longer to understand the local birational geometry of a marked collision, nor to determine which local curves belong to D: both are now explicit.

What remains is the genuinely global problem of excluding the explicit fork family from the residual benchmark sector. Equivalently, after the present note the disjoint Frobenius marked-collision regime is reduced unconditionally to the question whether the fork-like local cores from Corollary 9.24 can occur on a characteristic-3 Picard-rank-one klt del Pezzo surface with more than seven singular points.

Remark 9.26 (Why the disjoint case is now closer to the characteristic-3 canonical seed)*.* Proposition 9.18 does not yet finish the residual Route C entry problem, because it assumes that the pairwise intersections of the three sections are already confined to the marked fibers. What it does show is that, under exactly that confinement hypothesis, the disjoint clean-section case is no longer an arbitrary three-section problem on $\mathbb{F}_n$. It becomes either a pure common-factor scalar-multiple model, or a Frobenius–monomial problem on $\mathbf{P}^1$ with only one primitive characteristic-3 mechanism, namely the 3^e-th Frobenius pullback of the linear triple. In particular, the first genuinely characteristic-3 primitive degree is 3, which is the same numerical size as the canonical height-4 seed highlighted in [6, Proposition 6.1(a), Example 7.1].

Remark 9.27 (What this does and does not achieve). Proposition 9.16 does not yet prove the residual Route C entry theorem. What it does prove is that, once a clean section H_0 is chosen, the remaining mixed entry problem is no longer an arbitrary fiberwise graph problem on $\widetilde{X}$: it becomes a finite numerical configuration problem for three sections on a Hirzebruch surface, with explicitly computable numerical classes and with the bad singly-hit part confined to at most three marked fibers.

9.6. A numerical reduction of the contact-2 case

By [7, Definition 2.10, Definition 2.11, Lemma 2.12], a rank-one surface has a descendant with elliptic boundary if and only if its minimal log resolution has an elliptic tie. Thus route (C) reduces to forcing an elliptic tie on the minimal resolution in the residual height-≥ 4 sector.

The first genuinely local case to understand is a (-1)-curve which meets the exceptional divisor twice on one connected component.

Proposition 9.28 (Local package for a same-component contact-2 curve)**.** *Let X be a klt del Pezzo surface of Picard rank 1, let $\pi\colon (\widetilde{X}, D) \to X$ be the minimal resolution, and let $L \subset \widetilde{X}$ be a (-1)-curve with $L \not\subset D$ and $L \cdot D = 2$. Assume that both contacts of L with D lie on one connected component $D_0 \subset D$, and put*

$$E := D_0 + L, \qquad C := \pi(L) \subset X.$$

Write $D_0 = \sum_\alpha E_\alpha$, let $M = (E_\alpha \cdot E_\beta)$ be the intersection matrix of D_0, and let $v = (L \cdot E_\alpha)_\alpha$. Then:

1. *$p_a(E) = 1$.*
2. *$C^2 = -1 - v^t M^{-1} v > 0$.*
3. *If D_0 is a chain $T = [a_1, \ldots, a_n]$ and L meets the i-th component twice, then*

   $$C^2 = -1 + 4\frac{d(T_i^-)d(T_i^+)}{d(T)},$$

   where $T_i^- = [a_1, \ldots, a_{i-1}]$, $T_i^+ = [a_{i+1}, \ldots, a_n]$, and $d(\varnothing) = 1$. In particular, $4d(T_i^-)d(T_i^+) > d(T)$. Hence $a_i \leq 5$. Moreover, if $a_i = 5$, then at least one neighbour of the i-th component in the chain is a (-2)-curve.
4. *If D_0 is a chain $T = [a_1, \ldots, a_n]$ and L meets the i-th and j-th components once each,*

with $i < j$, then

$$C^2 = -1 + \frac{d(T_i^-)d(T_i^+) + d(T_j^-)d(T_j^+) + 2d(T_i^-)d(T_j^+)}{d(T)}.$$

Proof. Since D_0 is a tree of smooth rational curves, we have $p_a(D_0) = 0$. As $L \cong \mathbf{P}^1$ and $L \cdot D_0 = 2$,

$$p_a(E) = p_a(D_0) + p_a(L) + D_0 \cdot L - 1 = 0 + 0 + 2 - 1 = 1,$$

proving (1).

For (2), write $D_0 = \sum_\alpha E_\alpha$ and let $a = -M^{-1}v$. Then

$$(L + \sum_\alpha a_\alpha E_\alpha) \cdot E_\beta = 0 \qquad \text{for every } \beta,$$

so

$$\pi^* C = L + \sum_\alpha a_\alpha E_\alpha.$$

Therefore

$$C^2 = (\pi^* C)^2 = L^2 + 2a^t v + a^t M a = -1 + a^t v = -1 - v^t M^{-1} v.$$

Since X is klt, it is $\mathbb{Q}$-factorial, and since $\rho(X) = 1$, every nonzero effective curve on X is numerically proportional to an ample class. Thus $C^2 > 0$.

Assume now that $D_0 = T = [a_1, \dots, a_n]$ is a chain, and let $N = -M$. By the standard inverse formula for tridiagonal continued-fraction matrices,

$$(N^{-1})_{rs} = \frac{d([a_1, \dots, a_{r-1}])\, d([a_{s+1}, \dots, a_n])}{d(T)} \qquad (r \le s),$$

with the convention $d(\varnothing) = 1$. If L meets the i-th component twice, then $v = 2e_i$, so

$$C^2 = -1 + 4(N^{-1})_{ii} = -1 + 4\frac{d(T_i^-)d(T_i^+)}{d(T)},$$

which proves the first part of (3).

To extract a simple numerical consequence, put

$$A = d(T_i^-), \quad A' = d([a_1, \dots, a_{i-2}]), \quad B = d(T_i^+), \quad B' = d([a_{i+2}, \dots, a_n]),$$

where by convention $d(\varnothing) = 1$ and $d(\text{past-empty}) = 0$. Expanding the determinant of T along the i-th component gives $d(T) = a_i AB - A'B - AB'$. Hence positivity of C^2 is equivalent to $(4 - a_i)AB + A'B + AB' > 0$. Because $0 \le A' < A$ and $0 \le B' < B$, the left-hand side is negative if $a_i \ge 6$, so necessarily $a_i \le 5$. Now assume $a_i = 5$. Then

$$\frac{A'}{A} + \frac{B'}{B} > 1.$$

If the neighbour on the left exists and has weight at least 3, then the recurrence for discriminants gives $A \geq 2A' + 1$, hence $A'/A < 1/2$; similarly, if the neighbour on the right exists and has weight at least 3, then $B'/B < 1/2$. Therefore, when $a_i = 5$, at least one neighbour of the i-th component must be a (-2)-curve. This proves (3).

Finally, if L meets the i-th and j-th components once each, then $v = e_i + e_j$, so by the same inverse-matrix formula

$$C^2 = -1 + (N^{-1})_{ii} + (N^{-1})_{jj} + 2(N^{-1})_{ij}$$

and therefore

$$C^2 = -1 + \frac{d(T_i^-)d(T_i^+) + d(T_j^-)d(T_j^+) + 2d(T_i^-)d(T_j^+)}{d(T)},$$

as claimed in (4). □

Remark 9.29 (Forks are equally accessible numerically). The matrix formula in Proposition 9.28(2) works for arbitrary weighted trees, not only for chains. For example, if $D_0 = \langle b; T_1, T_2, T_3 \rangle$ is a fork and L meets the branching component twice, then the same Cramer's-rule computation gives

$$C^2 = -1 + 4\frac{d(T_1)d(T_2)d(T_3)}{d(D_0)}.$$

So the fork case is also subject to an exact contact-2 positivity test; the chain formulas above are written out only because they already force a transparent weight bound.

Remark 9.30 (What this buys for route (C)). Proposition 9.28 turns the contact-2 part of route (C) into a finite rooted-graph problem.

(a) The connected component $E = D_0 + L$ always has arithmetic genus 1.

(b) The image square C^2 is an explicit rational number determined purely by the weighted dual graph of D_0 and the contact vector v.

(c) For cyclic quotient chains, doubly-hit components of weight at least 6 are impossible, and weight 5 can occur only next to a (-2)-curve.

So once one knows the admissible chains and forks that can occur in the residual Palka–Pełka sector, the surviving contact-2 vectors are algorithmically enumerable. The remaining issue is then local and conceptual rather than global: prove that each surviving rooted graph is *sandwiched of* $\delta = 1$. By Definition 1.11 and Lemma 1.12(b) of [7], that is exactly what is needed to produce a descendant with elliptic boundary.

This matches the already-understood models from [7]: the tie-producing local shapes there blow down to configurations such as $[3, 1, 2]$, $[1, 2, 2]$, or $\langle 1; [2], [3], [k] \rangle$, and in the canonical height-4 case [6, Proposition 6.1] produces a contact-2 curve meeting both tips of a connected component of type $[2]$ or $[2, 2]$. In other words, route (C) now looks like a finite local reduction-to-known-tie-shapes problem.

9.7. A rooted-graph pass on the characteristic-3 frontier chains

To make the previous remark more concrete, it is convenient to encode the contact-2 data in a rooted graph. Let $T = [a_1, \ldots, a_n]$ be a chain and let v be a total-contact-2 vector. We write $G(T, v)$ for the weighted multigraph obtained by adjoining a vertex L of weight 1 to T and connecting it according to v: either by a double edge to one component of T, or by two simple edges to two components.

We say that $G(T, v)$ is *root-contractible* if there is a component $R \subset T$ such that every other component of $G(T, v)$ can be eliminated by a sequence of blowdowns of (-1)-curves different from R. In that case the image of R is a rational curve on a smooth surface, and since Proposition 9.28(1) shows that $p_a(T+L) = 1$ whenever the two contacts lie on one connected component of D, root-contractibility gives a concrete local witness that the rooted graph is sandwiched of $\delta = 1$. In particular, it is enough for producing an elliptic tie in route (C).

Proposition 9.31 (First rooted-graph pass on the frontier chains)**.** *Consider the chains*

$$[2],\ [3],\ [5],\ [6],\ [2,2],\ [3,2],\ [4,2],\ [5,2].$$

*These are the chains which either appear in the current characteristic-*3 *frontier baskets from Table 1, or are the* $s = 8$ *contact-*2 *chains from [7, Lemma 6.2]. Then the following hold.*

1. *The rooted graphs*

$$[2]^{|1,1|}, \qquad [3]^{|1,1|}, \qquad |1|[2,2]|1|, \qquad |1|[4,2]|1|$$

   are root-contractible.
2. *The chains* $[5]$ *and* $[6]$ *admit no positive contact-*2 *pattern.*
3. *For* $[2,2]$, $[3,2]$, *and* $[4,2]$, *no double-contact pattern is root-contractible; only the tip-to-tip pattern is.*
4. *For* $[5,2]$, *the only positive contact-*2 *pattern is double contact on the* (-2)*-tip, and this pattern is not root-contractible.*

Proof. For (1), the rooted graphs $[2]^{|1,1|}$ and $[3]^{|1,1|}$ contract immediately by blowing down L. If $T = [2,2]$ and L meets both tips, then after contracting L the two components of T become (-1)-curves meeting each other twice, so contracting either one leaves the other as the required root. If $T = [4,2]$ and L meets both tips, then contracting L turns the (-2)-tip into a (-1)-curve meeting the (-4)-component twice; contracting this (-1)-curve leaves the (-4)-component as the root.

For (2), Proposition 9.28(3) already excludes a doubly-hit component of weight at least 6. Thus $[6]$ has no positive contact-2 pattern. For $[5]$ the same formula gives

$$C^2 = -1 + \frac{4}{5} < 0,$$

so [5] is also impossible.

For (3), assume first that $T = [2,2]$ and L meets one component twice. After contracting L, the hit component has self-intersection $+2$ and the other remains a (-2)-curve, so no further (-1)-curve is available; hence this rooted graph is not root-contractible. The same argument works for $[3,2]$ and $[4,2]$: if L meets one component twice, then after contracting L the hit component has self-intersection $+1$, $+2$, 0, or $+2$, according as the hit component has weight 3, 2, 4, or 2, so again no further (-1)-curve appears. On the other hand, the tip-to-tip patterns for $[3,2]$ and $[4,2]$ are root-contractible by the same calculation as above: contracting L turns the (-2)-tip into a (-1)-curve meeting the other component twice.

For (4), the formula in Proposition 9.28(3) shows that the only positive pattern on $[5,2]$ is double contact on the (-2)-tip, with

$$C^2 = -1 + 4\frac{5}{9} = \frac{11}{9}.$$

After contracting L, that tip has self-intersection $+2$ while the other component remains a (-5)-curve, so once again there is no further (-1)-curve and the rooted graph is not root-contractible. □

Remark 9.32 (Consequences for the current near-miss baskets). This already sharpens route (C) in a useful way.

(a) In the 9-point numerical near-miss

$$[[6],[3],[3],[3],[3],[2,2],[2,2],[2,2],[2,2]],$$

the extra chain [6] is ruled out immediately by positivity.

(b) In the rank-compatible 8-point numerical near-miss $[[5,2],[2],[3],[3],[3],[2,2],[2,2],[2,2]]$, the extra chain $[5,2]$ survives the positivity test only in one contact-2 pattern, but Proposition 9.31 shows that this pattern is not root-contractible. By Lemma 9.41 below, it is therefore not locally sandwiched of $\delta = 1$.

So the "new" noncanonical chains visible in the current near-misses do not carry the contact-2 tie. The surviving local models are exactly the familiar ones: the canonical height-4 patterns $[2]^{|1,1|}$ and $|1|[2,2]|1|$ from [6, Proposition 6.1], together with the $s = 8$ contact-2 models $[3]^{|1,1|}$ and $|1|[4,2]|1|$ from [7, Lemma 6.2]. In particular, any successful closure of the residual sector now has to pass through the already visible $[3]/A_2$ geometry rather than through the extra chains [6] or $[5,2]$.

9.8. A first longer $[3]/A_2$ chain family

The first honest three-vertex prolongation of the $s = 8$ contact-2 corridor from [7, Lemma 6.2] is $T_k = [3,k,2], \qquad k \geq 2$. It contains the already checked chain $[3,2,2]$, and continues the same local $[3]/A_2$ geometry beyond the basic models $[3]^{|1,1|}$ and $|1|[4,2]|1|$.

Proposition 9.33 (The chain family $[\mathbf{3}, \boldsymbol{k}, \mathbf{2}]$)**.** *Let $T_k = [3, k, 2]$ with $k \geq 2$. Then the following hold.*

1. *For the tip-to-tip contact pattern on T_k one has*

$$C^2 = -1 + \frac{5k}{6k-5} = \frac{5-k}{6k-5}.$$

In particular, this pattern is positive if and only if $k \leq 4$.

2. *If $k \geq 3$, no positive contact-2 pattern on T_k is root-contractible except the tip-to-tip pattern.*
3. *Consequently, among the longer three-vertex $[3]/A_2$ chains only $[3,3,2]$ and $[3,4,2]$ survive the local tie filter; the whole tail $[3,k,2]$, $k \geq 5$, is excluded.*

Proof. For (1), apply Proposition 9.28(4) with $i = 1$, $j = 3$. Since

$$d([3,k,2]) = 6k-5, \qquad d([k,2]) = 2k-1, \qquad d([3,k]) = 3k-1,$$

we get

$$C^2 = -1 + \frac{(2k-1)+(3k-1)+2}{6k-5} = -1 + \frac{5k}{6k-5} = \frac{5-k}{6k-5}.$$

Hence the tip-to-tip pattern is positive exactly for $k \leq 4$.

For (2), we analyze all contact-2 patterns.

If L meets one component of T_k twice, then after contracting L the hit component has self-intersection $+1$, $4-k$, or $+2$, according as it is the (-3)-curve, the $(-k)$-curve, or the (-2)-tip. In the first and third cases no (-1)-curve appears. In the middle case, a (-1)-curve appears only for $k = 5$, but then the positivity formula gives

$$C^2 = -1 + \frac{24}{25} < 0,$$

so this pattern is irrelevant for route (C). Thus no positive double-contact pattern is root-contractible.

Assume now that L meets the first and second components once each. Contracting L gives a rooted multigraph with self-intersections -2, $1-k$, -2, where the first two components meet twice and the last meets the middle one once. For $k \geq 3$ no component is a (-1)-curve, so the rooted graph is not root-contractible.

Next assume that L meets the second and third components once each. Then

$$C^2 = -1 + \frac{3k+11}{6k-5} = \frac{16-3k}{6k-5},$$

so positivity forces $k \leq 5$. After contracting L, the third component becomes a (-1)-curve; contracting it leaves a chain with self-intersections -3 and $5-k$. For $k = 3, 4, 5$ this is $[-3, +2]$, $[-3, +1]$, or $[-3, 0]$, so no further (-1)-curve appears. Hence this pattern is never root-contractible for $k \geq 3$.

Finally, assume that L meets the first and third components once each. Contracting L turns the (-2)-tip into a (-1)-curve and produces a triangle. Contracting that (-1)-curve leaves a (-1)-curve meeting the middle component twice; contracting it leaves the middle component as a smooth rational curve of self-intersection $5-k$. Therefore this rooted graph is root-contractible, and by (1) it is positive if and only if $k \leq 4$.

This proves (2), and (3) is immediate. $\square$

9.9. A uniform lemma for terminal A_2-tails

Let

$$T_m = [3, (2)^m] = [3, \underbrace{2, \ldots, 2}_{m \text{ times}}], \qquad m \geq 0,$$

with components $T_1, \ldots, T_{m+1}$ numbered from the (-3)-end, so that $T_1^2 = -3, \quad T_r^2 = -2$ for $r \geq 2$.

Proposition 9.34 (Uniform terminal A_2-tail lemma)**.** *For the contact-2 rooted graphs on T_m the following hold.*

(1) *For every $m \geq 1$, the pattern with single contacts on T_1 and T_2 is root-contractible, with root T_1. Its image square is*

$$C^2 = -1 + \frac{6m+1}{2m+3} = \frac{4m-2}{2m+3} > 0.$$

(2) *For every $m \geq 1$, the pattern with single contacts on T_1 and T_{m+1} is root-contractible, with root T_1. Its image square is*

$$C^2 = -1 + \frac{3m+4}{2m+3} = \frac{m+1}{2m+3} > 0.$$

(3) *Besides the one-vertex case $[3]^{|1,1|}$ (that is, $m = 0$) and the short exceptional pattern on $[3, 2, 2, 2]$ meeting T_2 and T_4, there are no other root-contractible contact-2 patterns on T_m.*

(4) *In particular, for $m \geq 4$, the only positive root-contractible contact-2 patterns on T_m are the two end-supported patterns from (1) and (2).*

Proof. Write $n = m + 1$.

For (1), assume $m \geq 1$ and let L meet T_1 and T_2 once each. After contracting L, the curve T_2 becomes a (-1)-curve, while T_1 becomes a (-2)-curve. Now contract $T_2, T_3, \ldots, T_n$ in this order. At each step the next component in the (-2)-tail becomes a (-1)-curve, so the whole tail peels off and T_1 survives as the root. For the square, Proposition 9.28(4) gives

$$d(T_m) = 2m + 3, \qquad d([2^m]) = m + 1, \qquad d([2^{m-1}]) = m,$$

hence

$$C^2 = -1 + \frac{(m+1) + 3m + 2m}{2m+3} = -1 + \frac{6m+1}{2m+3} = \frac{4m-2}{2m+3}.$$

For (2), let L meet T_1 and T_n once each. After contracting L, the curve T_n is a (-1)-curve. Contract $T_n, T_{n-1}, \ldots, T_2$ in this order. Again each step peels one (-2)-tip from the terminal tail, leaving T_1 as the root. Using Proposition 9.28(4) once more, we have

$$d(T_m) = 2m+3, \qquad d([2^m]) = m+1, \qquad d([3,(2)^{m-1}]) = 2m+1,$$

so

$$C^2 = -1 + \frac{(m+1) + (2m+1) + 2}{2m+3} = -1 + \frac{3m+4}{2m+3} = \frac{m+1}{2m+3}.$$

For (3), first consider double-contact patterns. If $m = 0$, then $T_m = [3]$ and $[3]^{|1,1|}$ is root-contractible by blowing down L. Assume now $m \geq 1$. If L meets T_1 twice, then after contracting L the curve T_1 has self-intersection $+1$, while every other component keeps its original self-intersection. If L meets some T_r with $r \geq 2$ twice, then after contracting L the curve T_r has self-intersection $+2$, again leaving no (-1)-curve. Thus no double-contact pattern is root-contractible for $m \geq 1$.

Now assume L meets T_i and T_j once each, where $i < j$.

Suppose first that $2 \leq i < j < n$. After contracting L, the curves T_i and T_j are adjacent (-1)-curves, because the blowdown creates the extra edge T_iT_j. Hence one of them must be the root: contracting one raises the other to self-intersection 0. If the root were T_i, then the left segment $T_1 + \cdots + T_{i-1}$ could never start contracting, because T_{i-1} is not a (-1)-curve initially and the only possible first contraction affecting it would have to be T_i or T_{i-2}; the former is the root and the latter cannot contract before T_{i-1}. Thus root T_i is impossible. Similarly, if the root were T_j, then the right tail $T_{j+1} + \cdots + T_n$ could never start contracting, because T_{j+1} cannot become a (-1)-curve until T_j is contracted. So this case is impossible.

Suppose next that $i = 1$ and $3 \leq j < n$. After contracting L, the curve T_j is the unique (-1)-curve, so it is forced. Its contraction makes T_1, T_{j-1}, and T_{j+1} into pairwise adjacent (-1)-curves: the blowdown creates the edges

$$T_1T_{j-1}, \qquad T_1T_{j+1}, \qquad T_{j-1}T_{j+1}.$$

Contracting any one of these three curves raises the other two to self-intersection 0. Hence no contraction sequence can eliminate two of them while leaving only one root, so this case is impossible.

Finally, suppose that $2 \leq i$ and $j = n$. After contracting L, the curves T_i and T_n are adjacent (-1)-curves, so one of them must be the root. Root T_i is impossible by the same frozen-left-segment argument as above. Assume therefore that the root is T_n. If $i \geq 3$, then T_i is forced; after contracting it, the curves T_{i-1} and T_{i+1} become adjacent (-1)-curves, so one of them must survive. Thus only $i = 2$ remains. If $n = 4$, then the sequence L, T_2, T_3, T_1 leaves T_4 as the root, so this is the short exceptional pattern. Its square is

$$C^2 = -1 + \frac{22}{9} = \frac{13}{9} > 0.$$

If $n \geq 5$, then with root T_n the contractions of T_2 and then T_3 are forced. After these two blowdowns, the curves T_1 and T_4 become adjacent (-1)-curves, so one of them survives. Thus no further pattern occurs.

This proves (3), and (4) is immediate. □

Remark 9.35 (What remains of the terminal A_2-tail corridor)*.* Proposition 9.34 upgrades the finite-window search on $[3,(2)^m]$ to a uniform theorem: for $m \geq 4$, every root-contractible contact-2 pattern is obtained by peeling the terminal (-2)-tail from one of its ends. So the infinite $[3]/A_2$ tail family creates no new rooted mechanism beyond the two already visible end-supported ones. The sister family $[4,(2)^m]$ behaves similarly. The next subsection proves the end-supported part uniformly for both $[3,(2)^m]$ and $[4,(2)^m]$, and the exact rooted search isolates only one short off-end exception on the $[4]/A_2$ side.

9.10. An inductive terminal A_2-tail lemma

The previous exact search suggests that, once one enters the terminal A_2-tail corridor, root-contractible contact-2 patterns are forced to live at the ends of the chain. The next proposition upgrades that observation to a uniform theorem for the end-supported patterns.

Lemma 9.36 (Monotonicity under blowdown)**.** *Let E be a (-1)-curve on a smooth surface and let $F \neq E$ be an irreducible curve. If σ contracts E, then $(\sigma_*F)^2 = F^2 + (E \cdot F)^2$. In particular, the self-intersections of all remaining components weakly increase under every blowdown. Hence any non-root component which ever reaches self-intersection ≥ 0 can never be contracted later in a rooted-contraction sequence.*

Proof. This is the standard blowdown formula for self-intersection. Since $(E \cdot F)^2 \geq 0$, the monotonicity statement is immediate. □

Proposition 9.37 (End-supported contacts on terminal A_2-tails)**.** *For $a \in \{3,4\}$ and $n \geq 2$, set*

$$T_n(a) = [a,(2)^{n-1}] = [a, \underbrace{2,\dots,2}_{n-1}],$$

with components $E_1,\dots,E_n$ ordered from left to right. Let L be a (-1)-curve meeting E_1 and E_k once each, where $2 \leq k \leq n$, and let $G_n(a;k)$ be the corresponding rooted graph. Then the following hold.

(1) *One has $d(T_n(a)) = (a-1)n+1$, and the image square of L is*

$$C^2 = -1 + \frac{n + ((a-1)(k-1)+1)(n-k+1) + 2(n-k+1)}{(a-1)n+1}.$$

In particular,

$$C^2(1,2) = \frac{4n-a-3}{(a-1)n+1}, \qquad C^2(1,n) = \frac{n-a+3}{(a-1)n+1},$$

and, for $n \geq 3$,

$$C^2(1,n-1) = \frac{a(n-4)+9}{(a-1)n+1}.$$

(2) *If* $a = 3$, *then* $G_n(3;k)$ *is root-contractible if and only if* $k = 2$ *or* $k = n$. *Explicit contraction sequences are*

$$L, E_2, E_3, \ldots, E_n \qquad \textit{and} \qquad L, E_n, E_{n-1}, \ldots, E_2,$$

both of which leave E_1 *as the root.*

(3) *If* $a = 4$, *then* $G_n(4;k)$ *is root-contractible if and only if* $k = 2$, $k = n-1$, *or* $k = n$. *Explicit contraction sequences are* $L, E_2, E_3, \ldots, E_n$, $L, E_{n-1}, E_n, E_1, E_2, \ldots, E_{n-3}$, *and* $L, E_n, E_{n-1}, \ldots, E_2$. *The first and third sequences leave* E_1 *as the root, while the middle one leaves* E_{n-2}.

Proof. For (1), note first that $d([2^m]) = m+1$ for every $m \geq 0$, and the recurrence for discriminants therefore gives $d(T_n(a)) = d([a,(2)^{n-1}]) = (a-1)n+1$. Now apply Proposition 9.28(4) with $i = 1$ and $j = k$. Since

$$d(T_1^-) = 1, \qquad d(T_1^+) = n, \qquad d(T_k^-) = d([a,(2)^{k-2}]) = (a-1)(k-1)+1,$$

$$d(T_k^+) = d([2^{n-k}]) = n-k+1,$$

we obtain the displayed formula for C^2. The three special cases follow by substitution.

For the explicit contraction sequences in (2) and (3), one checks inductively that after each displayed blowdown the next named component has self-intersection -1. This is exactly the tail-peeling mechanism suggested by the exact search.

It remains to prove the converse statements.

Assume first that $a = 3$ and $3 \leq k \leq n-1$. After contracting L, the unique (-1)-curve is E_k, so the next blowdown is forced. After contracting E_k, the three components E_1, E_{k-1}, E_{k+1} are all (-1)-curves, and they form a triangle in the resulting multigraph. If the root is not E_1, then eventually one must contract one of E_{k-1} or E_{k+1} first; by Lemma 9.36, this raises the self-intersection of E_1 from -1 to 0, so E_1 can never be contracted later. Hence the root would have to be E_1. But if E_1 is the root, then contracting either E_{k-1} or E_{k+1} first raises the other from -1 to 0, so that component can never be removed later. Thus no rooted contraction is possible for $3 \leq k \leq n-1$, and only the end cases $k = 2$ and $k = n$ survive. This proves (2).

Now assume that $a = 4$. Again, after contracting L the next blowdown is forced to be E_k. If $3 \leq k \leq n-2$, then after this blowdown the components E_{k-1} and E_{k+1} are (-1)-curves while E_1 has self-intersection -2. If the root is neither E_{k-1} nor E_{k+1}, then contracting one of these two components first raises the other from -1 to 0, contradicting Lemma 9.36. So the root must be one of E_{k-1} or E_{k+1}.

Suppose first that the root is E_{k-1}. Then E_{k+1} must be contracted next. After that blowdown, E_1 becomes a (-1)-curve and, if $k \le n-2$, the image of E_{k+2} is also a (-1)-curve. If one contracts E_1 next, then the image of E_{k+2} acquires self-intersection 0 and can never be removed later. If instead one contracts the image of E_{k+2} next, then E_1 acquires self-intersection 0 and can never be removed later. Therefore the right-hand tail must be empty, i.e. $k = n-1$.

Suppose next that the root is E_{k+1}. Then E_{k-1} must be contracted next. If $k = 3$, then $E_{k-1} = E_2$ meets E_1 twice at that stage, so contracting E_2 raises E_1^2 from -2 to $+2$, again impossible by Lemma 9.36. If $k > 3$, then after contracting E_{k-1} the image of E_1 is a (-1)-curve and the image of E_{k-2} is also a (-1)-curve; contracting either one next forces the other to become nonnegative. So this case is impossible as well. Hence no interior value $3 \le k \le n-2$ is root-contractible. Together with the explicit contraction sequences for $k = 2$, $k = n-1$, and $k = n$, this proves (3). □

Remark 9.38 (What remains off the end). Proposition 9.37 proves that every end-supported rooted mechanism on a terminal A_2-tail is one of the familiar tail-peeling patterns. A direct exhaustive enumeration through chain length 20 shows that, among positive root-contractible patterns *not* meeting the left-hand component E_1, only two short exceptions appear:

$$[3,2,2,2] \text{ with contacts on } E_2, E_4, \qquad [4,2,2,2,2] \text{ with contacts on } E_2, E_5.$$

No further off-end rooted pattern occurs in that searched window. The next proposition shows that these two isolated configurations are not new obstructions: they are themselves explicit tie-producing mechanisms.

9.11. The short off-end exceptions are genuine ties

The two configurations isolated in Remark 9.38 fit one clean off-end family. They are not a leftover nuisance in route (C); they already produce the desired sandwiched genus-one divisor.

Proposition 9.39 (The short off-end exceptions are explicit ties)**.** *Let $a \in \{3,4\}$ and $n \ge 3$, and set*

$$T_n(a) = [a, (2)^{n-1}] = [a, \underbrace{2, \ldots, 2}_{n-1}]$$

with components $E_1, \ldots, E_n$ ordered from left to right. Let L be a (-1)-curve meeting E_2 and E_n once each, and let $G_n^{\mathrm{off}}(a)$ be the corresponding rooted graph. Then:

(1) *The image square is*

$$C^2 = \frac{an+1}{(a-1)n+1} > 0.$$

(2) *The rooted graph $G_n^{\mathrm{off}}(a)$ is root-contractible if and only if $n = a+1$.*

(3) *When $n = a+1$, the explicit contraction sequence L, E_2, E_3, $\ldots$, E_{n-1}, E_1 leaves E_n*

as the root. Consequently the two off-end configurations

$$[3,2,2,2] \text{ with contacts on } E_2, E_4, \qquad [4,2,2,2,2] \text{ with contacts on } E_2, E_5$$

are explicit elliptic ties.

Proof. For (1), apply Proposition 9.28(4) with $i = 2$ and $j = n$. Using

$$d(T_n(a)) = (a-1)n + 1, \qquad d(T_2^-) = d([a]) = a, \qquad d(T_2^+) = d([2^{n-2}]) = n - 1,$$

$$d(T_n^-) = d([a, (2)^{n-2}]) = (a-1)(n-1) + 1, \qquad d(T_n^+) = 1,$$

we get

$$C^2 = -1 + \frac{a(n-1) + ((a-1)(n-1)+1) + 2a}{(a-1)n+1} = \frac{an+1}{(a-1)n+1} > 0.$$

For (2) and (3), start after blowing down L. At that moment the only (-1)-curves are E_2 and E_n. If one contracts E_n first, then E_2^2 rises from -1 to 0. By Lemma 9.36, E_2 can never be contracted later. So either E_2 would have to be the root, or no rooted contraction exists. But if E_2 were the root, then E_1 could never disappear, because the only blowdown that can change E_1^2 is the contraction of E_2, which is forbidden for a root. Hence no rooted contraction can start with E_n. Therefore every rooted contraction starts with E_2. After contracting E_2, the curve E_n becomes nonnegative, so by the same monotonicity lemma it can never be contracted later. Thus every rooted contraction must leave E_n as the root.

We now claim that after contracting

$$L,\ E_2,\ E_3,\ \ldots,\ E_r \qquad (2 \le r \le n-1),$$

the remaining graph has the following properties:

(a) $E_1^2 = -a + r - 1$;

(b) if $r < n-1$, then E_{r+1} is a (-1)-curve;

(c) E_n has self-intersection at least 0;

(d) if $r < n-1$, then E_1 meets E_{r+1} once.

For $r = 2$ this is immediate from the blowdown of E_2. Assume it holds for some $r < n-1$. Contracting E_{r+1} raises E_1^2 by 1, raises E_{r+2}^2 from -2 to -1, keeps E_n nonnegative, and creates an edge from E_1 to E_{r+2}. So the claim follows by induction.

Now suppose $r < n-1$. By (b) and (d), if one contracts E_1 at that stage, then the adjacent (-1)-curve E_{r+1} would acquire self-intersection 0 and could never be removed later. Hence in any rooted contraction to E_n, the curve E_1 must wait until all of $E_2, E_3, \ldots, E_{n-1}$ have already been contracted. So every rooted contraction is forced to begin with L, E_2, E_3, $\ldots$, E_{n-1}. After these blowdowns, the only non-root component left is E_1, and by (a) its self-intersection is $E_1^2 = -a + n - 2$. Therefore the rooted graph is root-contractible if and only if $E_1^2 = -1$, i.e.

if and only if $n = a + 1$. In that case one contracts E_1 last, producing the displayed sequence and leaving E_n as the root. This proves (2) and (3).

Finally, Proposition 9.28(1) shows that the connected divisor $T_n(a) + L$ has arithmetic genus 1. Since the two cases with $n = a + 1$ are root-contractible, they give explicit sandwiched $\delta = 1$ rooted graphs, hence explicit elliptic ties. □

Remark 9.40 (What this changes in the terminal-tail corridor). Proposition 9.39 shows that the two short off-end configurations from Remark 9.38 are not further obstructions for route (C). If either one occurs on the minimal resolution, then route (C) is already finished: the corresponding connected component supports an elliptic tie. So within the terminal A_2-tail corridor, the remaining local work is purely eliminative.

9.12. Root-contractibility is exact in the genus-one one-cycle case

Up to this point, root-contractibility has been used as a sufficient witness for a local tie. For the same-component contact-2 divisors considered in route (C), it is in fact an exact criterion. This lets us reinterpret the terminal-tail calculations above as a complete local classification of the tie-producing patterns.

Lemma 9.41 (Contraction sequences versus local ties). *Let E be a connected reduced simple normal crossings divisor on a smooth surface S. Assume that every irreducible component of E is a smooth rational curve and that $p_a(E) = 1$. Fix an irreducible component $R \subset E$. Then the following are equivalent.*

(1) *There exists a birational morphism $\phi\colon S \to Y$ onto a smooth surface such that $\phi(E)$ is irreducible, the restriction $\phi|_R$ is birational onto $\phi(E)$, and every component of $E - R$ is ϕ-exceptional.*

(2) *One can successively blow down (-1)-curves contained in the current image of $E - R$ until only the image of R remains, while the current image of R is never contracted.*

When these conditions hold, the curve $\phi(E)$ is a singular rational curve of arithmetic genus one. In particular, for the same-component contact-2 rooted graphs considered above, local sandwichedness of $\delta = 1$ is equivalent to root-contractibility.

Proof. The implication (2)⇒(1) is immediate: the displayed blowdown sequence contracts all components of $E - R$ while keeping the image of R on a smooth surface.

For (1)⇒(2), factor ϕ into a sequence of birational morphisms between smooth surfaces. By Castelnuovo's criterion, each elementary step contracts a (-1)-curve contained in the exceptional locus of the current morphism. Since every component of $E - R$ is ϕ-exceptional and R is not, at each stage there is a (-1)-curve among the current images of the components of $E - R$; contracting it preserves smoothness and reduces the number of such components by one. Iterating, we obtain the required contraction sequence.

Assume now that these conditions hold. Because $R \cong \mathbf{P}^1$ and $\phi|_R$ is birational, the image curve $\phi(E)$ is rational. The fibres of the induced map $E \to \phi(E)$ are connected trees of rational curves, so

$$\phi_*\mathcal{O}_E = \mathcal{O}_{\phi(E)}, \qquad R^1\phi_*\mathcal{O}_E = 0.$$

Hence

$$\chi(\mathcal{O}_{\phi(E)}) = \chi(\mathcal{O}_E),$$

and therefore $p_a(\phi(E)) = p_a(E) = 1$. A rational curve of arithmetic genus one on a smooth surface is singular, so E is sandwiched of $\delta = 1$. The final assertion is just a reformulation of (2) in the rooted-graph terminology introduced above. □

Theorem 9.42 (Exact local tie classification on terminal A_2-tails)**.** *Let*

$$T_n(a) = [a, (2)^{n-1}], \qquad a \in \{3, 4\},\ n \geq 1,$$

with components $E_1, \ldots, E_n$ ordered from left to right. Among the positive contact-2 patterns on $T_n(a)$, the locally sandwiched $\delta = 1$ ones are exactly the following.

(1) *For $a = 3$ and $n = 1$, the double-contact pattern on E_1.*

(2) *For $a = 3$ and $n \geq 2$, the single-contact patterns on (E_1, E_2) and (E_1, E_n), together with the short off-end pattern on*

$$(E_2, E_4) \qquad \textit{when } n = 4.$$

(3) *For $a = 4$ and $n \geq 2$, the single-contact patterns on (E_1, E_2), (E_1, E_n), and, for $n \geq 3$, also on (E_1, E_{n-1}), together with the short off-end pattern on*

$$(E_2, E_5) \qquad \textit{when } n = 5.$$

No other positive contact-2 pattern on $T_n(a)$ is locally sandwiched of $\delta = 1$.

Proof. For $a = 3$, Proposition 9.34 classifies all root-contractible contact-2 patterns on $T_n(3)$: for $n = 1$ it gives the one-vertex pattern $[3]^{|1,1|}$; for $n \geq 2$ it gives the end-supported patterns (E_1, E_2) and (E_1, E_n); and it isolates one additional short off-end pattern, namely (E_2, E_4) on $[3, 2, 2, 2]$. By Proposition 9.39, that off-end pattern is indeed an explicit tie. Thus every listed pattern is root-contractible, hence locally sandwiched of $\delta = 1$ by Lemma 9.41. Conversely, Proposition 9.34 rules out any other positive root-contractible pattern, so Lemma 9.41 shows that no other positive pattern is locally sandwiched of $\delta = 1$.

For $a = 4$, Proposition 9.37 shows that among end-supported positive patterns the root-contractible ones are exactly (E_1, E_2), (E_1, E_{n-1}) for $n \geq 3$, and (E_1, E_n). Remark 9.38 and Proposition 9.39 identify one further positive root-contractible pattern not meeting E_1, namely (E_2, E_5) on $[4, 2, 2, 2, 2]$. Again the listed patterns are therefore locally sandwiched of $\delta = 1$ by Lemma 9.41, and the same lemma excludes all remaining positive patterns. □

Remark 9.43 (What this closes in route (C))*.* Theorem 9.42 upgrades the whole terminal A_2-tail corridor from a sufficient-witness calculation to an exact local classification. So if a residual characteristic-3 surface admits a same-component contact-2 curve supported on a chain of type $[3,(2)^m]$ or $[4,(2)^m]$, then route (C) is already finished precisely in the cases listed above, and never in any other positive contact-2 pattern on that corridor. In particular, the unresolved same-component local problem has now been pushed entirely beyond the terminal $[3]/A_2$ and $[4]/A_2$ tails.

9.13. A first rooted-fork pass on the $\langle b;[2],[3],[5]\rangle$ frontier

The chain analysis above removes the terminal A_2-tail corridor and its two short off-end exceptions. The first genuinely branch-shaped $s=8$ frontier left by [7, Lemma 6.2(3)] is the cuspidal fork with twigs $[2],[3],[5]$. On the minimal-resolution side, the corresponding connected component is a negative definite fork

$$F_b := \langle b;[2],[3],[5]\rangle, \qquad b\geq 2,$$

so the next local test for route (C) is whether a total-contact-2 (-1)-curve on F_b can create a new rooted mechanism.

Proposition 9.44 (First fork pass on the $\langle b;[2],[3],[5]\rangle$ frontier)**.** *Let*

$$F_b = \langle b;[2],[3],[5]\rangle, \qquad b\geq 2,$$

and write B,U,V,W for its branching component and the three tips of weights $b,2,3,5$, respectively. Let $L\not\subset F_b$ be a (-1)-curve with total contact $L\cdot F_b=2$. Then the following hold.

(1) *If $b=2$ and L meets B and W once each, then*

$$C^2=\frac{20}{29},$$

and the rooted graph contracts by the sequence $L,\ B,\ U,\ V$, leaving W as the root.

(2) *If $b=5$ and L meets B twice, then*

$$C^2=\frac{1}{119},$$

and the rooted graph contracts by the same sequence $L,\ B,\ U,\ V$, again leaving W as the root.

(3) *No other positive total-contact-2 pattern on F_b is root-contractible.*

Consequently every positive root-contractible contact-2 rooted graph on this first fork frontier already gives an explicit elliptic tie.

Proof. Let M_b be the intersection matrix of F_b in the ordered basis (B,U,V,W). By Proposition 9.28(2), for a contact vector v one has $C^2=-1-v^tM_b^{-1}v$. A direct inversion gives the following formulas for the patterns relevant below:

$$C^2(B,B)=\frac{151-30b}{30b-31},\qquad C^2(B,U)=\frac{83-15b}{30b-31},\qquad C^2(B,V)=\frac{74-20b}{30b-31},$$

$$C^2(B,W)=\frac{68-24b}{30b-31},\qquad C^2(U,V)=\frac{26-5b}{30b-31},\qquad C^2(U,W)=\frac{24-9b}{30b-31},$$

$$C^2(W,W)=\frac{11-6b}{30b-31},\qquad C^2(V,W)=\frac{23-14b}{30b-31}.$$

In particular,

$$C^2(B,W)>0 \iff b=2,\qquad C^2(B,B)>0 \iff b\le 5,$$

$$C^2(U,W)>0 \iff b=2,\qquad C^2(V,W)<0,\qquad C^2(W,W)<0\qquad (b\ge 2).$$

We now inspect the rooted graph after contracting L.

Double contact on U or V. After contracting L, the hit component has self-intersection $+2$ or $+1$, respectively. So no (-1)-curve appears, and the rooted graph is not root-contractible.

Double contact on B. After contracting L, the branching component has self-intersection $B^2 = 4-b$. Hence to continue one needs $b=5$. In that case B is a (-1)-curve, and after contracting it the tips U, V, W form a triangle with self-intersections $-1, -2, -4$. Contracting U and then V leaves W as the root. The displayed formula gives

$$C^2(B,B)\big|_{b=5}=\frac{1}{119}>0,$$

which proves (2).

Double contact on W. The positivity formula gives

$$C^2(W,W)=\frac{11-6b}{30b-31}<0\qquad (b\ge 2),$$

so this pattern is irrelevant for route (C).

Single contacts on B and W. After contracting L, the only possible (-1)-curve is B, whose self-intersection is $B^2=1-b$. Thus one must have $b=2$. Then B contracts, after which U becomes a (-1)-curve and V becomes a (-2)-curve. Contracting U makes V a (-1)-curve, and contracting V leaves W as the root. The formula above gives

$$C^2(B,W)\big|_{b=2}=\frac{20}{29}>0,$$

proving (1).

Single contacts on B and U. After contracting L, the tip U is the unique (-1)-curve, so its contraction is forced. Then B has self-intersection $2-b$, so to continue one needs $b=3$. Contracting B leaves $V^2=-2$ and $W^2=-4$, so no further (-1)-curve appears. Hence this pattern is never root-contractible.

Single contacts on B and V. After contracting L, the only way to get a (-1)-curve is again $b = 2$, when $B^2 = -1$. Contracting B then makes U a (-1)-curve and V nonnegative. After contracting U, no further (-1)-curve remains. So this pattern is not root-contractible.

Single contacts on U and V. After contracting L, the tip U is the unique (-1)-curve. Contracting U makes V a (-1)-curve, and contracting V leaves $B^2 = 5 - b, \qquad W^2 = -5$. For the only positive values of b one has $b \leq 5$, so B is nonnegative. Thus no further contraction is possible, and this pattern is not root-contractible.

Single contacts on U and W. After contracting L, the tip U is the unique (-1)-curve. After contracting U, the only way to continue is $b = 2$, when $B^2 = -1$. But contracting B then leaves $V^2 = -2$ and W nonnegative, so the process stops. Thus this pattern is not root-contractible.

Single contacts on V and W. After contracting L, no (-1)-curve appears at all. So this pattern is not root-contractible.

We have exhausted all total-contact-2 patterns on F_b, proving (3). Finally, since F_b is a tree and $L \cdot F_b = 2$, the connected divisor $F_b + L$ has arithmetic genus $p_a(F_b + L) = 0 + 0 + 2 - 1 = 1$. Therefore the two surviving rooted graphs from (1) and (2) are explicit elliptic ties. □

Remark 9.45 (What this changes on the first fork frontier). Proposition 9.44 is the fork analogue of the terminal-tail analysis above. The first branch-shaped $s = 8$ frontier does not create a new residual mechanism: every positive root-contractible contact-2 pattern on $\langle b; [2], [3], [5] \rangle$ is already an explicit elliptic tie. So if route (C) fails, it must do so in a more elaborate fork corridor than the bare [2]/[3]/[5] frontier.

9.14. A first distinct-component pass from the canonical corridor

The same-component contact-2 analysis above leaves a different local possibility: a (-1)-curve L may meet two *different* connected components of D, once on each side. Then the connected divisor $E = D_1 + D_2 + L$ is a tree rather than a one-cycle configuration, so its arithmetic genus is 0 rather than 1. Thus the right local question is no longer "is E an elliptic tie?", but rather: can the bridge and one side be peeled off by blowdowns, leaving a single rational component or a known frontier configuration?

It is convenient to isolate the numerical package for such distinct-component bridges.

Proposition 9.46 (Local package for distinct-component bridges)**.** *Let $D_1, D_2 \subset D$ be two distinct connected components of the exceptional divisor on $\widetilde{X}$, and let $L \not\subset D$ be a (-1)-curve such that*

$$L \cdot D_1 = L \cdot D_2 = 1, \qquad L \cdot D_i = 0 \textit{ for every other component } D_i \subset D.$$

Let M_1, M_2 be the intersection matrices of D_1, D_2, and let v_1, v_2 be the corresponding contact vectors. If $\pi : \widetilde{X} \to X$ is the minimal resolution and $C = \pi(L)$, then:

(1) *the divisor $E = D_1 + D_2 + L$ has arithmetic genus $p_a(E) = 0$;*

(2) *one has the exact square formula*

$$C^2 = -1 - v_1^t M_1^{-1} v_1 - v_2^t M_2^{-1} v_2 > 0;$$

(3) *if* $D_1 = A = [a_1, \dots, a_r]$ *and* $D_2 = B = [b_1, \dots, b_s]$ *are chains, and* L *meets the* i*-th component of* A *and the* j*-th component of* B*, then*

$$C^2 = -1 + \frac{d(A_i^-)d(A_i^+)}{d(A)} + \frac{d(B_j^-)d(B_j^+)}{d(B)}.$$

Proof. Since D_1, D_2, and L are all trees of smooth rational curves, and L joins D_1 to D_2 by one transverse point on each side, the dual graph of E is itself a tree. Hence $p_a(E) = 0$, proving (1).

Write

$$\pi^* C = L + \sum_\alpha a_\alpha E_\alpha^{(1)} + \sum_\beta b_\beta E_\beta^{(2)},$$

where $E_\alpha^{(1)} \subset D_1$ and $E_\beta^{(2)} \subset D_2$. Orthogonality to all exceptional components gives

$$M_1 a = -v_1, \qquad M_2 b = -v_2,$$

so

$$a = -M_1^{-1} v_1, \qquad b = -M_2^{-1} v_2.$$

Since D_1 and D_2 are disjoint, there are no mixed terms. Therefore

$$C^2 = (\pi^* C)^2 = -1 + a^t v_1 + b^t v_2 = -1 - v_1^t M_1^{-1} v_1 - v_2^t M_2^{-1} v_2,$$

which proves (2). Positivity follows exactly as in Proposition 9.28(2), because C is a nonzero effective curve on the Picard-rank-one surface X.

If A and B are chains, the standard inverse formula for tridiagonal continued-fraction matrices gives

$$-(M_A^{-1})_{ii} = \frac{d(A_i^-)d(A_i^+)}{d(A)}, \qquad -(M_B^{-1})_{jj} = \frac{d(B_j^-)d(B_j^+)}{d(B)},$$

and substituting these into (2) proves (3). □

Proposition 9.47 (Anticanonical contact formula for chain bridges)**.** *In the chain setup of Proposition 9.46(3), define*

$$\mu(A, i) := \frac{d(A_i^-) + d(A_i^+)}{d(A)}, \qquad \mu(B, j) := \frac{d(B_j^-) + d(B_j^+)}{d(B)}.$$

Then the image curve $C = \pi(L)$ *satisfies the exact anticanonical contact formula* $-K_X \cdot C = -1 + \mu(A, i) + \mu(B, j)$.

Proof. Write

$$\pi^*C = L + \Gamma_A + \Gamma_B, \qquad \Gamma_A = \sum_{k=1}^{r} x_k A_k, \qquad \Gamma_B = \sum_{\ell=1}^{s} y_\ell B_\ell.$$

Since $\pi^*C \cdot A_k = 0$ for every k, one has

$$\delta_{k,i} - a_k x_k + x_{k-1} + x_{k+1} = 0 \qquad (1 \le k \le r),$$

where by convention $x_0 = x_{r+1} = 0$. Therefore

$$(2 - a_k)x_k = -\delta_{k,i} + 2x_k - x_{k-1} - x_{k+1}.$$

Summing over k gives

$$-K_{\widetilde{X}} \cdot \Gamma_A = \sum_{k=1}^{r} (2 - a_k)x_k = -1 + x_1 + x_r,$$

because the discrete Laplacian term telescopes to $x_1 + x_r$. By the standard inverse formula for tridiagonal continued-fraction matrices,

$$x_1 = -(M_A^{-1})_{1i} = \frac{d(A_i^+)}{d(A)}, \qquad x_r = -(M_A^{-1})_{ri} = \frac{d(A_i^-)}{d(A)}.$$

Hence $-K_{\widetilde{X}} \cdot \Gamma_A = -1 + \mu(A, i)$. The same argument on the B-side gives $-K_{\widetilde{X}} \cdot \Gamma_B = -1 + \mu(B, j)$. Finally,

$$-K_X \cdot C = -\pi^*K_X \cdot \pi^*C = -K_{\widetilde{X}} \cdot \pi^*C,$$

because $\pi^*K_X - K_{\widetilde{X}}$ is exceptional and π^*C is orthogonal to every exceptional component. Therefore

$$-K_X \cdot C = -K_{\widetilde{X}} \cdot L - K_{\widetilde{X}} \cdot \Gamma_A - K_{\widetilde{X}} \cdot \Gamma_B = 1 + (-1 + \mu(A, i)) + (-1 + \mu(B, j)),$$

which is the claimed formula. □

We now test the first cross-link corridor that is geometrically visible from the canonical side, namely bridges from a component of type $[2]$ or $[2, 2]$ into the terminal A_2-tails

$$T_n(a) := [a, (2)^{n-1}], \qquad a \in \{3, 4\}, \ n \ge 2.$$

Write $E_1, \dots, E_n$ for the components of $T_n(a)$, with $E_1^2 = -a$.

Proposition 9.48 (Explicit cross-link survivors against the terminal A_2-tails)**.** *The first distinct-component cross-links from the canonical corridor into the terminal A_2-tail corridor are as follows.*

1. *Let $Q = [2]$, with component U, and let L meet U and $E_2 \subset T_n(a)$ once each. Then the rooted graph is root-contractible if and only if $n = a$. In that case the contraction sequence*

is $L,\ E_2,\ E_3,\ \dots,\ E_n,\ E_1$*, leaving* U *as the root, and*

$$C^2 = \frac{(a+1)n - (2a+1)}{2((a-1)n+1)}.$$

Hence the only such cases are

$$[2] \dashv [3,2,2] \qquad \textit{and} \qquad [2] \dashv [4,2,2,2],$$

with

$$C^2 = \frac{5}{14} \qquad \textit{and} \qquad C^2 = \frac{11}{26},$$

respectively.

(2) *Let* $Q = [2,2]$*, with tips* U_1, U_2*, and let* L *meet* U_1 *and* $E_1 \subset T_n(a)$ *once each. Then the rooted graph is root-contractible if and only if* $a = 3$*. In that case the contraction sequence is* $L,\ U_1,\ E_1,\ E_2,\ \dots,\ E_n$*, leaving* U_2 *as the root, and*

$$C^2 = \frac{n-1}{3(2n+1)} > 0.$$

(3) *An exact rooted-graph search through*

$$2 \le n \le 20, \qquad a \in \{3,4\},$$

finds no other positive root-contractible cross-link from $[2]$ *or* $[2,2]$ *into* $T_n(a)$*.*

Proof. For (1), contract L first. Then $U^2 = -1$, the hit component E_2 becomes a (-1)-curve, and every other component of $T_n(a)$ keeps its original weight. If one insists on preserving U as the root, then the first blowdown inside the tail is forced: one must contract E_2. After that, E_3 becomes a (-1)-curve, so the tail continues to peel in the forced order $E_2,\ E_3,\ \dots,\ E_n$. At the end of this forced peeling, the only remaining component of $T_n(a)$ is E_1, and its self-intersection is $E_1^2 = -a + (n-1) = n - a - 1$. Thus E_1 is contractible if and only if $n = a$, proving the root-contractibility criterion. The displayed value of C^2 follows from Proposition 9.46(3), because

$$d(T_n(a)) = (a-1)n+1, \qquad \frac{d(T_n(a)^-_{E_2})d(T_n(a)^+_{E_2})}{d(T_n(a))} = \frac{a(n-1)}{(a-1)n+1}.$$

Substituting the [2]-contribution 1/2 gives the stated formula.

For (2), contract L first. Then $U_1^2 = -1$ and $E_1^2 = -(a-1)$. If one preserves U_2 as the root, then U_1 is the unique forced blowdown on the $[2,2]$ side. After contracting U_1, the component E_1 has self-intersection $E_1^2 = -(a-2)$. Hence E_1 is contractible if and only if $a = 3$. In that case the remaining tail is $U_2 - E_1 - E_2 - \cdots - E_n$ with $E_1^2 = -1$ and all later $E_i^2 = -2$, so the peeling order $E_1,\ E_2,\ \dots,\ E_n$ is forced and leaves U_2 as the root. The square formula follows from Proposition 9.46(3): the $[2,2]$-tip contributes $2/3$, the heavy end of $T_n(3)$ contributes $n/(2n+1)$, and therefore

$$C^2 = -1 + \frac{2}{3} + \frac{n}{2n+1} = \frac{n-1}{3(2n+1)}.$$

Finally, (3) follows from a direct exhaustive enumeration of every contact pattern from [2] or [2, 2] to the chains $T_n(3)$ and $T_n(4)$ for $2 \leq n \leq 20$, which computes the exact square using Proposition 9.46(3), and then performs a complete root-contractibility search on the resulting weighted multigraph. □

Remark 9.49 (What this changes in the distinct-component corridor). Proposition 9.48 isolates the first genuine residual cross-link family. The two [2]-cross-links are short and explicit, while the only infinite survivor in the first canonical corridor is the peelable genus-0 family $[2,2] \dashv [3,(2)^m], \qquad m \geq 1.$ obtained by attaching a tip of [2, 2] to the heavy end of the tail. In particular, the first distinct-component corridor does *not* open a new characteristic-3 mechanism on the $[4]/A_2$ side. The next subsection turns the auxiliary fork scan into an exact classification on the first branch-shaped frontier.

9.15. Exact canonical cross-links on the first fork frontier

We now upgrade the auxiliary fork scan above into a complete local classification on the first branch-shaped $s = 8$ frontier from [7, Lemma 6.2(3)]. Let

$$F_b = \langle b; [2], [3], [5] \rangle, \qquad b \geq 2,$$

with branching component B and twig tips U, V, W of self-intersection $-2, -3, -5$, respectively.

Proposition 9.50 (Exact canonical cross-links on the first fork frontier). *Let Q be either* [2] *or* [2, 2]*, and let L be a distinct-component bridge from Q to one component of F_b. Then the rooted graph $Q + L + F_b$ has positive image square and is root-contractible if and only if one of the following holds.*

1. *$Q = [2]$, the curve L meets the unique component $A \subset [2]$ and the branching component $B \subset F_b$, and $b = 3$. In that case*
$$C^2 = \frac{1}{118},$$
and the contraction sequence $L,\ A,\ B,\ U,\ V$ leaves W as the root.
2. *$Q = [2, 2]$, the curve L meets one tip $A_1 \subset [2, 2]$ and the branching component $B \subset F_b$, and $b = 4$. Writing A_2 for the other component of* [2, 2]*, one has*
$$C^2 = \frac{1}{267},$$
and the contraction sequence $L,\ A_1,\ A_2,\ B,\ U,\ V$ leaves W as the root.

No other positive root-contractible cross-link from [2] *or* [2, 2] *to F_b exists.*

Proof. A direct inversion of the intersection matrix of F_b gives

$$-(M_{F_b}^{-1})_{BB} = \frac{30}{30b - 31}, \qquad -(M_{F_b}^{-1})_{UU} = \frac{15b - 8}{30b - 31},$$

$$-(M_{F_b}^{-1})_{VV} = \frac{10b-7}{30b-31}, \qquad -(M_{F_b}^{-1})_{WW} = \frac{6b-5}{30b-31}.$$

If $Q = [2]$, the unique diagonal contribution from the canonical side is $1/2$, so

$$C^2_{[2],B} = \frac{91-30b}{2(30b-31)}, \quad C^2_{[2],U} = \frac{15}{2(30b-31)}, \quad C^2_{[2],V} = \frac{17-10b}{2(30b-31)}, \quad C^2_{[2],W} = \frac{3(7-6b)}{2(30b-31)}.$$

If $Q = [2,2]$, a tip contributes $2/3$, so

$$C^2_{[2,2],B} = \frac{121-30b}{3(30b-31)}, \quad C^2_{[2,2],U} = \frac{15b+7}{3(30b-31)}, \quad C^2_{[2,2],V} = \frac{10}{3(30b-31)}, \quad C^2_{[2,2],W} = \frac{4(4-3b)}{3(30b-31)}.$$

All of these are immediate from Proposition 9.46(2).

We now analyze root-contractibility case by case.

For $Q = [2]$, the V- and W-attachments are never positive for $b \geq 2$, while the U-attachment is always positive. Let A be the unique component of $[2]$. If L meets U, then after contracting L the only (-1)-curves are A and U. Contracting A first makes $U^2 = 0$, so no further blowdown is possible. Contracting U first makes $A^2 = 0$ and $B^2 = -(b-1)$; for $b \geq 3$ there is again no (-1)-curve, and for $b = 2$ the only possible next blowdown is B, after which the remaining components have self-intersection $A^2 = 1$, $V^2 = -2$, $W^2 = -4$. Thus the U-attachment is never root-contractible.

It remains to consider the B-attachment. The square formula shows that positivity holds only for $b = 2, 3$. If $b \geq 4$, then after contracting L the curve A is the unique (-1)-curve, and blowing it down changes B^2 to $-(b-2) \leq -2$, so the process stops. If $b = 2$, then after contracting L both A and B are (-1)-curves. Blowing down A first gives $B^2 = 0$. Blowing down B first gives $A^2 = 0$, $U^2 = -1$, $V^2 = -2$, $W^2 = -4$; then blowing down U gives $A^2 = 1$, $V^2 = -1$, $W^2 = -3$, and blowing down V leaves two components, so no root-contractibility is possible. For $b = 3$, on the other hand, the contraction sequence L, A, B, U, V exists and leaves W as the root. Substituting $b = 3$ into $C^2_{[2],B}$ gives $C^2 = 1/118$.

Now let $Q = [2,2]$, and write A_1 for the component hit by L and A_2 for the other component. The W-attachment is never positive for $b \geq 2$. If L meets U, then after contracting L the only (-1)-curves are A_1 and U. Blowing down A_1 makes $A_2^2 = -1$ and $U^2 = 0$, so one cannot continue to a single root. Blowing down U first makes $A_1^2 = 0$ and $B^2 = -(b-1)$; for $b \geq 3$ there is no further (-1)-curve, and for $b = 2$ the only next blowdown is B, after which the remaining components have self-intersection $A_1^2 = 1$, $V^2 = -2$, $W^2 = -4$. Hence the U-attachment is never root-contractible.

If L meets V, then after contracting L the curve A_1 is the unique (-1)-curve, so its blowdown is forced. After that, both A_2 and V are (-1)-curves. Blowing down A_2 makes $V^2 = 0$, so that branch stops. Blowing down V instead makes $A_2^2 = 0$ and $B^2 = -(b-1)$; for $b \geq 3$ there is again no further (-1)-curve, while for $b = 2$ the only next blowdown is B, after which one has $A_2^2 = 1$, $U^2 = -1$, $W^2 = -4$, and then blowing down U still leaves two components. So the V-attachment is never root-contractible.

Finally, consider the B-attachment. Positivity holds only for $b = 2, 3, 4$. If $b \geq 5$, then after contracting L and the forced (-1)-curve A_1, the curve A_2 is the unique (-1)-curve and $B^2 = -(b-2) \leq -3$; blowing down A_2 changes B^2 to $-(b-3) \leq -2$, while keeping A_2 as the root leaves no other (-1)-curve. Thus no root-contractibility is possible. If $b = 2$, then after contracting L the curves A_1 and B are both (-1)-curves. Blowing down A_1 gives $A_2^2 = -1$ and $B^2 = 0$, so the process stops. Blowing down B first gives $A_1^2 = 0$, $U^2 = -1$, $V^2 = -2$, $W^2 = -4$, and then blowing down U still leaves no route to a single root. If $b = 3$, then after contracting L and the forced curve A_1, one gets $A_2^2 = B^2 = -1$. Blowing down A_2 forces $B^2 = 0$, while blowing down B first gives $A_2^2 = 0$, $U^2 = -1$, $V^2 = -2$, $W^2 = -4$; after blowing down U and then V, two components still remain. Hence $b = 3$ also fails. For $b = 4$, however, the contraction sequence $L,\ A_1,\ A_2,\ B,\ U,\ V$ leaves W as the root, and substituting $b = 4$ into $C^2_{[2,2],B}$ gives $C^2 = 1/267$.

A direct exhaustive enumeration scans every bridge from [2] or [2, 2] into F_b for $2 \leq b \leq 200$, computes the exact square via Proposition 9.46(2), and finds exactly the two families listed above. □

Remark 9.51 (What remains after the first fork cross-link pass). Combining Proposition 9.48 with Proposition 9.50, the only infinite distinct-component survivor currently visible from the first canonical corridor is the peelable genus-0 family $[2, 2] \dashv [3, (2)^m], \qquad m \geq 1$. All other survivors on the first chain and fork frontiers are isolated short bridges. So after the exact local passes above, the next real bottleneck in route (C) is no longer the $[4]/A_2$ side or the bare fork $\langle b; [2], [3], [5]\rangle$, but rather the global behavior of that peelable family and whatever more elaborate heavy/fork configurations lie beyond the first frontier.

9.16. The peelable family has a single birational prototype

We now return to the only infinite distinct-component survivor left by the first exact chain and fork passes, namely the family

$$\Gamma_n := [2, 2] \dashv [3, (2)^{n-1}], \qquad n \geq 2,$$

from Proposition 9.48(2). Write $U_2 - U_1$ for the [2, 2]-chain, with U_2 the untouched tip and U_1 the tip hit by the bridge L, and write $E_1, \dots, E_n$ for the tail $[3, (2)^{n-1}]$, so that $U_2 - U_1 - L - E_1 - E_2 - \cdots - E_n$ has weights $2, 2, 1, 3, 2, \dots, 2$.

Proposition 9.52 (The peelable family is a determinant-one staircase). *Let* $S_n := U_1 + L + E_1 + \cdots + E_n = [2, 1, 3, (2)^{n-1}]$ *be the non-root part of* Γ_n. *Then:*

1. $d(S_n) = 1$.
2. *The chain* S_n *contracts to a smooth point by the forced blowdown sequence* $L,\ U_1,\ E_1,\ E_2,\ \dots,\ E_n$.
3. *If* Y_n *is the resulting smooth surface and* $R_n \subset Y_n$ *is the image of* U_2, *then*

$$R_n \cong \mathbf{P}^1, \qquad R_n^2 = n - 1.$$

(4) *Consequently every occurrence of* Γ_n *is obtained by a sequence of infinitely near blowups over a single smooth point of the curve* R_n*. In particular, the family* Γ_n *carries no new local singularity type beyond a smooth-point staircase over one surviving rational curve.*

Proof. The discriminant formulas

$$d([3,(2)^{n-1}]) = 2n+1, \qquad d([1,3,(2)^{n-1}]) = n+1$$

follow immediately by the standard recurrence for chains. Therefore $d(S_n) = d([2,1,3,(2)^{n-1}]) = 2(n+1) - (2n+1) = 1$, which proves (1).

For (2), contract L first. Then $U_1^2 = -1$ and every other component of S_n has self-intersection at most -2, so U_1 is forced. After contracting U_1, the component E_1 becomes a (-1)-curve. Contracting E_1 makes E_2 a (-1)-curve, and so on, so the displayed order is forced and contracts the whole chain S_n. Because $d(S_n) = 1$, the target point is smooth.

Statement (3) follows by tracing the self-intersection of U_2. It starts with $U_2^2 = -2$. The contraction of U_1 raises it by 1, and then each contraction of $E_1, \ldots, E_n$ raises it by another 1, because at each stage the current transform of E_i meets the current transform of U_2 once. Hence $R_n^2 = -2+1+n = n-1$. The curve remains smooth and rational because it is the image of the smooth rational curve U_2 under a contraction of curves disjoint from its smooth locus.

Finally, (4) is simply the inverse interpretation of (2): the birational morphism $\widetilde{X} \to Y_n$ contracts the chain S_n to one smooth point of R_n, so recovering Γ_n from Y_n amounts to a sequence of point blowups over that single point. ☐

The previous proposition removes the infinite combinatorics from this corridor. After the staircase S_n is contracted, the only surviving datum is the self-intersection $R_n^2 = n-1$ of one smooth rational curve disjoint from the rest of the boundary.

Corollary 9.53 (A single birational prototype controls the peelable family)**.** *Let* $B_n \subset Y_n$ *be the image of the rest of the exceptional divisor, so that* $B_n \cap R_n = \varnothing$*. Then the following hold.*

(1) *After blowing up* $n-2$ *general points of* $R_n \setminus B_n$*, the proper transform* R *of* R_n *has self-intersection* $R^2 = 1$*, and remains disjoint from the transformed boundary.*

(2) *Fix a point* $p \in R$ *away from the transformed boundary. Starting from the pair* (Y, R) *and blowing up in the following order:*

  (a) *the point* $p \in R$*;*

  (b) *the intersection of the new exceptional curve with the transform of* R*;*

  (c) *the intersection of the newest exceptional curve with the transform of* R*;*

  (d) *the intersection of the last two newly created curves,*

one obtains the rooted chain $[2,2] \dashv [3,2]$.

Hence, up to birational modifications supported away from the rest of the boundary, the whole infinite family $[2,2] \dashv [3,(2)^m], \qquad m \geq 1$, *has the single prototype* $[2,2] \dashv [3,2]$.

Proof. Part (1) is immediate: each blowup at a smooth point of R_n lowers the self-intersection by 1 and, by construction, does not meet the transformed boundary.

For (2), write $R_0 := R$. Blow up $p \in R_0$ and call the new exceptional curve A_2. Then R_0^2 drops from 1 to 0 and $A_2^2 = -1$. Blow up the intersection of the transform of R_0 with A_2 and call the new exceptional curve A_1. Now the transform of R_0 has self-intersection -1, the curve A_2 has self-intersection -2, and $A_1^2 = -1$. Blow up the intersection of the transform of R_0 with A_1 and call the new exceptional curve U_1. Then the transform of R_0 has self-intersection -2, the curve A_1 has self-intersection -2, and $U_1^2 = -1$. Finally, blow up the intersection of U_1 with A_1 and call the new exceptional curve L. This changes the two adjacent curves to self-intersection -2 and -3, respectively, and leaves $L^2 = -1$. Thus the final weighted chain is

$$R_0^{\mathrm{str}}(-2) - U_1(-2) - L(-1) - A_1(-3) - A_2(-2),$$

which is exactly the rooted graph $[2,2] \dashv [3,2]$ after renaming

$$U_2 := R_0^{\mathrm{str}}, \qquad E_1 := A_1, \qquad E_2 := A_2.$$

The whole construction was performed away from the transformed boundary, so it is birationally invisible to the rest of D. ☐

Remark 9.54 (What this changes in the peelable corridor). Corollary 9.53 does not yet give the desired contradiction, but it does remove the last infinite local freedom from the first distinct-component corridor. The family $[2,2] \dashv [3,(2)^m]$ is not an infinite supply of essentially different rooted graphs: after the smooth contraction of Proposition 9.52, all that remains is one smooth rational root curve, and after general blowups away from the rest boundary the unique non-trivial local staircase to analyze is the base prototype $[2,2] \dashv [3,2]$. In particular, any future contradiction for this corridor must now be genuinely global and can be sought already on the square-one prototype.

9.17. The square-one prototype has a global line-at-infinity normal form

The previous subsection reduced the only infinite distinct-component survivor in the first canonical corridor to the single local prototype $[2,2] \dashv [3,2]$. The next step is global: put that prototype on a birational plane model and record the ruling that comes for free from the line at infinity.

Proposition 9.55 (Square-one curves are lines on a birational plane model). *Keep the notation of Corollary 9.53: Y is a smooth rational surface, $B \subset Y$ is the image of the rest of the exceptional divisor, $R \subset Y$ is a smooth rational curve with*

$$R^2 = 1, \qquad B \cap R = \varnothing.$$

Then the complete linear system $|R|$ is base-point free of dimension 2. Its associated morphism $\tau := \phi_{|R|} \colon Y \to \mathbf{P}^2$ is birational, contracts only curves disjoint from R, and maps R isomorphi-

cally onto a line $L_\infty \subset \mathbf{P}^2$*. In particular,*

$$\tau(B) \subset \mathbf{P}^2 \setminus L_\infty \cong \mathbf{A}^2.$$

Proof. Consider the exact sequence

$$0 \to \mathcal{O}_Y \to \mathcal{O}_Y(R) \to \mathcal{O}_R(R) \to 0.$$

Since Y is rational, $H^1(Y, \mathcal{O}_Y) = 0$. Since $R \cong \mathbf{P}^1$ and $R^2 = 1$, we have $\mathcal{O}_R(R) \cong \mathcal{O}_{\mathbf{P}^1}(1)$, so the associated long exact sequence gives

$$h^0(Y, \mathcal{O}_Y(R)) = 1 + h^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(1)) = 3,$$

and the restriction map

$$H^0(Y, \mathcal{O}_Y(R)) \twoheadrightarrow H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(1))$$

is surjective. Therefore $|R|$ has no base point on R.

It also has no base point away from R, because R itself is a member of $|R|$. Hence $|R|$ is base-point free and has dimension 2, so it defines a morphism $\tau = \phi_{|R|} \colon Y \to \mathbf{P}^2$ with

$$\tau^* \mathcal{O}_{\mathbf{P}^2}(1) \cong \mathcal{O}_Y(R).$$

Consequently

$$\deg(\tau) \cdot \deg \tau(Y) = \left(\tau^* \mathcal{O}_{\mathbf{P}^2}(1)\right)^2 = R^2 = 1.$$

Thus $\tau(Y)$ is a plane of degree 1 and τ is birational.

If an irreducible curve $C \subset Y$ is contracted by τ, then

$$R \cdot C = \tau^* \mathcal{O}_{\mathbf{P}^2}(1) \cdot C = 0.$$

Since R is effective, smooth, and irreducible, this forces $C \cap R = \varnothing$. So every contracted curve is disjoint from R, and τ is an isomorphism in a neighborhood of R.

Finally, the restriction of $|R|$ to R is the complete linear system $|\mathcal{O}_{\mathbf{P}^1}(1)|$, so $\tau|_R$ is an isomorphism from R onto a line $L_\infty \subset \mathbf{P}^2$. Because $B \cap R = \varnothing$ and every curve contracted by τ is disjoint from R, the image $\tau(B)$ lies in the affine plane $\mathbf{P}^2 \setminus L_\infty \cong \mathbf{A}^2$. □

Proposition 9.56 (The square-one prototype carries a distinguished ruling)**.** *Keep the notation of Proposition 9.55. Choose a point* $p \in L_\infty$ *and perform the four blowups from Corollary 9.53(2) over* p*, producing the rooted chain* $U_2(-2) - U_1(-2) - L(-1) - E_1(-3) - E_2(-2)$*, where* U_2 *is the strict transform of* L_∞*. Let* $g \colon Z \to \mathbf{P}^2$ *denote the resulting birational morphism. Then the pencil of lines through* p *lifts to a morphism* $f \colon Z \to \mathbf{P}^1$*, with the following properties.*

1. *The outer curve* E_2 *on the* $[3, 2]$*-tail is a section of* f*.*
2. *The fiber of* f *corresponding to the line* L_∞ *has support* $U_2 + U_1 + L + E_1$*.*

(3) *If one further blows up any sequence of points lying over* $\tau(B) \subset \mathbf{A}^2$*, then the same morphism to* $\mathbf{P}^1$ *persists by composition, all new centers remain away from the line-at-infinity cluster, and the prototype continues to sit over one distinguished fiber with the same section* E_2*.*

Proof. On $\mathbf{P}^2$, consider the pencil of lines through p. After the first blowup at p, this pencil becomes a morphism to $\mathbf{P}^1$; the first exceptional curve is the usual section of that ruling. The next three blowups in Corollary 9.53(2) are performed at points on the special fiber corresponding to L_∞ or on curves created inside that same fiber. Therefore the composition with the ruling remains a morphism throughout, so the final surface Z carries a morphism $f\colon Z \to \mathbf{P}^1$ lifting the line pencil through p.

The first exceptional curve is precisely the curve called E_2 in the rooted chain above. Since only one point of it is ever used in the subsequent prototype blowups, its strict transform still maps isomorphically onto the base, so E_2 is a section of f.

The only vertical curves created in the four-step cluster are the strict transform U_2 of L_∞ together with the three later curves U_1, L, and E_1. Hence the fiber corresponding to L_∞ has support $U_2 + U_1 + L + E_1$, while E_2 meets that fiber as a section.

Finally, Proposition 9.55 places the rest boundary inside $\mathbf{A}^2 = \mathbf{P}^2 \setminus L_\infty$. So any later blowups performed over $\tau(B)$ occur away from the line-at-infinity cluster. Composing the previous morphism with such blowups still gives a morphism to $\mathbf{P}^1$, and the prototype remains concentrated over the same special fiber at infinity. ☐

Remark 9.57 (What the line-at-infinity normal form isolates)*.* Propositions 9.55 and 9.56 isolate the geometry of the square-one prototype very explicitly: up to birational contraction of curves disjoint from R, it lives on a pair $(\mathbf{P}^2, L_\infty)$ with exactly one four-step boundary-free cluster over a point of L_∞ and the rest of the Palka–Pełka boundary inside $\mathbf{A}^2$. What this does *not* yet prove is that the peelable corridor is impossible in the many-singularity range. What it does prove is that the whole infinite family $[2,2] \dashv [3,(2)^m]$ is controlled by one square-one one-point-at-infinity model, together with blowups away from the rest boundary. Thus the remaining work on that corridor is global rather than local.

9.18. A formal Step 2 on the first analyzed frontier

The calculations above now admit a clean synthesis. They do not yet prove that *every* residual contact-2 rooted configuration enters the first chain/fork/cross-link corridor analyzed in this report, but once one is on that frontier the local analysis is essentially complete. The only remaining logical input for a full Step 2 theorem is therefore a frontier-completeness statement saying that every residual positive root-contractible contact-2 configuration either already gives an elliptic tie or belongs to one of the first frontiers treated above.

Theorem 9.58 (First-frontier exhaustion up to a finite shortlist)**.** *Let* X *be a klt del Pezzo surface of Picard rank* 1 *over an algebraically closed field of characteristic* 3*, let* $\pi\colon \widetilde{X} \to X$ *be* *the minimal resolution, and let D be the reduced exceptional divisor. Let $L \not\subset D$ be a (-1)-curve such that $L \cdot D = 2$, and assume that the corresponding contact-2 rooted configuration is positive and root-contractible. Assume moreover that the connected component(s) of D met by L lie in the first analyzed chain/fork/cross-link frontier, in the following precise sense:*

(a) *either both contacts lie on one connected component of type $T_n(a) = [a, (2)^{n-1}], \qquad a \in \{3, 4\}$, or on a fork*

$$F_b = \langle b; [2], [3], [5] \rangle, \qquad b \geq 2;$$

(b) *or L is a distinct-component bridge from $Q = [2]$ or $Q = [2, 2]$ to one component of $T_n(a)$ or F_b.*

Then exactly one of the following occurs.

(i) *The divisor supported on the rooted configuration contains an elliptic tie. In particular, X has a descendant with elliptic boundary.*

(ii) *The configuration is one of the peelable bridges $[2, 2] \dashv [3, (2)^m], \qquad m \geq 1$.*

(iii) *The configuration is one of the four isolated short bridges*

$$[2] \dashv [3, 2, 2], \qquad [2] \dashv [4, 2, 2, 2], \qquad [2] \to F_3, \qquad [2, 2] \to F_4.$$

In particular, on the first analyzed frontier every positive root-contractible contact-2 configuration is either an elliptic tie, a member of the peelable family, or one of the four isolated short bridges.

Proof. Assume first that both contacts lie on one connected component of D. If that component is a terminal A_2-tail $T_n(a)$ with $a \in \{3, 4\}$, then Theorem 9.42 gives the exact local classification of the positive contact-2 tie-producing patterns on that corridor. Since the rooted configuration is assumed root-contractible, Lemma 9.41 identifies it with a local sandwiched $\delta = 1$ divisor, hence with an elliptic tie. So alternative (i) holds in the chain case.

If instead the touched component is the fork $F_b = \langle b; [2], [3], [5] \rangle$, then Proposition 9.44 shows that every positive root-contractible contact-2 pattern on that frontier is already an explicit elliptic tie. So alternative (i) also holds in the first fork case.

Assume next that L is a distinct-component bridge. If the target side is a terminal A_2-tail $T_n(a)$ and the other side is $[2]$ or $[2, 2]$, then Proposition 9.48 gives the complete list of positive root-contractible cross-links on that corridor. It consists precisely of the two isolated short bridges $[2] \dashv [3, 2, 2], \qquad [2] \dashv [4, 2, 2, 2]$, and the infinite peelable family $[2, 2] \dashv [3, (2)^m], \qquad m \geq 1$. Thus alternatives (ii) and (iii) account for all terminal-tail cross-links.

If the target side is the fork F_b and the other side is $[2]$ or $[2, 2]$, then Proposition 9.50 gives the complete list of positive root-contractible bridges on that frontier. It consists exactly of

$$[2] \to F_3 \qquad \text{and} \qquad [2, 2] \to F_4,$$

which are the two remaining configurations listed in alternative (iii). This proves the trichotomy and the final summary sentence. □

Corollary 9.59 (Formal Step 2 modulo frontier completeness)**.** *Assume the boxed residual Route C entry statement of Corollary 9.4, and assume the following frontier-completeness input:*

> *Every positive root-contractible contact-2 configuration in the residual height-≥ 4, no-descendant sector either already supports an elliptic tie or belongs to the first analyzed chain/fork/cross-link frontier.*

Then every characteristic-3 Picard-rank-one klt del Pezzo surface X with $\#\operatorname{Sing}(X) \geq 8$ contains, on its minimal resolution, either the peelable family $[2,2] \dashv [3,(2)^m], \qquad m \geq 1$, or one of the four isolated short bridges from Theorem 9.58(iii). In particular, to prove that no such surface exists it is enough to rule out the peelable family and the four short bridges globally.

Proof. By Theorem 9.1, every benchmark counterexample already lies in the residual height-≥ 4, no-descendant sector. Under the boxed statement of Corollary 9.4, such a surface admits a positive root-contractible contact-2 configuration. If that configuration already supports an elliptic tie, then the equivalence between elliptic ties and descendants with elliptic boundary contradicts the residual no-descendant assumption. So the frontier-completeness hypothesis forces the configuration onto the first analyzed chain/fork/cross-link frontier. Theorem 9.58 then applies, and its alternatives are exactly an elliptic tie, the peelable family, or one of the four isolated short bridges. The elliptic-tie case is already excluded, so only the remaining five non-tie local possibilities survive. □

Remark 9.60 (What remains after the formal Step 2 package)*.* Theorem 9.58 is unconditional and packages all of the exact local work already proved in this paper. Corollary 9.59 is a logical reduction: it shows that, once the residual Route C entry theorem and frontier completeness are available, the many-singularity problem reduces to the peelable family together with four isolated short bridges. The next subsection turns that still-infinite list into a finite global prototype problem on $(\mathbf{P}^2, L_\infty)$.

9.19. A formal Step 3: finite global prototype reduction

The corrected global output of the first Route C frontier is not a direct contradiction but a finite prototype problem. The peelable family already has one square-one prototype by Corollary 9.53. We now put the four isolated short bridges into the same framework.

Proposition 9.61 (The short bridges peel to positive roots)**.** *Let X be a Picard-rank-one klt del Pezzo surface over an algebraically closed field of characteristic 3, let $\pi\colon \widetilde{X} \to X$ be the minimal resolution, and let D be the reduced exceptional divisor. Assume that $\widetilde{X}$ contains one of the four short bridges from Theorem 9.58(iii). Let $B \subset D$ be the union of the connected components of D disjoint from that bridge. Then, after contracting the non-root part of the bridge along the explicit root-contraction sequence, one obtains a smooth rational surface Y carrying the image*

$R \subset Y$ of the root and the image $B_Y \subset Y$ of B, with $B_Y \cap R = \varnothing$, and R a smooth rational curve. More precisely, the surviving root and its self-intersection are as follows:

bridge	*contraction sequence*	R^2
$[2] \dashv [3,2,2]$	$L,\ E_2,\ E_3,\ E_1$	5
$[2] \dashv [4,2,2,2]$	$L,\ E_2,\ E_3,\ E_4,\ E_1$	11
$[2] \to F_3$	$L,\ A,\ B,\ U,\ V$	1
$[2,2] \to F_4$	$L,\ A_1,\ A_2,\ B,\ U,\ V$	1

In the first two cases the root is the $[2]$-component U, and in the last two cases it is the $[5]$-tip W of the fork.

Proof. The contraction sequences are exactly the ones recorded in Propositions 9.48 and 9.50. Since all curves in B are disjoint from the rooted configuration, their image B_Y remains disjoint from the surviving root on the resulting smooth surface Y.

The final self-intersections are obtained by tracking the root under the multigraph blowdown rule $C^2 \longmapsto C^2 + (C \cdot E)^2$ for the contraction of a (-1)-curve E. For $[2] \dashv [3,2,2]$, the root U starts with self-intersection -2; contracting L, E_2, and E_3 raises it to -1, 0, and 1, and the last curve E_1 meets the root twice, so its contraction raises the self-intersection to 5. For $[2] \dashv [4,2,2,2]$, the same computation gives the chain of values $-2, -1, 0, 1, 2, 11$. For $[2] \to F_3$, the root W stays at -5 through the contractions of L and A, then moves to -4 and -3 after contracting B and U, and finally V meets it twice, so the last contraction raises W^2 to 1. The case $[2,2] \to F_4$ is identical after the extra contraction of A_2: again the last curve meets the root twice and the final self-intersection is 1. $\square$

Proposition 9.62 (Each short bridge has a finite line-at-infinity model)**.** *Keep the notation of Proposition 9.61.*

1. *For the fork bridges $[2] \to F_3$ and $[2,2] \to F_4$, one has $R^2 = 1$. Therefore Proposition 9.55 applies: there is a birational morphism $\tau\colon Y \to \mathbf{P}^2$ sending R isomorphically onto a line $L_\infty \subset \mathbf{P}^2$ and mapping B_Y into $\mathbf{P}^2 \setminus L_\infty \cong \mathbf{A}^2$. Reversing the contraction sequences from Proposition 9.61 reconstructs the short bridge by a boundary-free cluster over a single point $p \in L_\infty$.*
2. *For the tail bridges $[2] \dashv [3,2,2]$ and $[2] \dashv [4,2,2,2]$, one has $R^2 = 5$ and $R^2 = 11$, respectively. Blow up 4 or 10 general points of R, chosen away from B_Y and away from the local bridge cluster. The strict transform R' then satisfies*

$$(R')^2 = 1, \qquad B' \cap R' = \varnothing,$$

so Proposition 9.55 again gives a birational morphism to $\mathbf{P}^2$ sending R' to a line L_∞. Reversing the contraction sequence reconstructs the short bridge over one point $p \in L_\infty$, while the auxiliary blowups become 4 or 10 free marked points on L_∞ away from p.

In particular, the four short bridges reduce to four explicit global prototypes on a pair $(\mathbf{P}^2, L_\infty)$.

Proof. Part (1) is immediate from Proposition 9.61 and the square-one plane model of Proposition 9.55. The inverse of each listed contraction sequence is a succession of blowups whose first center lies on R and whose later centers lie on the total transform created by the previous local blowups, so all local centers lie over one point $p \in R$. Transporting that picture through τ shows that the whole short bridge lives over one point of the line at infinity.

For part (2), each auxiliary blowup lowers the self-intersection of the strict transform of R by 1 and, by construction, is disjoint from both B_Y and the local short-bridge cluster. After 4 or 10 such blowups one is in the square-one situation, so Proposition 9.55 applies exactly as in part (1). The same inverse-blowup argument again shows that the local short bridge is concentrated over one point of the resulting line at infinity, while the extra general blowups remain as free points on that line away from the local cluster. □

Theorem 9.63 (Formal Step 3: finite prototype reduction)**.** *Assume the boxed residual Route C entry statement of Corollary 9.4 and the frontier-completeness hypothesis of Corollary 9.59. Then every characteristic-*3 *Picard-rank-one klt del Pezzo surface* X *with* $\#\operatorname{Sing}(X) \geq 8$ *reduces, after the local peeling operations of Corollary 9.53 and Proposition 9.61 and after possibly blowing up finitely many general points on the surviving positive curve away from the rest boundary, to one of the following five explicit global prototypes on a pair* $(\mathbf{P}^2, L_\infty)$*:*

(i) *the peelable square-one prototype* $[2,2] \dashv [3,2]$*;*

(ii) *the short-bridge prototype* $[2] \dashv [3,2,2]$ *together with* 4 *free points on* L_∞*;*

(iii) *the short-bridge prototype* $[2] \dashv [4,2,2,2]$ *together with* 10 *free points on* L_∞*;*

(iv) *the square-one short-bridge prototype* $[2] \to F_3$*;*

(v) *the square-one short-bridge prototype* $[2,2] \to F_4$*.*

*In particular, to prove that no characteristic-*3 *Picard-rank-one klt del Pezzo surface with at least* 8 *singular points exists, it is enough to rule out these five explicit global prototypes.*

Proof. By Corollary 9.59, every benchmark counterexample reduces first to the peelable family or to one of the four isolated short bridges. Corollary 9.53 and Propositions 9.55–9.56 turn the peelable family into the single prototype (i). Proposition 9.62 does the same for the four short bridges, giving prototypes (ii)–(v). This is exactly the claimed finite reduction. □

Remark 9.64 (What the corrected Step 3 leaves open)*.* The corrected global output of the first Route C frontier is a finite prototype problem, not yet a contradiction. In particular, the earlier temptation to conclude directly that the peelable family or the short bridges force $\#\operatorname{Sing}(X) \leq 3$ is stronger than the current proof supports. What is proved rigorously is that every first-frontier non-tie mechanism reduces to one of five explicit global models on $(\mathbf{P}^2, L_\infty)$. Thus the remaining theoretical bottleneck is now very concrete: either show that the residual sector never reaches the first frontier at all, or exclude those five prototypes globally.

9.20. The two square-one fork prototypes already leave the residual sector

The finite prototype reduction from Theorem 9.63 is not optimal. The two square-one fork prototypes can already be recognized as elliptic-boundary local models, so they do not survive inside the residual no-descendant sector.

Proposition 9.65 (The square-one fork prototypes force descendants)**.** *Let X be a characteristic-3 klt del Pezzo surface of Picard rank 1, let $\pi\colon \widetilde{X} \to X$ be the minimal resolution, and let D be the reduced exceptional divisor. Assume that $\widetilde{X}$ contains one of the two short bridges*

$$[2] \to F_3 \qquad \textit{or} \qquad [2,2] \to F_4$$

from Theorem 9.58(iii)*. Then X has a descendant with elliptic boundary.*

More precisely, with the notation of Proposition 9.50, let B be the branching component of the fork, let U, V, W be the [2]-*,* [3]-*,* [5]*-tips, and let Q denote the bridge side, with component(s) A or $A_1 - A_2$. Then the bridge staircase $Q + L$ contracts to a smooth point, and on the resulting smooth surface Z the image of $Q+L+F_b$ is the fork* $\langle 1; [2], [3], [5]\rangle$*, with $W =$* [5] *the distinguished twig. By [7, Lemma 6.2(b)*(3) *and its proof], this is exactly the three-blowup cuspidal local model for an elliptic boundary with $s = 8$. Hence the image of W on the further local contraction is an irreducible cuspidal curve of arithmetic genus one, contained in the smooth locus of the target.*

Proof. Consider first the bridge $[2] \to F_3$. Write A for the unique component of the [2]-side. By Proposition 9.50, the bridge curve L meets A and the branching component B. Contracting L turns A into a (-1)-curve and changes B^2 from -3 to -2. Contracting A next changes B^2 from -2 to -1. Thus on the resulting smooth surface Z the image of the local divisor is the fork

$$U(-2) - B(-1) - V(-3), \qquad B \text{ also meeting } W(-5),$$

that is, the weighted graph $\langle 1; [2], [3], [5]\rangle$ with W the [5]-twig.

Now consider the bridge $[2,2] \to F_4$. Here L meets one tip A_1 of the chain $A_1 - A_2 = [2,2]$ and the branching component B of F_4. Contracting L changes A_1^2 to -1 and B^2 to -3. Contracting A_1 changes A_2^2 to -1 and B^2 to -2. Contracting A_2 then changes B^2 to -1. Again we obtain on a smooth surface the same fork $\langle 1; [2], [3], [5]\rangle$ with twigs $U = [2]$, $V = [3]$, $W = [5]$.

By [7, Lemma 6.2(b)(3) and the construction in its proof], the fork $\langle 1; [2], [3], [5]\rangle$ is precisely the local total-transform graph obtained after three blowups over the cusp of a singular elliptic boundary T with $s = 8$; in that model the [5]-twig is the proper transform of T, while the other three components are exceptional over the cusp. Therefore the divisor $B + U + V$ contracts locally to a smooth point of a normal surface Y_0, the image T_0 of W is an irreducible cuspidal curve with $p_a(T_0) = 1$, and $T_0 \subset (Y_0)_{\mathrm{reg}}$.

Let D_{rest} be the union of the connected components of D disjoint from the local bridge. These components stay disjoint from the local cusp model throughout the above contractions. Contracting their images on Z to quotient singularities gives a further birational morphism $\eta\colon Z \to Y$ whose exceptional locus is disjoint from T_0. Writing $T := \eta_* T_0$, we obtain a birational morphism $\varphi\colon (\widetilde{X}, D) \to (Y, T)$ such that $\varphi_* D = T$, with $T \subset Y_{\mathrm{reg}}$ irreducible and $p_a(T) = 1$. By Definition 1.11 of [7], (Y, T) is a descendant of X, and its boundary is elliptic. □

Corollary 9.66 (The fork prototypes are not residual)**.** *Assume the hypotheses of Theorem 9.63. Then the two square-one fork prototypes*

$$[2] \to F_3 \qquad \textit{and} \qquad [2,2] \to F_4$$

cannot occur on a hypothetical benchmark counterexample. Equivalently, inside the residual height-≥ 4, no-descendant sector, the first-frontier non-tie survivors reduce to the following three tail-type prototypes on $(\mathbf{P}^2, L_\infty)$:

(i) *the peelable square-one prototype* $[2,2] \dashv [3,2]$*;*

(ii) *the short-bridge prototype* $[2] \dashv [3,2,2]$ *together with* 4 *free points on* L_∞*;*

(iii) *the short-bridge prototype* $[2] \dashv [4,2,2,2]$ *together with* 10 *free points on* L_∞*.*

Proof. Under the hypotheses of Theorem 9.63, every benchmark counterexample lies in the residual height-≥ 4, no-descendant sector by Theorem 9.1, and its first-frontier non-tie configuration reduces to one of the five prototypes listed in Theorem 9.63. By Proposition 9.65, the two fork prototypes instead force descendants with elliptic boundary, contradicting Theorem 9.1(2). Therefore only the three stated tail-type prototypes remain. □

Remark 9.67 (What Step 3 now leaves open)*.* After eliminating the two fork prototypes, the first analyzed Route C frontier is no longer a five-model problem in the residual sector. It has shrunk to three explicit tail-type prototypes on $(\mathbf{P}^2, L_\infty)$: one square-one peelable prototype and two higher-square short bridges with auxiliary free points on the line at infinity. Thus any further global contradiction on the first frontier has to come from these tail models, not from the fork side.

9.21. The explicit square-one peelable prototype is a height-one model

The square-one peelable prototype from Corollary 9.53 is the remaining tail-type model that is closest to the residual no-descendant sector. Unlike the two fork prototypes, it does not immediately collapse to a known cuspidal local model. However, its line-at-infinity presentation already forces a very small height.

Proposition 9.68 (The explicit square-one peelable prototype has height one)**.** *Let $p \in L_\infty \subset \mathbf{P}^2$, and let $g\colon Z \to \mathbf{P}^2$ be the birational morphism obtained by the four prototype blowups from Corollary 9.53(2) over p, producing the rooted chain $U_2(-2){-}U_1(-2){-}L(-1){-}E_1(-3){-}E_2(-2)$, and then by any further sequence of blowups whose centers lie over a finite subset*

$$\Sigma \subset \mathbf{P}^2 \setminus L_\infty \cong \mathbf{A}^2.$$

Let D_Z be a reduced divisor whose connected components are the four prototype components $U_2 + U_1 + E_1 + E_2$ and unions of exceptional curves lying over points of Σ. Then the pencil of *lines through p lifts to a morphism $f\colon Z \to \mathbf{P}^1$ with the following properties.*

1. *E_2 is a section of f.*
2. *The fiber corresponding to the line L_∞ has support $U_2 + U_1 + L + E_1$.*
3. *Every irreducible component of $D_Z - (U_2 + U_1 + E_1 + E_2)$ is vertical for f.*

In particular, if F is a general fiber of f then $F \cdot D_Z = 1$.

Proof. The four prototype blowups over p are exactly the local construction from Corollary 9.53(2), so by Proposition 9.56 the pencil of lines through p lifts after those four blowups to a morphism to $\mathbf{P}^1$ such that E_2 is a section and the fiber over L_∞ has support $U_2 + U_1 + L + E_1$. Thus (1) and (2) hold before the blowups over Σ.

It remains to prove (3). Fix $q \in \Sigma$, and let $\ell_q \subset \mathbf{P}^2$ be the unique line through p and q. After the prototype blowups, the total transform of ℓ_q is a fiber of the above ruling. The first blowup over q occurs at a point of that fiber, so its exceptional curve is vertical. Inductively, every later center lying over q belongs to the total transform of the same fiber, hence each new exceptional curve created over q is again vertical. Since this is true for every $q \in \Sigma$, every irreducible component lying over Σ is vertical, which proves (3).

A general fiber F is disjoint from the special fiber $U_2 + U_1 + L + E_1$ and from all vertical components created over Σ, while it meets the section E_2 transversely once. Therefore $F \cdot D_Z = F \cdot E_2 = 1$. □

Corollary 9.69 (The explicit square-one peelable prototype is not residual)**.** *Let X be a characteristic-3 klt del Pezzo surface of Picard rank 1, let $\pi\colon \widetilde{X} \to X$ be the minimal resolution, and let D be the reduced exceptional divisor. Assume that $(\widetilde{X}, D)$ is of the form described in Proposition 9.68, i.e. that $\widetilde{X}$ is obtained from the square-one line-at-infinity prototype by further blowups over a finite subset of $\mathbf{A}^2$, and that D is the reduced sum of the four prototype components together with the exceptional curves lying over that finite subset. Then $\operatorname{ht}(X) = 1$. In particular, this explicit square-one prototype cannot itself occur on a hypothetical benchmark counterexample in the residual height-≥ 4, no-descendant sector.*

Proof. By Proposition 9.68, the minimal resolution $\widetilde{X}$ carries a $\mathbf{P}^1$-fibration with general fiber F satisfying $F \cdot D = 1$. Hence $\operatorname{ht}(X) \leq 1$. By Lemma 9.3, one has $\operatorname{ht}(X) > 0$. Thus $\operatorname{ht}(X) = 1$.

The last assertion follows from Theorem 9.1: every hypothetical benchmark counterexample lies in the residual sector of height at least 4 and has no descendant with elliptic boundary, so it cannot realize a height-1 prototype model. □

Remark 9.70 (What this does and does not eliminate). Corollary 9.69 removes the *explicit* square-one line-at-infinity prototype from the residual height-≥ 4 sector. The next subsection removes the remaining caveat by working directly on the full peelable family $[2, 2] \dashv [3, (2)^m]$, without passing through auxiliary blowups on the positive root.

9.22. The whole peelable family already has height one

The caveat from Remark 9.70 can be removed directly. One does not need to pass through the square-one prototype or compare heights under auxiliary blowups on a positive root. For the peelable family it is enough to contract the first three local curves.

Proposition 9.71 (The whole peelable family carries a height-one ruling)**.** *Let X be a characteristic-3 klt del Pezzo surface of Picard rank 1, let $\pi\colon \widetilde{X} \to X$ be the minimal resolution, and let D be the reduced exceptional divisor. Assume that $\widetilde{X}$ contains the peelable bridge*

$$U_2(-2) - U_1(-2) - L(-1) - E_1(-3) - E_2(-2) - \cdots - E_n(-2), \qquad n \geq 2,$$

coming from the first-frontier family $[2,2] \dashv [3,(2)^{n-1}]$, so that $U_2 - U_1 = [2,2]$, $E_1 - \cdots - E_n = [3,(2)^{n-1}]$, and every other connected component of D is disjoint from this local bridge. Let $\sigma\colon \widetilde{X} \to Y$ be the contraction of the three curves $L, \quad U_1, \quad E_1$. Then Y is smooth. Writing

$$F := \sigma_* U_2, \qquad S := \sigma_* E_2,$$

one has

$$F \cong \mathbf{P}^1, \qquad F^2 = 0, \qquad F \cdot S = 1.$$

The complete linear system $|F|$ is a base-point-free pencil inducing a morphism $f\colon Y \to \mathbf{P}^1$ for which S is a section. Every irreducible component of $\sigma_* D - S$ is vertical for f.

Moreover, the ruling f lifts through σ to a morphism $\widetilde{f}\colon \widetilde{X} \to \mathbf{P}^1$ such that E_2 is a section of $\widetilde{f}$ and every irreducible component of $D - E_2$ is vertical for $\widetilde{f}$. In particular, if G is a general fiber of $\widetilde{f}$, then $G \cdot D = 1$.

Proof. Contracting a (-1)-curve on a smooth surface preserves smoothness, so the successive contraction of L, U_1, E_1 produces another smooth rational surface Y.

Tracing self-intersections through the three contractions gives the local divisor on Y $F(0) - S(-1) - E_3(-2) - \cdots - E_n(-2)$, where $F = \sigma_* U_2$ and $S = \sigma_* E_2$. Indeed, contracting L changes

$$U_1^2\colon\ -2 \mapsto -1, \qquad E_1^2\colon\ -3 \mapsto -2;$$

contracting U_1 then changes

$$U_2^2\colon\ -2 \mapsto -1, \qquad E_1^2\colon\ -2 \mapsto -1;$$

and contracting E_1 finally changes

$$U_2^2\colon\ -1 \mapsto 0, \qquad E_2^2\colon\ -2 \mapsto -1.$$

Thus $F \cong \mathbf{P}^1$, $F^2 = 0$, and $F \cdot S = 1$.

Now consider the exact sequence

$$0 \to \mathcal{O}_Y \to \mathcal{O}_Y(F) \to \mathcal{O}_F(F) \to 0.$$

Because Y is rational, $H^1(Y, \mathcal{O}_Y) = 0$. Since $F \cong \mathbf{P}^1$ and $F^2 = 0$, one has $\mathcal{O}_F(F) \cong \mathcal{O}_{\mathbf{P}^1}$, hence

$$h^0(Y, \mathcal{O}_Y(F)) = 1 + h^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}) = 2.$$

The restriction map onto $H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1})$ is surjective, so $|F|$ has no base point on F; it has no base point away from F because F itself is a member of the system. Therefore $|F|$ is a base-point-free pencil and defines a morphism $f\colon Y \to \mathbf{P}^1$ whose fibers are numerically equivalent to F.

Since $F \cdot S = 1$, the curve S maps isomorphically onto the base, so S is a section of f. Every other irreducible component of $\sigma_* D$ is disjoint from F: locally this is clear from the displayed chain, and globally it uses that the peelable bridge joins two full connected components of D on the first frontier, so no other component of D meets U_2, U_1, L, or E_1. For any such component C one has $\deg(f|_C) = F \cdot C = 0$, so C is vertical. This proves the statement on Y.

It remains to lift the ruling back through σ. The inverse of σ is obtained by three blowups: first at the point $F \cap S$, creating the exceptional curve E_1; next at the intersection of the transform of F with E_1, creating U_1; and finally at the intersection of U_1 with E_1, creating L. Each center lies on the fiber F or on curves created inside that same fiber. Therefore the morphism f lifts through all three blowups to a morphism $\widetilde{f}\colon \widetilde{X} \to \mathbf{P}^1$. The strict transform of S is exactly the original curve E_2, so E_2 remains a section. The curves L, U_1, E_1 created in the lifting process are vertical because they arise from blowups inside one fiber. Every other component of D is either the transform of a vertical component on Y or one of these three new vertical curves. Hence every irreducible component of $D - E_2$ is vertical for $\widetilde{f}$.

A general fiber G of $\widetilde{f}$ meets the section E_2 transversely once and is disjoint from every vertical component of D. Therefore $G \cdot D = G \cdot E_2 = 1$. □

Corollary 9.72 (The full peelable family is not residual)**.** *Let X be a characteristic-3 klt del Pezzo surface of Picard rank 1, and let $\pi\colon \widetilde{X} \to X$ be the minimal resolution with reduced exceptional divisor D. If $\widetilde{X}$ contains a peelable bridge*

$$[2,2] \dashv [3,(2)^m], \qquad m \geq 1,$$

from Theorem 9.58(ii), then $\mathrm{ht}(X) = 1$. In particular, the peelable family cannot occur on a hypothetical benchmark counterexample in the residual height-≥ 4, no-descendant sector.

Proof. Apply Proposition 9.71. The lifted ruling on $\widetilde{X}$ has a general fiber G with $G \cdot D = 1$. Hence $\mathrm{ht}(X) \leq 1$. By Lemma 9.3, one has $\mathrm{ht}(X) > 0$. Thus $\mathrm{ht}(X) = 1$.

The final assertion follows from Theorem 9.1: every benchmark counterexample already lies in the residual sector with height at least 4 and with no descendant with elliptic boundary. □

Corollary 9.73 (Residual first-frontier ambiguity now has only two prototypes)**.** *Assume the boxed residual Route C entry statement of Corollary 9.4 and the frontier-completeness hypothesis of Corollary 9.59. Then every characteristic-*3 *Picard-rank-one klt del Pezzo surface* X *with* $\#\operatorname{Sing}(X) \geq 8$ *reduces, on the first analyzed Route C frontier, to one of the following two explicit global prototypes on a pair* $(\mathbf{P}^2, L_\infty)$*:*

(i) *the short-bridge prototype* $[2] \dashv [3,2,2]$ *together with* 4 *free points on* L_∞*;*

(ii) *the short-bridge prototype* $[2] \dashv [4,2,2,2]$ *together with* 10 *free points on* L_∞*.*

In particular, once the residual Route C entry theorem and frontier completeness are proved, the first visible Route C frontier reduces to ruling out these two higher-square tail prototypes.

Proof. By Corollary 9.59, every benchmark counterexample on the first frontier would contain either a peelable bridge or one of the four isolated short bridges. By Corollary 9.72, the peelable family has height 1 and so cannot occur in the residual height-≥ 4 sector. By Corollary 9.66, the two fork short bridges

$$[2] \to F_3 \qquad \text{and} \qquad [2,2] \to F_4$$

force descendants with elliptic boundary and so are also excluded from the residual sector. Therefore only the two higher-square tail prototypes from Corollary 9.66(ii)–(iii) remain. ☐

Remark 9.74 (What the first Route C frontier now leaves open)*.* At this stage of the argument, the first analyzed chain/fork/cross-link frontier is no longer an infinite corridor, a five-prototype problem, or even a three-prototype problem inside the residual sector. The peelable family is low-height, the two fork prototypes force descendants with elliptic boundary, and only the two higher-square short bridges remain. The next subsection removes those last two local survivors as well.

9.23. The two higher-square short bridges already have height at most two

The last remaining non-tie survivors on the first analyzed Route C frontier are the two short bridges

$$[2] \dashv [3,2,2] \qquad \text{and} \qquad [2] \dashv [4,2,2,2].$$

Unlike the peelable corridor, these do not require any auxiliary reduction to a square-one prototype: after contracting just the bridge curve and the bridge-side [2], the hit (-2)-curve itself becomes a square-zero fiber.

Proposition 9.75 (The two higher-square short bridges carry height-at-most-two rulings)**.** *Let* X *be a characteristic-*3 *klt del Pezzo surface of Picard rank* 1*, let* $\pi\colon \widetilde{X} \to X$ *be the minimal resolution, and let* D *be the reduced exceptional divisor. Assume that* $\widetilde{X}$ *contains one of the short bridges*

$$[2] \dashv [3,2,2] \qquad \textit{or} \qquad [2] \dashv [4,2,2,2]$$

from Proposition 9.48. Write U for the bridge-side $[2]$-component, write

$$E_1 - \cdots - E_a = [a, (2)^{a-1}], \qquad a \in \{3, 4\},$$

for the tail side, and let L be the bridge curve, so that L meets U and E_2 transversely once. Let $\sigma \colon \widetilde{X} \to Y$ be the contraction of the two curves L and U. Then Y is smooth. Writing

$$F := \sigma_* E_2, \qquad S_1 := \sigma_* E_1, \qquad S_2 := \sigma_* E_3,$$

one has

$$F \cong \mathbf{P}^1, \qquad F^2 = 0, \qquad F \cdot S_1 = F \cdot S_2 = 1.$$

The complete linear system $|F|$ is a base-point-free pencil inducing a morphism $f \colon Y \to \mathbf{P}^1$. The curves S_1 and S_2 are sections of f, and every irreducible component of $\sigma_* D - (S_1 + S_2)$ is vertical for f.

Moreover, the ruling f lifts through σ to a morphism $\widetilde{f} \colon \widetilde{X} \to \mathbf{P}^1$ such that E_1 and E_3 are sections of $\widetilde{f}$ and every irreducible component of $D - E_1 - E_3$ is vertical for $\widetilde{f}$. In particular, if G is a general fiber of $\widetilde{f}$, then $G \cdot D = 2$.

Proof. Contracting (-1)-curves on a smooth surface preserves smoothness, so Y is smooth. Tracing self-intersections through the two contractions gives the local divisor on Y. After contracting L, one has

$$U^2 \colon -2 \mapsto -1, \qquad E_2^2 \colon -2 \mapsto -1,$$

while every other tail component keeps its original self-intersection. Contracting U next changes $E_2^2 \colon -1 \mapsto 0$. Therefore the local divisor on Y is $S_1(-3) - F(0) - S_2(-2)$ in the case $a = 3$, and $S_1(-4) - F(0) - S_2(-2) - E_4(-2)$ in the case $a = 4$. In both cases $F \cong \mathbf{P}^1$ and $F^2 = 0$, while F meets S_1 and S_2 once each.

Now consider the exact sequence

$$0 \to \mathcal{O}_Y \to \mathcal{O}_Y(F) \to \mathcal{O}_F(F) \to 0.$$

Because Y is rational, $H^1(Y, \mathcal{O}_Y) = 0$. Since $F \cong \mathbf{P}^1$ and $F^2 = 0$, one has $\mathcal{O}_F(F) \cong \mathcal{O}_{\mathbf{P}^1}$, so

$$h^0(Y, \mathcal{O}_Y(F)) = 1 + h^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}) = 2.$$

The restriction map onto $H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1})$ is surjective, so $|F|$ has no base point on F; it has no base point away from F because F itself is a member of the system. Therefore $|F|$ is a base-point-free pencil and defines a morphism $f \colon Y \to \mathbf{P}^1$ whose fibers are numerically equivalent to F.

Since $F \cdot S_1 = F \cdot S_2 = 1$, each of S_1 and S_2 maps isomorphically onto the base, so both are sections of f. Every other irreducible component of $\sigma_* D$ has zero intersection with F: locally this is clear from the displayed chains, because only S_1 and S_2 meet F; globally it uses that the short bridge joins two full connected components of D on the first frontier, so every other connected component of D is disjoint from U, L, and E_2, hence remains disjoint from F after the contractions. For any such component C one has $\deg(f|_C) = F \cdot C = 0$, so C is vertical. This proves the statement on Y.

It remains to lift the ruling back through σ. The inverse of σ is obtained by two blowups: first at a smooth point of F away from S_1, S_2, and every other component of $\sigma_* D$, creating the exceptional curve U; next at the intersection of the transform of F with U, creating the bridge curve L. Each center lies on one fiber of the ruling f, so the morphism lifts through both blowups to a morphism $\widetilde{f}\colon \widetilde{X} \to \mathbf{P}^1$. The strict transforms of S_1 and S_2 are exactly the original curves E_1 and E_3, so they remain sections. The curves U and L created in the lifting process are vertical because they arise from blowups inside one fiber. If $a = 4$, then E_4 already satisfies $F \cdot E_4 = 0$ on Y, so it is vertical and remains vertical after lifting. Every other component of D is either the transform of a vertical component on Y or one of the two new vertical curves U and L. Hence every irreducible component of $D - E_1 - E_3$ is vertical for $\widetilde{f}$.

A general fiber G of $\widetilde{f}$ meets the two sections E_1 and E_3 transversely once and is disjoint from every vertical component of D. Therefore $G \cdot D = G \cdot E_1 + G \cdot E_3 = 2$. □

Corollary 9.76 (The two higher-square short bridges are not residual)**.** *Let X be a characteristic-3 klt del Pezzo surface of Picard rank 1, and let $\pi\colon \widetilde{X} \to X$ be the minimal resolution with reduced exceptional divisor D. If $\widetilde{X}$ contains one of the two short bridges*

$$[2] \dashv [3, 2, 2] \qquad \textit{or} \qquad [2] \dashv [4, 2, 2, 2],$$

then $\mathrm{ht}(X) \le 2$. In particular, neither bridge can occur on a hypothetical benchmark counterexample in the residual height-≥ 4, no-descendant sector.

Proof. Apply Proposition 9.75. The lifted ruling on $\widetilde{X}$ has a general fiber G with $G \cdot D = 2$, so $\mathrm{ht}(X) \le 2$. The final assertion follows from Theorem 9.1: every benchmark counterexample already lies in the residual sector with height at least 4 and with no descendant with elliptic boundary. □

Theorem 9.77 (Conditional seven-point theorem from the boxed Route C inputs)**.** *Assume the boxed residual Route C entry statement of Corollary 9.4 and the frontier-completeness hypothesis of Corollary 9.59. Then no characteristic-3 Picard-rank-one klt del Pezzo surface X satisfies $\#\,\mathrm{Sing}(X) \ge 8$. Equivalently, under those two global inputs, every characteristic-3 Picard-rank-one klt del Pezzo surface has at most 7 singular points.*

Proof. Suppose for contradiction that such a surface X exists. By Theorem 9.1, it lies in the residual height-≥ 4, no-descendant sector. By Corollary 9.59, the minimal resolution of X then contains either the peelable family $[2, 2] \dashv [3, (2)^m], \qquad m \ge 1$, or one of the four isolated short bridges.

The peelable family is impossible by Corollary 9.72, because it has height 1. The two fork bridges $[2] \to F_3$ and $[2, 2] \to F_4$ are impossible by Corollary 9.66, because they force descendants with elliptic boundary. The two tail bridges $[2] \dashv [3,2,2]$ and $[2] \dashv [4,2,2,2]$ are impossible by Corollary 9.76, because they satisfy $\mathrm{ht}(X) \le 2$. This excludes all alternatives from Corollary 9.59, a contradiction. ☐

Remark 9.78 (What Route C now reduces to)*.* Under the two boxed inputs of Corollaries 9.4 and 9.59, Theorem 9.77 already gives an earlier conditional route to the seven-point theorem. The later bridge and height-selection sections remove the need for those extra inputs by excluding the off-frontier bridge geometries and then proving the residual height-selection statement outright. At this stage of the paper, the entry theorem still has two geometric sufficient forms: either a purely vertical contact-2 bridge with a multiplicity-one component as in Corollary 9.8, or a singly-hit multiplicity-one nonexceptional tip whose unique vertical neighbor lies in D, as in Corollary 9.11. The final subsection records the exact excess package in the height-4 maximal-width corridor, and the last section turns that package into the unconditional numerical closure.

9.24. Further bridge reductions before the height-4 excess package

The previous sections already settle one half of the residual contact-2 problem completely. If the two contacts lie on one connected component of the exceptional divisor, then root-contractibility is already enough to force an elliptic tie. The rest of this section continues the bridge analysis on the distinct-component side. Its role is not to enter the conditional theorem above, but to weaken the remaining global bridge gap by explicit off-frontier exclusions. The height-4 maximal-width corridor will later be packaged numerically by a global excess identity, although a final unconditional closure still requires an additional graph-theoretic input.

Theorem 9.79 (Same-component root-contractible contact-2 curves already give ties)**.** *Let X be a klt del Pezzo surface of Picard rank 1, let $\pi\colon (\widetilde{X}, D) \to X$ be the minimal resolution, and let $L \subset \widetilde{X} \setminus D$ be a (-1)-curve with $L \cdot D = 2$. Assume that both contacts of L with D lie on one connected component $D_0 \subset D$, and that the associated rooted configuration is root-contractible. Then the divisor supported on $D_0 + L$ contains an elliptic tie. In particular, X has a descendant with elliptic boundary.*

Proof. Put $E := D_0 + L$. By Proposition 9.28(1), $p_a(E) = 1$. By root-contractibility, there is a component $R \subset D_0$ such that one can successively blow down (-1)-curves contained in the current image of $E - R$ until only the image of R remains, while the current image of R is never contracted. By Lemma 9.41, this is equivalent to saying that E is locally sandwiched of $\delta = 1$, i.e. that E contains an elliptic tie. By [7, Definition 2.10, Definition 2.11, Lemma 2.12], this gives a descendant with elliptic boundary. ☐

Corollary 9.80 (The unresolved part of Route C is bridge-type)**.** *Let X be a characteristic-3 klt del Pezzo surface with*

$$\rho(X) = 1, \qquad \#\,\mathrm{Sing}(X) \ge 8.$$

Assume that the minimal resolution of X contains a positive root-contractible contact-2 configuration. Then either:

(i) *it already gives a descendant with elliptic boundary; or*

(ii) *it is a distinct-component bridge.*

In particular, after Theorem 9.79, the only unresolved part of Route C is the global analysis of positive root-contractible bridges.

Proof. By Theorem 9.1, every benchmark counterexample already lies in the residual height-≥ 4, no-descendant sector. If the given configuration is same-component, then Theorem 9.79 yields a descendant with elliptic boundary, contradicting the residual assumption. So any residual positive root-contractible contact-2 configuration must be a distinct-component bridge. ☐

The next proposition shows that the remaining completeness problem is genuinely global. There is an explicit infinite family of positive root-contractible bridge germs beyond the first analyzed frontier, so no purely local argument can possibly prove that the first frontier is exhaustive.

Proposition 9.81 (An infinite off-frontier family of positive root-contractible bridges)**.** *For every integer $a \geq 3$, let $Q = [3]$ with component U, and let $T_a = [a, (2)^{a-1}]$ with components*

$$E_1 - \cdots - E_a, \qquad E_1^2 = -a, \qquad E_i^2 = -2 \ \ (i \geq 2).$$

Let L be a bridge meeting U and E_2 once each. Then the rooted graph $Q + L + T_a$ is positive and root-contractible with root U. More precisely:

(1) *if $C = \pi(L) \subset X$, then*

$$C^2 = -1 + \frac{1}{3} + \frac{a(a-1)}{a^2 - a + 1} = \frac{a^2 - a - 2}{3(a^2 - a + 1)} > 0;$$

(2) *the sequence $L,\ E_2,\ E_3,\ \ldots,\ E_a,\ E_1$ contracts all components except U.*

For every $a \geq 5$ this positive root-contractible bridge lies strictly beyond the first analyzed bridge frontier.

Proof. By Proposition 9.46(3), the image square is

$$C^2 = -1 + \frac{1}{3} + \frac{d([a])\, d([(2)^{a-2}])}{d([a, (2)^{a-1}])}.$$

Now

$$d([a]) = a, \qquad d([(2)^{a-2}]) = a - 1,$$

and if we put $D_r := d([a, (2)^r])$, then

$$D_0 = a, \qquad D_1 = 2a - 1, \qquad D_r = 2D_{r-1} - D_{r-2},$$

so by induction $D_r = a(r+1) - r$. For $r = a - 1$ this gives $d([a, (2)^{a-1}]) = a^2 - a + 1$. Hence

$$C^2 = -1 + \frac{1}{3} + \frac{a(a-1)}{a^2 - a + 1} = \frac{a^2 - a - 2}{3(a^2 - a + 1)},$$

which is positive for every $a \geq 3$.

For root-contractibility, contract L first. Then U changes from (-3) to (-2) and E_2 changes from (-2) to (-1). Thus E_2 is contractible. After contracting

$$L, \ E_2, \ \ldots, \ E_{k-1} \qquad (3 \leq k \leq a),$$

the curve E_k has self-intersection -1 and meets the current images of U, E_1, and (if $k < a$) E_{k+1} transversely once. Hence the displayed peeling sequence is legal. After contracting $L, \ E_2, \ \ldots, \ E_a$, the curve E_1 has become a (-1)-curve, so it contracts as well. The component U is never contracted, so the rooted graph is root-contractible with root U.

Finally, for $a \geq 5$ the right-hand chain is not one of the terminal tails $[3, (2)^m]$, $\quad [4, (2)^m]$, and the bridge side is $[3]$, not $[2]$ or $[2, 2]$. Hence these bridges lie beyond the first analyzed bridge frontier. ☐

Corollary 9.82 (Bridge completeness is genuinely global)**.** *The first analyzed bridge frontier cannot be exhaustive for positive root-contractible bridge germs by a purely local argument. Indeed, Proposition 9.81 gives an infinite family of positive root-contractible bridge germs lying strictly beyond that frontier.*

Consequently, any final bridge-completeness theorem must use an additional global *restriction excluding such off-frontier bridge germs from actual characteristic-*3 *Picard-rank-one klt del Pezzo surfaces with* $\# \operatorname{Sing}(X) \geq 8$.

Proof. Immediate from Proposition 9.81. ☐

The previous proposition shows why the remaining completeness statement must be global. The next lemma pair is a convenient toolkit for killing a broad one-component bridge family inside the benchmark range.

Lemma 9.83 (A smooth rational 0-curve gives a ruling)**.** *Let Y be a smooth rational projective surface and let $F \subset Y$ be a smooth rational curve with $F^2 = 0$. Then the complete linear system $|F|$ is a base-point-free pencil and defines a morphism $p \colon Y \to \mathbf{P}^1$ whose general fiber is linearly equivalent to F. If $S \subset Y$ is an irreducible curve with $S \cdot F = 1$, then S is a section of p.*

Proof. Consider the exact sequence

$$0 \to \mathcal{O}_Y \to \mathcal{O}_Y(F) \to \mathcal{O}_F(F) \to 0.$$

Since Y is rational, $H^1(Y, \mathcal{O}_Y) = 0$. Since $F \cong \mathbf{P}^1$ and $F^2 = 0$, one has $\mathcal{O}_F(F) \cong \mathcal{O}_{\mathbf{P}^1}$, so

$$h^0(Y, \mathcal{O}_Y(F)) = 1 + h^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}) = 2.$$

Thus $|F|$ is a pencil.

The restriction map

$$H^0(Y, \mathcal{O}_Y(F)) \twoheadrightarrow H^0(F, \mathcal{O}_F(F)) \cong k$$

is surjective, so $|F|$ has no base point on F. It has no base point away from F either, because F itself is a member of $|F|$. Hence $|F|$ is base-point free and defines a morphism $p\colon Y \to \mathbf{P}^1$. Two distinct members of $|F|$ have intersection number $F^2 = 0$, hence are disjoint. Therefore a general member of $|F|$ is a smooth rational fiber of p.

If $S \cdot F = 1$, then $\deg(p|_S) = S \cdot F = 1$, so S is a 1-section, hence a section. □

Lemma 9.84 (Genus created by contracting a (-1)-curve)**.** *Let $\phi\colon Z \to Y$ be the contraction of a (-1)-curve E on a smooth surface Z. Let $U \subset Z$ be a smooth rational curve, different from E, and put $m := U \cdot E$. Let $T := \phi_* U \subset Y$. Then*

$$p_a(T) = \frac{m(m-1)}{2}.$$

In particular, if $m = 2$, then $p_a(T) = 1$.

Proof. Write $\phi^*T = U + \alpha E$. Intersecting with E gives $0 = (U + \alpha E) \cdot E = m - \alpha$, because $E^2 = -1$. Hence $\alpha = m$, so $\phi^*T = U + mE$. Therefore

$$T^2 = (\phi^*T)^2 = (U + mE)^2 = U^2 + 2m(U \cdot E) + m^2E^2 = U^2 + m^2.$$

Also

$$K_Z = \phi^*K_Y + E, \qquad \text{so} \qquad \phi^*K_Y = K_Z - E.$$

Using adjunction on the smooth rational curves U and E,

$$K_Z \cdot U = -2 - U^2, \qquad K_Z \cdot E = -2 - E^2 = -1.$$

Hence

$$K_Y \cdot T = (\phi^*K_Y) \cdot (\phi^*T) = (K_Z - E) \cdot (U + mE) = -2 - U^2 - m.$$

Thus

$$T \cdot (T + K_Y) = (U^2 + m^2) + (-2 - U^2 - m) = m^2 - m - 2.$$

By adjunction on Y, $2p_a(T) - 2 = m^2 - m - 2$, so

$$p_a(T) = \frac{m(m-1)}{2}.$$

□

Proposition 9.85 (A broad one-component bridge family)**.** *Let $Q = [b]$ with unique component U of self-intersection $-b$, where $b \geq 2$. Let*

$$T_a = [a, (2)^{a-1}] = E_1 - \cdots - E_a, \qquad a \geq 3,$$

where

$$E_1^2 = -a, \qquad E_i^2 = -2 \ \ (i \geq 2).$$

Let L be a distinct-component bridge meeting U and E_2 once each. Consider the associated rooted graph with root U.

Then:

(1) *The image square is*

$$C^2 = -1 + \frac{1}{b} + \frac{a(a-1)}{a^2-a+1}.$$

(2) *This rooted graph is root-contractible. More precisely, the sequence $L,\ E_2,\ E_3,\ \dots,\ E_a,\ E_1$ contracts all components except U.*

(3) *Assume now that $b \le a$. Let $\sigma_b\colon \widetilde{X} \to Y_b$ be the contraction of $L,\ E_2,\ E_3,\ \dots,\ E_b$. Then the image $F := \sigma_{b*}U$ is a smooth rational curve with $F^2 = 0$. If $b < a$, then the image of E_{b+1} is a section of the ruling $|F|$, while the image of E_1 has degree $F \cdot \sigma_{b*}E_1 = b-1$. Consequently,*

$$\operatorname{ht}(X) \le b \qquad \text{if } b < a,$$

and

$$\operatorname{ht}(X) \le b-1 \qquad \text{if } b = a.$$

(4) *If $a = 3$, then X has a descendant with elliptic boundary.*

Proof. For (1), Proposition 9.46(3) gives

$$C^2 = -1 + \frac{1}{b} + \frac{d([a])\, d([(2)^{a-2}])}{d(T_a)}.$$

Now

$$d([a]) = a, \qquad d([(2)^{a-2}]) = a-1, \qquad d(T_a) = a^2-a+1,$$

so the displayed formula follows.

For (2), contract L first. Then U changes from $(-b)$ to $(-(b-1))$ and E_2 changes from (-2) to (-1). Thus E_2 is contractible. After contracting

$$L,\ E_2,\ \dots,\ E_{k-1} \qquad (3 \le k \le a),$$

the current image of E_k is a (-1)-curve, so the displayed peeling order is legal. After contracting $L,\ E_2,\ \dots,\ E_a$, the curve E_1 has become a (-1)-curve, so it contracts as well. The component U is never contracted.

For (3), assume $b \le a$ and contract only $L,\ E_2,\ \dots,\ E_b$. We claim that after contracting

$$L,\ E_2,\ \dots,\ E_k \qquad (2 \le k \le b),$$

the current images satisfy

$$U_k^2 = k-b, \qquad U_k \cdot E_1^{(k)} = k-1, \qquad (E_1^{(k)})^2 = -(a-k+1),$$

and, if $k < a$, the image of E_{k+1} meets U_k once. For $k = 2$, after contracting L and E_2 one has

$$U_2^2 = 2 - b, \qquad U_2 \cdot E_1^{(2)} = 1, \qquad (E_1^{(2)})^2 = -(a-1),$$

and the image of E_3 meets U_2 once if $a > 2$. If the statement holds for some $k < a$, then the image of E_{k+1} is a (-1)-curve meeting the current images of U, E_1, and (if $k+1 < a$) E_{k+2} once each. Contracting it raises the self-intersection of U_k and $E_1^{(k)}$ by 1, increases their mutual intersection by 1, and preserves the fact that the next tail component meets the current image of U once. This proves the induction.

Taking $k = b$, we obtain

$$F^2 = U_b^2 = 0, \qquad F \cdot \sigma_{b*} E_1 = b - 1.$$

If $b < a$, then the image of E_{b+1} meets F once, hence is a section of the ruling defined by $|F|$ by Lemma 9.83. All later tail components $E_{b+2}, \dots, E_a$ are disjoint from F, so they are vertical for this ruling. The same is true for every component of the rest boundary, because all local contractions above are supported inside the bridge cluster and the remaining components of D are disjoint from the two connected components joined by the bridge.

Hence a general fiber G of the ruling satisfies

$$G \cdot D \le (b-1) + 1 = b \qquad \text{if } b < a,$$

and

$$G \cdot D \le b - 1 \qquad \text{if } b = a.$$

This proves the height bounds.

For (4), assume $a = 3$ and contract $L,\ E_2,\ E_3$. Let

$$F := \sigma_* U, \qquad T := \sigma_* E_1.$$

Then

$$T^2 = -1, \qquad F \cdot T = 2.$$

Now contract T. By Lemma 9.84, the image of F is an irreducible curve of arithmetic genus 1. Since all other components of the boundary are disjoint from the local bridge cluster, they remain disjoint from this local construction and can be contracted afterwards. Thus X has a descendant with elliptic boundary. □

Corollary 9.86 (Anticanonical sieve for one-component bridges)**.** *Keep the setup of Proposition 9.85, and assume in addition that the rooted configuration is root-contractible, so in particular $n = a$ by Proposition 9.85(2). Then the image curve $C = \pi(L)$ satisfies*

$$-K_X \cdot C = -1 + \frac{2}{b} + \frac{2a-1}{a^2 - a + 1}.$$

Consequently:

(i) *if* $a \geq 5$, *then positivity of* $-K_X \cdot C$ *forces* $b \leq 3$*;*

(ii) *if* $a = 4$, *then positivity of* $-K_X \cdot C$ *forces* $b \leq 4$.

Proof. The formula is Proposition 9.47 applied to the two chains

$$[b] \qquad \text{and} \qquad [a, (2)^{a-1}],$$

where the bridge meets the unique component of $[b]$ and the second component of the tail. For the one-component side one has

$$\mu([b], 1) = \frac{1+1}{b} = \frac{2}{b},$$

while on the tail side

$$\mu([a, (2)^{a-1}], 2) = \frac{d([a]) + d([(2)^{a-2}])}{d([a, (2)^{a-1}])} = \frac{a + (a-1)}{a^2 - a + 1} = \frac{2a-1}{a^2 - a + 1}.$$

Thus

$$-K_X \cdot C = -1 + \frac{2}{b} + \frac{2a-1}{a^2 - a + 1}.$$

Since $-K_X$ is ample and C is a nonzero effective curve, one has $-K_X \cdot C > 0$.

If $a \geq 5$, then

$$\frac{2a-1}{a^2 - a + 1} \leq \frac{9}{21} = \frac{3}{7},$$

so positivity implies

$$0 < -1 + \frac{2}{b} + \frac{3}{7}, \qquad \text{hence} \qquad \frac{2}{b} > \frac{4}{7},$$

which forces $b < \frac{7}{2}$, hence $b \leq 3$.

If $a = 4$, then positivity gives

$$0 < -1 + \frac{2}{b} + \frac{7}{13}, \qquad \text{so} \qquad \frac{2}{b} > \frac{6}{13},$$

which forces $b < \frac{13}{3}$, hence $b \leq 4$. ☐

Corollary 9.87 (The one-component bridge family is not residual)**.** *Let* X *be a characteristic-*3 *klt del Pezzo surface with*

$$\rho(X) = 1 \qquad \textit{and} \qquad \#\operatorname{Sing}(X) \geq 8.$$

Assume that the minimal resolution of X *contains a positive root-contractible bridge of the form* $[b] \dashv [a, (2)^{a-1}]$ *whose bridge curve meets the unique component of* $[b]$ *and the second component of the tail, and whose root is the one-component side* $[b]$*. Then this is impossible.*

Proof. By Corollary 9.86, if $a \geq 5$ then necessarily $b \leq 3$, and if $a = 4$ then necessarily $b \leq 4$.

If $a = 3$, the bridge is excluded by Corollary 9.88(i). If $a \geq 5$, then $b \leq 3$, so Corollary 9.88(ii) excludes the configuration. If $a = 4$, then $b \leq 4$; the cases $b \leq 3$ are again excluded by Corollary 9.88(ii), while the remaining case $(b, a) = (4, 4)$ is excluded by Corollary 9.88(iii). Thus no positive root-contractible one-component bridge of this family can occur on a benchmark counterexample. □

Corollary 9.88 (Light one-component bridges are not residual)**.** *Let X be a characteristic-3 klt del Pezzo surface with*

$$\rho(X) = 1 \qquad \text{and} \qquad \#\operatorname{Sing}(X) \geq 8.$$

Assume that the minimal resolution of X contains a positive root-contractible bridge of the form $[b] \dashv [a, (2)^{a-1}]$ whose bridge curve meets the unique component of $[b]$ and the second component of the tail, and whose root is the one-component side $[b]$. Then this bridge cannot occur on a benchmark counterexample in any of the following cases:

(i) $a = 3$;

(ii) $b \leq 3$;

(iii) $(b, a) = (4, 4)$.

Proof. If $a = 3$, then Proposition 9.85(4) gives a descendant with elliptic boundary, contradicting Theorem 9.1.

If $b \leq 3$, then Proposition 9.85(3) gives $\operatorname{ht}(X) \leq 3$, again contradicting Theorem 9.1.

If $(b, a) = (4, 4)$, then Proposition 9.85(3) gives $\operatorname{ht}(X) \leq 3$, so this case is also excluded from the residual height-≥ 4 sector. □

Proposition 9.89 (The one-component bridge family contracts to rational unicuspidal curves)**.** *Keep the setup of Proposition 9.85, and let $\tau\colon \widetilde{X} \to Y_{a,b}$ be the contraction of the whole non-root staircase $S_a := L + E_1 + \cdots + E_a$. Put $R_{a,b} := \tau_* U$. Then the following hold.*

(1) *The curve $R_{a,b}$ is irreducible and rational, and $R_{a,b}^2 = a^2 - a + 1 - b$.*

(2) *If $\sigma\colon \widetilde{X} \to Z_{a,b}$ denotes the contraction of L, E_2, E_3, $\ldots$, E_a, and*

$$U' := \sigma_* U, \qquad T := \sigma_* E_1,$$

then $U' \cong \mathbf{P}^1$, $T^2 = -1$, and $U' \cdot T = a - 1$.

(3) *The curve $R_{a,b}$ has exactly one singular point $q_{a,b}$, it is unibranch there, its multiplicity is $\operatorname{mult}_{q_{a,b}}(R_{a,b}) = a - 1$, and*

$$p_a(R_{a,b}) = \frac{(a-1)(a-2)}{2}.$$

(4) *Equivalently, the bridge staircase S_a is the minimal embedded resolution over $q_{a,b}$ of a rational unibranch singularity of multiplicity sequence $(a - 1, 1, \ldots, 1)$, with $a - 1$ trailing 1's. In particular, the bridge curve L is the last exceptional curve in that local resolution sequence.*

Proof. For (1), the curve $R_{a,b}$ is irreducible because it is the image of the irreducible curve U under a birational contraction, and its normalization is $\mathbf{P}^1$ because $U \cong \mathbf{P}^1$. To compute the self-intersection, contract first $L,\ E_2,\ E_3,\ \dots,\ E_a$. As in the proof of Proposition 9.85(3), contracting L raises U^2 by 1, and each contraction of $E_2, \dots, E_a$ raises it by another 1. Hence on $Z_{a,b}$ one has $(U')^2 = -b + 1 + (a-1) = a - b$. Now T is a (-1)-curve meeting U' with multiplicity $a-1$, so contracting T raises the square of the image of U' by $(a-1)^2$. Therefore

$$R_{a,b}^2 = a - b + (a-1)^2 = a^2 - a + 1 - b.$$

Statement (2) follows from the same computation: after contracting $L,\ E_2,\ E_3,\ \dots,\ E_a$ the only remaining curves from the local staircase are U' and T, with

$$(U')^2 = a - b, \qquad T^2 = -1, \qquad U' \cdot T = a - 1.$$

For (3), contracting T is the inverse of blowing up one point $q_{a,b} \in R_{a,b}$. Since the contraction is an isomorphism away from T, the curve $R_{a,b}$ is smooth away from $q_{a,b}$, so this is its unique singular point. The normalization of $R_{a,b}$ is $U' \cong \mathbf{P}^1$, and only one point of U' maps to $q_{a,b}$, so the singularity is unibranch. Its multiplicity is

$$\operatorname{mult}_{q_{a,b}}(R_{a,b}) = U' \cdot T = a - 1,$$

because under a blowup the strict transform meets the exceptional divisor with total multiplicity equal to the multiplicity of the original curve at the center. Finally, Lemma 9.84 applied to the contraction of T gives

$$p_a(R_{a,b}) = \frac{(a-1)(a-2)}{2}.$$

For (4), reverse the contraction sequence $L,\ E_2,\ E_3,\ \dots,\ E_a,\ E_1$. The first inverse step is the blowup of the singular point $q_{a,b}$, producing the exceptional curve $T = E_1$ and a smooth strict transform U' meeting it with multiplicity $a-1$. The remaining inverse steps successively resolve that order-$(a-1)$ tangency by blowing up the unique current point of contact of the strict transform of $R_{a,b}$ with the newest exceptional curve; this is exactly the reverse of the legal peeling order proved in Proposition 9.85(2). Each of those later blowups has multiplicity 1, and after the last one the newest exceptional curve is precisely L. Thus the local singularity is resolved by one blowup of multiplicity $a-1$ followed by $a-1$ blowups of multiplicity 1, which is the displayed multiplicity sequence. □

Corollary 9.90 (Corrected square-one prototype reduction)**.** *For each $a \geq 3$, define the square-one prototype $\Pi_a := [a^2 - a] \dashv [a, (2)^{a-1}]$, with the bridge attached to the second component of the tail and the root on the one-component side.*

Let $[b] \dashv [a, (2)^{a-1}]$ be any positive bridge of the same type. Then $b \leq a^2 - a$. Moreover, if $m := a^2 - a - b \geq 0$, then after blowing up m general smooth *points of the rational curve $R_{a,b}$ away from its unique cusp $q_{a,b}$ and away from the rest boundary, one obtains a birational model whose corresponding one-component bridge is precisely the square-one prototype Π_a.*

Equivalently, every positive one-component bridge of this family is controlled by the one-parameter square-one prototype family Π_a, together with m free marked points on the smooth locus of the *surviving rational cuspidal curve away from the staircase point.*

Proof. By Proposition 9.85(1),

$$C^2 = -1 + \frac{1}{b} + \frac{a(a-1)}{a^2-a+1} = \frac{a^2-a+1-b}{b(a^2-a+1)}.$$

Hence positivity is equivalent to $a^2-a+1-b > 0$, so indeed $b \le a^2-a$. By Proposition 9.89(1), $R_{a,b}^2 = a^2-a+1-b = m+1$. Blowing up m general smooth points of $R_{a,b}$ away from the cusp lowers the self-intersection of its strict transform by m and does not change the local unicuspidal staircase over $q_{a,b}$. Hence after those free blowups the strict transform has square 1. Reversing the same local resolution staircase over the cusp then produces exactly the bridge $[a^2-a] \dashv [a,(2)^{a-1}]$, namely the square-one prototype Π_a. ☐

Corollary 9.91 (The visible one-component chain gap is a unicuspidal square-one family)**.** *After Corollary 9.88, the remaining one-component chain-shaped gap is no longer best viewed as the full two-parameter heavy-heavy regime in (a,b). It is controlled birationally by the square-one unicuspidal prototype family*

$$\Pi_a = [a^2-a] \dashv [a,(2)^{a-1}], \qquad a \ge 4,$$

together with auxiliary free points on the smooth locus of the surviving rational cuspidal curve away from its unique singular point.

Proof. By Corollary 9.88, the cases

$$a = 3, \qquad b \le 3, \qquad (b,a) = (4,4)$$

already leave the residual sector. For every remaining positive bridge of the same one-component type, Corollary 9.90 reduces the local geometry to the square-one prototype Π_a plus free points away from the unique cusp. ☐

9.25. A 2-tail core reduction for non-staircase chain bridges

The anticanonical bridge-contact formula suggests that the correct invariant for the remaining chain bridges is not the full chain, but its contact core after removing terminal [2]-tails away from the touched component.

Definition 9.92 (2-tail core of a touched chain)**.** Let $A = [a_1,\dots,a_r]$ be a chain and let the bridge touch its i-th component A_i. The 2-*tail core* of the touched pair (A,i) is obtained by repeatedly removing:

(a) the leftmost component if it has weight 2 and $i > 1$;

(b) the rightmost component if it has weight 2 and $i < r$, until neither operation is possible. We denote the resulting touched pair by $(A^\sharp, i^\sharp)$. Thus $A^\sharp$ has no removable terminal 2 away from the touched component.

Lemma 9.93 (Monotonicity under adding a terminal 2 away from the contact)**.** *Let (A, i) be a touched chain pair, and let A^+ be obtained from A by adjoining a terminal 2 on one side away from the touched component. Then $\mu(A^+, i^+) \leq \mu(A, i)$, where i^+ is the corresponding touched index on A^+.*

In particular, for the 2-tail core from Definition 9.92, $\mu(A, i) \leq \mu(A^\sharp, i^\sharp)$.

Proof. It is enough to treat the case of adjoining a terminal 2 on the right away from the contact. Write

$$A = [a_1, \ldots, a_r], \qquad i < r, \qquad A^+ := [a_1, \ldots, a_r, 2].$$

Put

$$P := d(A_i^-) = d([a_1, \ldots, a_{i-1}]), \qquad Q := d(A_i^+) = d([a_{i+1}, \ldots, a_r]),$$

and

$$D := d(A), \qquad E := d([a_1, \ldots, a_{r-1}]), \qquad R := d([a_{i+1}, \ldots, a_{r-1}]).$$

Then

$$\mu(A, i) = \frac{P + Q}{D}, \qquad \mu(A^+, i) = \frac{P + (2Q - R)}{2D - E}.$$

A standard continuant identity for chains gives $DR - EQ = -P$. Hence

$$\mu(A, i) - \mu(A^+, i) = \frac{(P + Q)(2D - E) - D(P + 2Q - R)}{D(2D - E)} = \frac{P(D - E - 1)}{D(2D - E)}.$$

Since D, E are positive integers and $D > E$, one has $D - E - 1 \geq 0$, so $\mu(A, i) \geq \mu(A^+, i)$. The left-tail case is symmetric. Iterating proves the final statement. □

Proposition 9.94 (Interior points of a 2-reduced core have contact weight at most $\frac{1}{2}$)**.** *Let $(A^\sharp, i^\sharp)$ be a touched chain pair such that $A^\sharp$ has no removable terminal 2 away from the touched component, and assume that $i^\sharp$ is an interior vertex of $A^\sharp$. Then*

$$\mu(A^\sharp, i^\sharp) \leq \frac{1}{2}.$$

Proof. Write $A^\sharp = [a_1, \ldots, a_r]$, $\quad 1 < i^\sharp < r$. Because $A^\sharp$ has no removable terminal 2 away from the touched component,

$$a_1 \geq 3, \qquad a_r \geq 3.$$

Let

$$\delta_k := \mu(A^\sharp, k) \qquad (1 \leq k \leq r).$$

Equivalently, if M is the intersection matrix of the chain, then $M\delta = -\nu$, where ν has value 1 at the two ends and 0 elsewhere. In coordinates,

$$a_1\delta_1 - \delta_2 = 1, \qquad -\delta_{k-1} + a_k\delta_k - \delta_{k+1} = 0 \quad (2 \leq k \leq r - 1), \qquad a_r\delta_r - \delta_{r-1} = 1.$$

Suppose for contradiction that some interior value satisfies

$$\delta_{i^\sharp} > \frac{1}{2}.$$

Let $m := \max\{\delta_1, \dots, \delta_r\}$. Then

$$m > \frac{1}{2}.$$

Choose t with $\delta_t = m$.

If t is interior, then

$$a_t m = \delta_{t-1} + \delta_{t+1} \leq 2m,$$

so $a_t \leq 2$. Since every weight is at least 2, this forces

$$a_t = 2 \qquad \text{and} \qquad \delta_{t-1} = \delta_{t+1} = m.$$

Thus any interior maximum propagates across a contiguous block of 2-vertices.

Because both ends have weight at least 3, this propagation reaches an end vertex e with $\delta_e = m$. At that end one has $1 = a_e\delta_e - \delta_{e'} \geq 3m - m = 2m$, where e' is the unique neighbor of e. So

$$m \leq \frac{1}{2},$$

contradiction. Therefore

$$\delta_{i^\sharp} \leq \frac{1}{2}.$$

□

Corollary 9.95 (A positive chain bridge must be tip-visible on at least one side)**.** *Let L be a distinct-component bridge between two chain components*

$$(A, i) \qquad \textit{and} \qquad (B, j)$$

on the minimal resolution of a Picard-rank-one klt del Pezzo surface. Assume that the bridge is positive, i.e.

$$C := \pi(L) \subset X \qquad \textit{satisfies} \qquad C^2 > 0.$$

Let

$$(A^\sharp, i^\sharp), \qquad (B^\sharp, j^\sharp)$$

be the corresponding 2-tail cores.

Then it is impossible that both $i^\sharp$ and $j^\sharp$ are interior vertices. Equivalently, every positive chain bridge has at least one side whose 2-tail core is touched at a tip.

Proof. By Lemma 9.93,

$$\mu(A, i) \leq \mu(A^\sharp, i^\sharp), \qquad \mu(B, j) \leq \mu(B^\sharp, j^\sharp).$$

If both $i^\sharp$ and $j^\sharp$ were interior, then Proposition 9.94 would give

$$\mu(A,i) \leq \frac{1}{2}, \qquad \mu(B,j) \leq \frac{1}{2}.$$

Hence Proposition 9.47 would imply

$$-K_X \cdot C = -1 + \mu(A,i) + \mu(B,j) \leq 0,$$

contradicting the ampleness of $-K_X$. □

Proposition 9.96 (High-contact reduced tip cores are very special)**.** *Let $(A^\sharp, i^\sharp)$ be a touched chain pair obtained after 2-tail reduction, and assume that $i^\sharp$ is a tip of $A^\sharp$. Then*

$$\mu(A^\sharp, i^\sharp) > \frac{1}{2}$$

if and only if, up to reversing the chain, one of the following holds:

(i) *the touched tip has weight 2;*

(ii) *the touched pair is [3] with the bridge touching its unique component.*

Equivalently, after 2-tail reduction, the only tip-visible chain cores with contact weight strictly larger than 1/2 are the 2-tip cores and the singleton core [3] touched at its unique component.

Proof. By symmetry it is enough to treat the case where the bridge touches the left tip. Write $A^\sharp = [a_1, \ldots, a_r]$ with touched tip the component of weight a_1.

If $r = 1$, then

$$\mu(A^\sharp, 1) = \frac{2}{a_1}.$$

Hence

$$\mu(A^\sharp, 1) > \frac{1}{2}$$

if and only if $a_1 = 2$ or 3, which is exactly the stated classification.

Assume now that $r \geq 2$ and put

$$B := [a_2, \ldots, a_r], \qquad B' := [a_3, \ldots, a_r],$$

with the convention $d(\varnothing) = 1$. Then $d(A^\sharp) = a_1\, d(B) - d(B')$, and

$$\mu(A^\sharp, 1) = \frac{1 + d(B)}{d(A^\sharp)} = \frac{1 + d(B)}{a_1\, d(B) - d(B')}.$$

Case 1: $a_1 = 2$. Then

$$\mu(A^\sharp, 1) = \frac{1 + d(B)}{2d(B) - d(B')}.$$

Since $d(B') \geq 0$, one has $2d(B) - d(B') < 2d(B) + 2$, hence

$$\mu(A^\sharp, 1) > \frac{1}{2}.$$

Case 2: $a_1 \geq 4$. Since B is nonempty, one has $d(B') \leq d(B) - 1$. Therefore

$$d(A^\sharp) = a_1 d(B) - d(B') \geq 4d(B) - (d(B) - 1) = 3d(B) + 1.$$

Thus

$$\mu(A^\sharp, 1) \leq \frac{1 + d(B)}{3d(B) + 1} < \frac{1}{2}.$$

Case 3: $a_1 = 3$. Then

$$\mu(A^\sharp, 1) > \frac{1}{2} \iff 2(1 + d(B)) > 3d(B) - d(B') \iff 2 + d(B') > d(B).$$

Since $d(B) > d(B')$ and both are integers, this is equivalent to $d(B) - d(B') = 1$.

We now claim that for any chain $\Gamma = [c_1, \ldots, c_s]$, if $\Gamma' := [c_2, \ldots, c_s]$, then $d(\Gamma) - d(\Gamma') \geq 1$, with equality if and only if $\Gamma = [2, \ldots, 2]$. This is proved by induction on s. For $s = 1$ one has $d([c_1]) - d(\varnothing) = c_1 - 1$, so equality holds exactly when $c_1 = 2$. For $s \geq 2$, writing

$$\Gamma_1 = [c_2, \ldots, c_s], \qquad \Gamma_2 = [c_3, \ldots, c_s],$$

one gets

$$d(\Gamma) - d(\Gamma') = c_1 d(\Gamma_1) - d(\Gamma_2) - d(\Gamma_1) = (c_1 - 1)d(\Gamma_1) - d(\Gamma_2) = (c_1 - 2)d(\Gamma_1) + \big(d(\Gamma_1) - d(\Gamma_2)\big).$$

Since $d(\Gamma_1) > 0$, equality $= 1$ holds if and only if

$$c_1 = 2 \qquad \text{and} \qquad d(\Gamma_1) - d(\Gamma_2) = 1.$$

The induction hypothesis then gives $\Gamma_1 = [2, \ldots, 2]$, hence $\Gamma = [2, \ldots, 2]$. This proves the claim.

Applying the claim to B, we conclude that $d(B) - d(B') = 1$ holds if and only if $B = [2, \ldots, 2]$. Thus, in the case $a_1 = 3$,

$$\mu(A^\sharp, 1) > \frac{1}{2}$$

if and only if $A^\sharp = [3, (2)^m]$ for some $m \geq 0$, touched at the heavy tip. But $A^\sharp$ is already a 2-tail core, so no terminal 2 away from the touched component can remain. Hence necessarily $m = 0$, that is, $A^\sharp = [3]$.

Combining the three cases proves the proposition. □

Corollary 9.97 (Every positive chain bridge has a special tip-visible side)**.** *Let L be a positive distinct-component bridge between two chain components*

$$(A, i) \qquad \textit{and} \qquad (B, j)$$

on the minimal resolution of a Picard-rank-one klt del Pezzo surface. Let

$$(A^\sharp, i^\sharp), \qquad (B^\sharp, j^\sharp)$$

be the corresponding 2*-tail cores.*

Then at least one of the following holds:

(i) *one of the two cores is touched at a tip of weight* 2*;*

(ii) *one of the two cores is* [3] *touched at its unique component.*

In particular, every positive chain bridge has at least one special *tip-visible side after* 2*-tail reduction.*

Proof. By Corollary 9.95, at least one of the two reduced cores is touched at a tip. Suppose, for contradiction, that neither reduced core satisfies (i) or (ii).

If both reduced cores are tip-visible, then Proposition 9.96 gives

$$\mu(A^\sharp, i^\sharp) \le \frac{1}{2}, \qquad \mu(B^\sharp, j^\sharp) \le \frac{1}{2}.$$

By Lemma 9.93,

$$\mu(A, i) \le \mu(A^\sharp, i^\sharp) \le \frac{1}{2}, \qquad \mu(B, j) \le \mu(B^\sharp, j^\sharp) \le \frac{1}{2}.$$

Hence Proposition 9.47 yields

$$-K_X \cdot C = -1 + \mu(A, i) + \mu(B, j) \le 0,$$

contradicting the ampleness of $-K_X$.

So exactly one reduced core is tip-visible and the other is interior. Without loss of generality, assume that $(A^\sharp, i^\sharp)$ is tip-visible and $(B^\sharp, j^\sharp)$ is interior. Then Proposition 9.96 gives

$$\mu(A^\sharp, i^\sharp) \le \frac{1}{2},$$

while Proposition 9.94 gives

$$\mu(B^\sharp, j^\sharp) \le \frac{1}{2}.$$

Again Lemma 9.93 implies

$$\mu(A, i) \le \frac{1}{2}, \qquad \mu(B, j) \le \frac{1}{2},$$

so Proposition 9.47 gives the same contradiction $-K_X \cdot C \le 0$.

Thus at least one reduced core must satisfy (i) or (ii). □

Remark 9.98 (What this does to the remaining chain bridge problem)*.* Corollary 9.97 sharpens the 2-tail reduction substantially. The remaining chain-shaped bridge geometry is no longer an arbitrary one-tip-visible problem.

Indeed, every positive chain bridge now has at least one special tip-visible side, and there are only two such possibilities:

(i) a side beginning with a touched (-2)-tip;

(ii) a side of type [3] touched at its unique component.

So the unresolved chain gap is reduced further to:

(a) opposite-root bridges with a touched (-2)-tip side;

(b) opposite-root bridges with a touched [3] side at its unique component;

(c) mixed-core variants obtained by attaching one of those two special sides to a branching core on the opposite side.

This is strictly smaller than the previous one-tip-visible gap.

Remark 9.99 (What this reduces the remaining chain bridge problem to). Corollary 9.95 already shows that the remaining non-staircase *chain* bridge problem is no longer an uncontrolled interior-hit problem. Corollary 9.97 sharpens this further: every surviving positive chain bridge has at least one *special* tip-visible side after 2-tail reduction.

Thus the unresolved chain-shaped bridge geometry is reduced to the following two classes:

(i) *opposite-root special-tip bridges*, where the surviving root is on the opposite side and the touched side is either a (-2)-tip or the singleton core [3];

(ii) *mixed-core special-tip bridges*, where one side is special-tip-visible and the opposite side carries the remaining core complexity.

In particular, the remaining chain bridge problem is now a one-special-tip problem, not a general interior-hit problem.

Remark 9.100 (What is now genuinely left). After Theorem 9.79, Corollaries 9.72, 9.66, 9.76, and 9.87, the remaining Route C gap is bridge-specific and global.

The same-component half is already closed, the first analyzed bridge frontier is already removed from the residual sector, and the visible off-frontier one-root staircase family $[b] \dashv [a,(2)^{a-1}]$ is now eliminated completely in the benchmark range by the anticanonical sieve together with the earlier low-height and elliptic-descendant exclusions.

Corollaries 9.95 and 9.97 then sharpen the remaining chain problem further: every surviving positive chain bridge has at least one *special* tip-visible side after stripping terminal [2]-tails away from the touched components. So the unresolved chain-shaped gap is no longer the general heavy-heavy or interior-hit regime. What remains is the genuinely global geometry of opposite-root special-tip bridges, mixed-core special-tip bridges, and branching off-frontier bridges not yet constrained by the same anticanonical argument.

Remark 9.101 (The bridge analysis before the height-4 excess package). After Theorem 9.79, the same-component case is closed, the first analyzed bridge frontier is already removed from the residual sector, the visible one-root staircase family is excluded by the anticanonical sieve, and Corollaries 9.95 and 9.97 reduce the remaining chain-shaped problem to one-special-tip cores. The clean-section Hirzebruch model from Proposition 9.16 still packages the surviving geometry efficiently. The next subsection records the exact excess identity in the height-4 maximal-width corridor and a corrected fiberwise graph bound, thereby isolating the remaining obstacle to a direct numerical closure.

9.26. A rigorous section-only reduction in the high-height regime

The height-4 maximal-width corridor is not a global endgame: the height of a rank-one del Pezzo surface is defined as the *minimum* of $F \cdot D$ over all $\mathbf{P}^1$-fibrations on the minimal resolution, and in characteristics 2 and 3 examples of height 6 do occur. Accordingly, we isolate here a genuinely unconditional reduction for the high-height case in which the witnessing fibration is section-only.

Definition 9.102 (Globally bad singly-hit tips)**.** Let $\pi\colon (\widetilde{X}, D) \to X$ be the minimal resolution of a singular klt del Pezzo surface of Picard rank 1, and let $p\colon \widetilde{X} \to \mathbf{P}^1$ be a $\mathbf{P}^1$-fibration. For a degenerate fiber F, let $\mathcal{B}_1(F)$ be the set of multiplicity-one components

$$L \subset F_{\mathrm{red}}, \qquad L \not\subset D,$$

such that $L \cdot D_{\mathrm{hor}} = 1$ and the unique component of $F_{\mathrm{red}} - L$ meeting L does *not* belong to D. Put $\mathcal{B}_1 := \bigcup_F \mathcal{B}_1(F)$, where the union runs over all degenerate fibers.

Proposition 9.103 (Global bad-tip bound for witnessing fibrations of height at least 3)**.** *Let X be a singular klt del Pezzo surface of Picard rank 1, and let $p\colon \widetilde{X} \to \mathbf{P}^1$ be a witnessing $\mathbf{P}^1$-fibration. Put $h := F \cdot D$ for a general fiber F, and assume that $h \geq 3$. Then*

$$\#\mathcal{B}_1 \leq \Sigma := \sum_F (\sigma(F) - 1) = \#D_{\mathrm{hor}} - 1.$$

Proof. By Proposition 9.12, no fiber of p is contained in D. Hence [7, Lemma 2.4] gives

$$\#D_{\mathrm{hor}} + \rho(\widetilde{X}) = \#D + 2 + \Sigma.$$

Since $\rho(X) = 1$, we have $\rho(\widetilde{X}) = 1 + \#D$, so indeed $\Sigma = \#D_{\mathrm{hor}} - 1$.

It remains to prove the fiberwise estimate $\#\mathcal{B}_1(F) \leq \sigma(F) - 1$ for each degenerate fiber F.

Fix such an F. Let $N_1, \ldots, N_r$ be the connected components of $F_{\mathrm{red}} - D$, and put

$$m_i := \#\{\text{irreducible components of } N_i\}.$$

Then

$$\sigma(F) = \sum_{i=1}^{r} m_i.$$

Every component of $\mathcal{B}_1(F)$ lies in some N_i with $m_i \geq 2$, because by definition its unique vertical neighbor is not contained in D.

Fix such an N_i.

If $m_i = 2$, then N_i is a chain of two irreducible components. At most one of them belongs to $\mathcal{B}_1(F)$. Indeed, if both did, then each would meet D only horizontally and would have as its unique vertical neighbor the other one, so no component of $D \cap F$ would meet N_i. Hence $F_{\mathrm{red}} = N_i = [1, 1]$. But then, because both components lie in $\mathcal{B}_1(F)$, each meets D_{hor} exactly once, so $F \cdot D_{\mathrm{hor}} = 2$, contrary to the assumption that p is witnessing of height $h \geq 3$. Thus N_i contributes at most $1 = m_i - 1$ components to $\mathcal{B}_1(F)$.

If $m_i \geq 3$, then the dual graph of N_i is a tree with m_i vertices, and every component of $\mathcal{B}_1(F)$ contained in N_i is a leaf of that tree. Therefore N_i contributes at most $m_i - 1$ components to $\mathcal{B}_1(F)$.

Summing over all i with $m_i \geq 2$, we obtain

$$\#\mathcal{B}_1(F) \leq \sum_{m_i \geq 2} (m_i - 1) = \sigma(F) - \#\{i\,;\, m_i \geq 2\} \leq \sigma(F) - 1.$$

Now summing over all degenerate fibers gives

$$\#\mathcal{B}_1 \leq \sum_F (\sigma(F) - 1) = \Sigma = \#D_{\mathrm{hor}} - 1.$$

☐

Corollary 9.104 (Clean section for section-only witnessing fibrations)**.** *Assume the setup of Proposition 9.103, and assume in addition that every horizontal component of D is a 1-section. Write*

$$D_{\mathrm{hor}} = H_1 + \cdots + H_h, \qquad h = F \cdot D.$$

Then there exists a horizontal section $H_0 \subset D_{\mathrm{hor}}$ which meets no component of $\mathcal{B}_1$.

Proof. Each component of $\mathcal{B}_1$ is met by exactly one horizontal 1-section. Since $\#\mathcal{B}_1 \leq \#D_{\mathrm{hor}} - 1 = h - 1$ by Proposition 9.103, at least one of the h sections $H_1, \ldots, H_h$ meets none of the bad tips. ☐

Definition 9.105 (Special horizontal sections)**.** Assume that $p\colon \widetilde{X} \to \mathbf{P}^1$ is a section-only witnessing $\mathbf{P}^1$-fibration, and let $H \subset D_{\mathrm{hor}}$ be a horizontal section.

We call H *special* if either:

(i) the connected component of D containing H is not a chain; or

(ii) the connected component of D containing H is a chain, and after applying the 2-tail reduction of Definition 9.92 to the touched pair (A, H), the reduced touched core is either a touched tip of weight 2 or the singleton chain [3].

If neither condition holds, we call H *ordinary.*

Lemma 9.106 (Discrepancy weight on a chain component)**.** *Let*

$$K_{\widetilde{X}} = \pi^* K_X + \sum_{E \subset D} (\mu_X(E) - 1)E.$$

Assume that a connected component of D is a chain

$$A = [A_1, \dots, A_r], \qquad A_j^2 = -a_j,$$

and fix $1 \le i \le r$. Then

$$\mu_X(A_i) = \mu(A, i) = \frac{d(A_i^-) + d(A_i^+)}{d(A)}.$$

Equivalently,

$$\mathrm{cf}_X(A_i) := 1 - \mu_X(A_i) = 1 - \mu(A, i).$$

Proof. Write $\mu_j := \mu_X(A_j)$. Since $\pi^* K_X \cdot A_j = 0$ for every exceptional curve A_j, intersecting the discrepancy formula with A_j gives

$$K_{\widetilde{X}} \cdot A_j = \sum_{k=1}^{r} (\mu_k - 1)(A_k \cdot A_j).$$

By adjunction, $K_{\widetilde{X}} \cdot A_j = -2 - A_j^2 = a_j - 2$. Hence

$$a_j - 2 = -a_j(\mu_j - 1) + \sum_{k \sim j} (\mu_k - 1),$$

where $k \sim j$ means that A_k is a neighbor of A_j in the chain. Rearranging, we obtain

$$a_j \mu_j - \sum_{k \sim j} \mu_k = 2 - \beta_A(A_j),$$

where $\beta_A(A_j)$ is the branching number of A_j inside the chain A. Therefore $a_1\mu_1 - \mu_2 = 1$,

$$-\mu_{j-1} + a_j \mu_j - \mu_{j+1} = 0 \qquad (1 < j < r),$$

$a_r \mu_r - \mu_{r-1} = 1$. This is exactly the standard boundary-value problem defining $\mu(A, i)$ on a chain, whose unique solution is given by the determinant formula

$$\mu_i = \frac{d(A_i^-) + d(A_i^+)}{d(A)}.$$

Thus $\mu_X(A_i) = \mu(A, i)$, as claimed. □

Lemma 9.107 (Ordinary horizontal sections have discrepancy coefficient at least one half)**.** *Assume the setup of Definition 9.105. If $H \subset D_{\mathrm{hor}}$ is ordinary, then*

$$\mu_X(H) \le \frac{1}{2} \qquad \textit{and hence} \qquad \mathrm{cf}_X(H) = 1 - \mu_X(H) \ge \frac{1}{2}.$$

Proof. By definition, H lies on a chain component A of D. Let $(A^\sharp, H^\sharp)$ be the touched pair obtained from (A, H) by 2-tail reduction.

If $H^\sharp$ is an interior vertex of $A^\sharp$, then Proposition 9.94 gives

$$\mu(A^\sharp, H^\sharp) \leq \frac{1}{2}.$$

If $H^\sharp$ is a tip, then H is ordinary precisely when the reduced touched core is neither a touched 2-tip nor the singleton [3]. In that case Proposition 9.96 gives again

$$\mu(A^\sharp, H^\sharp) \leq \frac{1}{2}.$$

Finally, Lemma 9.93 shows that removing terminal 2-tails away from the touched component can only increase the contact weight, so

$$\mu(A, H) \leq \mu(A^\sharp, H^\sharp) \leq \frac{1}{2}.$$

By Lemma 9.106, $\mu_X(H) = \mu(A, H)$, which proves the claim. □

Lemma 9.108 (Anticanonical degree of a witnessing fiber)**.** *Let X be a singular klt del Pezzo surface of Picard rank 1, let $\pi\colon (\widetilde{X}, D) \to X$ be the minimal resolution, and let $p\colon \widetilde{X} \to \mathbf{P}^1$ be a witnessing $\mathbf{P}^1$-fibration. If F is a general fiber, then*

$$-K_X \cdot \pi(F) = 2 - \sum_{H \subset D_{\mathrm{hor}}} \mathrm{cf}_X(H)\,(F \cdot H).$$

In particular,

$$\sum_{H \subset D_{\mathrm{hor}}} \mathrm{cf}_X(H)\,(F \cdot H) < 2.$$

Proof. Write

$$K_{\widetilde{X}} = \pi^* K_X + \sum_{E \subset D} (\mu_X(E) - 1)E, \qquad \mathrm{cf}_X(E) := 1 - \mu_X(E).$$

Then

$$\pi^* K_X = K_{\widetilde{X}} + \sum_{E \subset D} \mathrm{cf}_X(E) E.$$

Intersecting with a general fiber F, and using that only horizontal exceptional components meet F, gives

$$K_X \cdot \pi(F) = \pi^* K_X \cdot F = K_{\widetilde{X}} \cdot F + \sum_{H \subset D_{\mathrm{hor}}} \mathrm{cf}_X(H)(F \cdot H).$$

Since $F \cong \mathbf{P}^1$ is a fiber of a ruling on a smooth surface, $K_{\widetilde{X}} \cdot F = -2$. Hence

$$-K_X \cdot \pi(F) = 2 - \sum_{H \subset D_{\mathrm{hor}}} \mathrm{cf}_X(H)(F \cdot H).$$

Finally, $\pi(F)$ is a nonzero effective curve on the del Pezzo surface X, so $-K_X \cdot \pi(F) > 0$. This proves the strict inequality. □

Proposition 9.109 (Many special sections in the section-only high-height regime)**.** *Let X be* *a characteristic-*3 *klt del Pezzo surface with*

$$\rho(X) = 1, \qquad \#\operatorname{Sing}(X) \geq 8,$$

and let $p \colon \widetilde{X} \to \mathbf{P}^1$ *be a witnessing* $\mathbf{P}^1$*-fibration such that every horizontal component of* D *is a* 1*-section. Write*

$$D_{\mathrm{hor}} = H_1 + \cdots + H_h, \qquad h = \operatorname{ht}(X),$$

and let s_{sp} *be the number of special horizontal sections in the sense of Definition 9.105. Then* $s_{\mathrm{sp}} \geq h - 3$*. Equivalently, the number of ordinary horizontal sections is at most* 3*.*

Proof. For every ordinary horizontal section H_i, Lemma 9.107 gives

$$\operatorname{cf}_X(H_i) \geq \frac{1}{2}.$$

For a special section we only use the trivial bound $\operatorname{cf}_X(H_i) \geq 0$. Since the fibration is section-only, we have

$$F \cdot H_i = 1 \qquad \text{for every } i.$$

Hence Lemma 9.108 gives

$$2 > \sum_{i=1}^{h} \operatorname{cf}_X(H_i).$$

Splitting this sum into ordinary and special sections, we obtain

$$2 > \sum_{H_i \text{ ordinary}} \operatorname{cf}_X(H_i) + \sum_{H_i \text{ special}} \operatorname{cf}_X(H_i) \geq \frac{1}{2}(h - s_{\mathrm{sp}}).$$

Therefore $s_{\mathrm{sp}} > h - 4$. Since s_{sp} is an integer, it follows that $s_{\mathrm{sp}} \geq h - 3$. ☐

Corollary 9.110 (Clean section plus special sections)**.** *Assume the setup of Proposition 9.109. Then:*

1. *there exists a clean horizontal section* $H_0 \subset D_{\mathrm{hor}}$ *meeting no component of* $\mathcal{B}_1$*;*
2. *if* $h \geq 5$*, then there is at least one special horizontal section distinct from* H_0*;*
3. *if* $h \geq 6$*, then there are at least two special horizontal sections distinct from* H_0*.*

Proof. Part (1) is Corollary 9.104.

For (2) and (3), Proposition 9.109 gives $s_{\mathrm{sp}} \geq h - 3$. If H_0 is ordinary, then all s_{sp} special sections are distinct from H_0. If H_0 is special, then there are at least $s_{\mathrm{sp}} - 1 \geq h - 4$ special sections distinct from H_0. Hence for $h \geq 5$ there is at least one such section, and for $h \geq 6$ there are at least two. ☐

Remark 9.111 (What this reduction achieves)*.* The preceding results give a rigorous replacement for the false height-4 endgame in the section-only regime. If a residual counterexample admits a section-only witnessing fibration of height $h \geq 5$, then:

(i) one horizontal section is clean in the sense of Corollary 9.104; and

(ii) all but at most three horizontal sections are special in the sense of Definition 9.105.

Thus the section-only high-height problem is reduced to a ruled-surface collision problem for many special sections relative to a clean section. What still remains is a genuine geometric input showing that such a configuration yields Route C entry, a descendant with elliptic boundary, or one of the already excluded explicit low-height / fork / Frobenius models.

9.27. A stronger unconditional high-height reduction: all but five units of horizontal multiplicity are strong

The preceding section-only reduction shows that many horizontal sections are special. One can sharpen this globally by separating the chain case into the genuinely hard touched 2-tip cores, the intermediate singleton [3] case, and the remaining ordinary cores.

Definition 9.112 (Strong, mild, and ordinary horizontal components)**.** Let $p\colon \widetilde{X} \to \mathbf{P}^1$ be a witnessing $\mathbf{P}^1$-fibration, and let $H \subset D_{\mathrm{hor}}$ be an irreducible horizontal component.

We call H *strong* if either

(i) the connected component of D containing H is not a chain; or

(ii) the connected component of D containing H is a chain, and after 2-tail reduction of the touched chain pair relative to H, the reduced touched core is tip-visible and the touched tip has weight 2.

If the connected component of D containing H is a chain and its reduced touched core is [3] touched at its unique component, then we call H *mild*.

All remaining horizontal components are called *ordinary*.

Lemma 9.113 (Mild and ordinary chain components have controlled discrepancy)**.** *Assume the setup of Definition 9.112, and assume that $H \subset D_{\mathrm{hor}}$ lies on a chain component of D. Then:*

(i) *if H is mild, then*
$$\mathrm{cf}_X(H) \geq \frac{1}{3};$$

(ii) *if H is ordinary, then*
$$\mathrm{cf}_X(H) \geq \frac{1}{2}.$$

Proof. Let (A, H) be the touched chain pair determined by the connected chain component of D containing H, and let $(A^\sharp, H^\sharp)$ be its 2-tail reduction. By Lemma 9.106, $\mathrm{cf}_X(H) = 1 - \mu(A, H)$. By Lemma 9.93, removing terminal 2-tails away from the touched component can only increase μ, so $\mu(A, H) \leq \mu(A^\sharp, H^\sharp)$.

If H is mild, then by definition $(A^\sharp, H^\sharp) = [3]$ touched at its unique component. Hence

$$\mu(A^\sharp, H^\sharp) = \frac{2}{3},$$

so

$$\mu(A,H) \leq \frac{2}{3}.$$

Therefore

$$\mathrm{cf}_X(H) = 1 - \mu(A,H) \geq \frac{1}{3}.$$

If H is ordinary, then the reduced touched core is neither an interior core with $\mu > 1/2$, nor a touched 2-tip, nor the singleton [3]. Hence Proposition 9.94 and Proposition 9.96 give

$$\mu(A^\sharp, H^\sharp) \leq \frac{1}{2}.$$

Thus

$$\mu(A,H) \leq \frac{1}{2}, \qquad \text{hence} \qquad \mathrm{cf}_X(H) = 1 - \mu(A,H) \geq \frac{1}{2}.$$

□

Theorem 9.114 (All but five units of horizontal multiplicity are strong)**.** *Let X be a klt del Pezzo surface of Picard rank 1, let $\pi\colon (\widetilde{X}, D) \to X$ be the minimal resolution, and let $p\colon \widetilde{X} \to \mathbf{P}^1$ be a witnessing $\mathbf{P}^1$-fibration. Let F be a general fiber, and write*

$$D_{\mathrm{hor}} = \sum_i H_i, \qquad m_i := F \cdot H_i.$$

Put

$$h := \sum_i m_i = \mathrm{ht}(X).$$

Let

$$R := \sum_{H_i \text{ strong}} m_i$$

be the total horizontal multiplicity carried by strong horizontal components. Then $R \geq h - 5$. Equivalently, the total horizontal multiplicity carried by mild or ordinary horizontal components is at most 5.

Proof. Split the horizontal components into strong, mild, and ordinary ones. For strong components we only use the trivial inequality $\mathrm{cf}_X(H_i) \geq 0$. If H_i is mild, then Lemma 9.113 gives

$$\mathrm{cf}_X(H_i) \geq \frac{1}{3}.$$

If H_i is ordinary, the same lemma gives

$$\mathrm{cf}_X(H_i) \geq \frac{1}{2}.$$

Let

$$M := \sum_{H_i \text{ mild}} m_i, \qquad O := \sum_{H_i \text{ ordinary}} m_i.$$

Then $h = R + M + O$.

By Lemma 9.108,

$$\sum_i \mathrm{cf}_X(H_i)\, m_i < 2.$$

Therefore

$$2 > \sum_i \mathrm{cf}_X(H_i)\, m_i \geq \frac{1}{3}M + \frac{1}{2}O \geq \frac{1}{3}(M+O) = \frac{1}{3}(h-R).$$

Hence $h - R < 6$. Since $h - R$ is an integer, this yields

$$h - R \leq 5, \qquad \text{i.e.} \qquad R \geq h - 5.$$

□

Corollary 9.115 (Section-only consequences of the strong bound)**.** *In the setup of Theorem 9.114, assume moreover that every horizontal component is a 1-section. Then at least $h-5$ of the horizontal sections are strong.*

In particular:

(i) *if $h \geq 6$, there exists a strong horizontal section;*

(ii) *if $h \geq 7$, there exist at least two strong horizontal sections.*

Proof. In the section-only case one has $m_i = 1$ for every horizontal component, so Theorem 9.114 says precisely that at least $h-5$ of them are strong. □

Corollary 9.116 (Clean section plus strong sections)**.** *Assume the setup of Corollary 9.104, and assume that the witnessing fibration is section-only of height h. Then:*

(1) *there exists a clean section $H_0 \subset D_{\mathrm{hor}}$ meeting no component of $\mathcal{B}_1$;*

(2) *if $h \geq 7$, there exists at least one strong horizontal section distinct from H_0;*

(3) *if $h \geq 8$, there exist at least two strong horizontal sections distinct from H_0.*

Proof. Part (1) is Corollary 9.104. By Corollary 9.115, the total number of strong sections is at least $h-5$. If H_0 is not strong, then all of those strong sections are distinct from H_0. If H_0 is strong, then the number of strong sections distinct from H_0 is at least $(h-5)-1 = h-6$. Therefore for $h \geq 7$ there is at least one such section, and for $h \geq 8$ there are at least two. □

Remark 9.117 (What this strengthens in the section-only regime)*.* Theorem 9.114 is strictly stronger than the earlier special-section bound. It shows that the non-strong horizontal geometry has total multiplicity at most 5. Thus, in a section-only witnessing fibration of height h, at least $h-5$ sections are already of the genuinely hard kind: branching sections or sections whose reduced chain core is a touched 2-tip.

In particular, if $h \geq 8$, then after choosing a clean section one still has at least two strong sections distinct from it. The remaining geometric input is therefore reduced to the collision behaviour of strong sections relative to a clean section.

9.28. General clean-section geometry and section packets in the high-height regime

The strong-section bound reduces the section-only high-height regime to the collision geometry of many strong sections relative to a clean section. The next two propositions record the corresponding local and global ruled-surface models.

Proposition 9.118 (General clean-section Hirzebruch model for section-only witnesses)**.** *Let X be a singular klt del Pezzo surface of Picard rank 1, let $\pi\colon (\widetilde{X}, D) \to X$ be the minimal resolution, and let $p\colon \widetilde{X} \to \mathbf{P}^1$ be a witnessing $\mathbf{P}^1$-fibration such that every horizontal component of D is a 1-section. Choose a clean section $H_0 \subset D_{\mathrm{hor}}$ as in Corollary 9.104, and let $\upsilon\colon \widetilde{X} \to \mathbb{F}_n$ be the contraction of all vertical curves disjoint from H_0. Then:*

1. *$S_0 := \upsilon(H_0)$ is the negative section of $\mathbb{F}_n$, with $n = -H_0^2 \geq 2$.*
2. *For every other horizontal section $H \subset D_{\mathrm{hor}} - H_0$, the image $S := \upsilon(H)$ is a section of $\mathbb{F}_n$ satisfying*

$$S \sim S_0 + (n + H \cdot H_0)f, \qquad S^2 = n + 2(H \cdot H_0), \qquad S \cdot S_0 = H \cdot H_0.$$

3. *If $H, H' \subset D_{\mathrm{hor}} - H_0$ are distinct horizontal sections, then their images satisfy*

$$\upsilon(H) \cdot \upsilon(H') = n + (H \cdot H_0) + (H' \cdot H_0) > 0.$$

In particular, any two horizontal sections distinct from H_0 meet on $\mathbb{F}_n$.

Proof. The proof is the same as in Proposition 9.16. Contracting all vertical curves which do not meet H_0 produces a relatively minimal ruled surface $\mathbb{F}_n$, and $S_0 := \upsilon(H_0)$ is its unique negative section because $H_0^2 \leq -2$. Thus

$$S_0^2 = -n, \qquad n = -H_0^2 \geq 2.$$

If $H \subset D_{\mathrm{hor}} - H_0$ is another horizontal 1-section and $S := \upsilon(H)$, then S is again a section. Writing $S \sim S_0 + bf$, one has $S \cdot S_0 = b - n = H \cdot H_0$, so $b = n + H \cdot H_0$. Therefore

$$S \sim S_0 + (n + H \cdot H_0)f, \qquad S^2 = -n + 2(n + H \cdot H_0) = n + 2(H \cdot H_0).$$

This proves (1) and (2).

For (3), if $S = \upsilon(H)$ and $S' = \upsilon(H')$, then

$$S \cdot S' = (S_0 + (n + H \cdot H_0)f)(S_0 + (n + H' \cdot H_0)f) = -n + (n + H \cdot H_0) + (n + H' \cdot H_0),$$

which is the displayed formula. Since $n \geq 2$, this number is positive. □

Proposition 9.119 (Primitive isolated collisions of two sections)**.** *Assume the setup of Propo-*

sition 9.118, and let $H_i, H_j \subset D_{\mathrm{hor}} - H_0$ be two distinct horizontal sections with images

$$S_i := \upsilon(H_i), \qquad S_j := \upsilon(H_j)$$

on $\mathbb{F}_n$. Let $o \in S_i \cap S_j$ be a point through which no other horizontal image passes, and put $m := (S_i \cdot S_j)_o \geq 1$. Then the first m blowups of υ^{-1} over o are forced and coincide with the order-m two-section collision resolution of Lemma 9.20.

Consequently, at the primitive separation stage one obtains a local chain $F_1 - \cdots - F_m$ such that

$$F_k^2 = -2 \ \ (1 \leq k < m), \qquad F_m^2 = -1,$$

and the strict transforms of S_i and S_j both meet F_m transversely at distinct points. In particular:

(i) *if $m = 1$, then the primitive last exceptional curve F_1 is a primitive contact-2 curve meeting the two colliding sections;*

(ii) *if $m \geq 2$, then the primitive last exceptional curve F_m is the center of a primitive fork with three arms: the two colliding sections and the chain $F_{m-1} - \cdots - F_1$.*

Proof. Because no other horizontal image passes through o, before the strict transforms of S_i and S_j separate there is at each step a unique possible center: the current intersection point of those two strict transforms. Hence the first m blowups over o are forced.

In suitable local coordinates (t, z) centered at o with the fiber through o given by $\{t = 0\}$, the two sections are analytically equivalent to $z = 0, \qquad z = t^m$. Therefore Lemma 9.20 applies and gives the displayed chain $F_1 - \cdots - F_m$, its self-intersections, and the transverse contacts of the two strict transforms with F_m. If $m = 1$, there is no earlier exceptional curve, so F_1 is a primitive contact-2 curve. If $m \geq 2$, then F_m also meets the adjacent exceptional component F_{m-1}, so the primitive local core is a fork with the stated three arms. □

Proposition 9.120 (Ramification of a horizontal multisection gives a primitive fork)**.** *Let $q \in S \subset Y_0$ be a smooth point of a smooth horizontal curve on a relatively minimal ruled surface $Y_0 \to \mathbf{P}^1$, and assume that the induced map $S \to \mathbf{P}^1$ has local degree $e \geq 2$ at q. Then, after the first e forced blowups separating S from the fiber through q, the reduced fiber over q is $F' - E_e - E_{e-1} - \cdots - E_1$, with*

$$F'^2 = -e, \qquad E_k^2 = -2 \ \ (1 \leq k < e), \qquad E_e^2 = -1,$$

and the strict transform of S meets E_e transversely away from $F' \cap E_e$. In particular, the primitive last exceptional curve E_e is the center of a primitive fork with three arms:

$$S, \qquad F', \qquad E_{e-1} - \cdots - E_1.$$

Proof. Choose local coordinates (t, z) at q so that the fiber through q is $F_0 = \{t = 0\}$ and the multisection is $S = \{t = z^e\}$. At each stage the current strict transforms of S and the fiber meet in a unique point, so the first e blowups are forced. A direct induction shows that after the k-th blowup one obtains a chain $F_k - E_k - E_{k-1} - \cdots - E_1$ with

$$F_k^2 = -k, \qquad E_i^2 = -2 \ \ (1 \leq i < k), \qquad E_k^2 = -1,$$

and the strict transform of S meeting E_k transversely at a point away from $F_k \cap E_k$. Taking $k = e$ gives the stated configuration. Since $e \geq 2$, the last exceptional curve E_e meets exactly the three displayed branches, so it is a primitive fork center. □

Proposition 9.121 (Packet theorem for many sections with collisions on at most three fibers)**.** *Let $S_1, \ldots, S_r$ be sections of the same ruling class on $\mathbb{F}_n$, disjoint from the negative section S_0. Assume that every pairwise intersection of the S_i is supported on a set of at most three fibers. After an automorphism of the base, assume those fibers are $\{x = 0\}$, $\{y = 0\}$, $\{x - y = 0\}$. After translating by S_1, write*

$$S_1 = \{z = 0\}, \qquad S_i = \{z = q_i\} \quad (i \geq 2),$$

with $q_i \in H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(n))$. Then exactly one of the following holds.

(i) *All nonzero q_i are scalar multiples of one monomial $g = x^a y^b (x - y)^c$. Equivalently, the whole packet is a scalar-multiple star.*

(ii) *There exist nonnegative integers a, b, c, e with $a + b + c + 3^e = n$ such that every q_i belongs to*

$$\{0,\ gx^{3^e},\ gy^{3^e},\ g(x-y)^{3^e}\}, \qquad g = x^a y^b (x-y)^c.$$

In particular, in the non-scalar case the packet contains at most four sections total.

Proof. Since each pairwise intersection $S_i \cap S_j$ is supported on the three distinguished fibers, every difference $q_i - q_j$ has zero divisor supported on those fibers. In particular, every nonzero q_i itself has support on those fibers, so each q_i is a scalar multiple of a monomial in x, y, and $x - y$.

If all nonzero q_i are scalar multiples of one such monomial, we are in case (i). Assume from now on that this is not the case. Then there exist two nonzero sections, say q_2 and q_3, which are not scalar multiples of each other. Applying Proposition 9.18 to the triple 0, q_2, q_3 shows that, after renumbering, there exist nonnegative integers a, b, c, e with $a + b + c + 3^e = n$ and

$$q_2 = gx^{3^e}, \qquad q_3 = gy^{3^e}, \qquad q_2 - q_3 = g(x-y)^{3^e}, \qquad g = x^a y^b (x-y)^c.$$

Let $i \geq 4$ and suppose $q_i \neq 0$. Because $q_i - q_2$ has support on the three distinguished fibers, the triple 0, q_2, q_i again satisfies the hypotheses of Proposition 9.18. Since q_i is not a scalar multiple of q_2 in the non-scalar branch, the proof of that proposition shows that after dividing by the maximal common factor of q_i and q_2, the primitive pair consists of pure powers of two distinct linear forms and its difference is the corresponding third power. Because $q_2 = gx^{3^e}$, this forces the maximal common factor of q_i and q_2 to contain g. Repeating the same argument with the pair (q_i, q_3) forces the same conclusion. Hence $q_i = gM_i$ for some degree-3^e monomial

M_i in $x, y, x-y$.

Now both

$$M_i - x^{3^e} \qquad \text{and} \qquad M_i - y^{3^e}$$

have zero divisors supported on the three distinguished fibers. Applying Proposition 9.18 with common factor 1 shows that $M_i \in \{x^{3^e}, y^{3^e}, (x-y)^{3^e}\}$ up to a scalar. Finally, that scalar must equal 1: for example, if $M_i = \lambda x^{3^e}$, then $\lambda x^{3^e} - y^{3^e}$ vanishes on the fiber $x - \lambda^{-1/3^e} y = 0$, which belongs to the distinguished set only when $\lambda = 1$. Thus $q_i \in \{gx^{3^e}, gy^{3^e}, g(x-y)^{3^e}\}$. This proves case (ii). □

Lemma 9.122 (Elementary transform at a packet–negative-section point)**.** *Let*

$$\epsilon_p \colon \mathbb{F}_n \dashrightarrow \mathbb{F}_{n+1}$$

be the elementary transform at a point $p \in S_0$, where $S_0^2 = -n$ is the negative section of $\mathbb{F}_n$. Let $S \subset \mathbb{F}_n$ be a section, write

$$S \equiv S_0 + (n+\varepsilon)f, \qquad \varepsilon \in \{0,1\},$$

and put $m := \operatorname{mult}_p(S) \in \{0,1\}$. Let $S^+ \subset \mathbb{F}_{n+1}$ be the image of S. Then $S^+ \equiv S_0^+ + (n+1+\varepsilon-m)f^+$, where $(S_0^+)^2 = -(n+1)$ is the negative section of $\mathbb{F}_{n+1}$. In particular, $S^+ \cdot S_0^+ = \varepsilon - m \in \{0,1\}$. So the transform at p removes the unique packet–negative-section contact exactly for those sections passing through p, and does not create a new one for sections already disjoint from S_0. Moreover, if $S_1, S_2 \subset \mathbb{F}_n$ are two such sections, then every fiber supporting $S_1^+ \cap S_2^+$ already supported $S_1 \cap S_2$. In particular, the elementary transform at p does not create a new packet-collision fiber.

Proof. The class formula is exactly Lemma 11.63 with

$$a = 1, \qquad b = n+\varepsilon, \qquad m \in \{0,1\}.$$

Thus $S^+ \equiv S_0^+ + (n+1+\varepsilon-m)f^+$, and intersecting with S_0^+ gives $S^+ \cdot S_0^+ = \varepsilon - m$. This proves the first claim.

For the collision-fiber statement, it is enough to work locally over the fiber through p. Choose analytic coordinates (t, z) centered at p so that the fiber is $F_0 = \{t = 0\}$ and the negative section is $S_0 = \{z = 0\}$. Because every packet section meets S_0 with multiplicity at most one, any section through p has local equation

$$z = at + O(t^2), \qquad a \neq 0,$$

while a section not passing through p is given locally by

$$z = c + O(t), \qquad c \neq 0.$$

After blowing up p, in the chart $z = tw$ all sections through p have strict transforms $w =$

$a+O(t)$, so they meet the new exceptional curve at points determined by their first-order slopes, whereas sections not through p are disjoint from that exceptional curve. After contracting the strict transform of F_0, a pair of sections which was disjoint over F_0 remains disjoint over the transformed fiber, while a pair colliding at p may either remain on that fiber or separate there. Hence no new collision fiber is created. ☐

Proposition 9.123 (Preparing a clean-section packet by transforms along the negative section)**.** *Assume the setup of Proposition 9.118. Let $S_i := \upsilon(H_i)$ for the horizontal sections distinct from H_0, and let $T \subset S_0$ be the set of distinct points lying on at least one of the sections S_i. Write $t := \#T$. Perform successive elementary transforms at the points of T, always on the current negative section, and let*

$$\tau\colon \mathbb{F}_n \dashrightarrow \mathbb{F}_{n+t}$$

be the resulting birational map. Denote by $S_0^\dagger$ the negative section of the final Hirzebruch surface and by $S_i^\dagger$ the final transform of S_i. Then:

(i) *every $S_i^\dagger$ is disjoint from $S_0^\dagger$;*

(ii) *all final packet sections lie in one ruling class,*

$$S_i^\dagger \equiv S_0^\dagger + (n+t)f^\dagger \qquad \textit{for every } i;$$

(iii) *for every pair of indices $i \neq j$, every fiber supporting $S_i^\dagger \cap S_j^\dagger$ already supported $S_i \cap S_j$. In particular, the total set of packet-collision fibers does not increase under the preparation process.*

Proof. Because υ is an isomorphism near H_0, one has $S_i \cdot S_0 = H_i \cdot H_0 \in \{0,1\}$ for every packet section S_i, since distinct irreducible components of the reduced exceptional divisor meet at most once. Hence each S_i passes through at most one point of T.

Apply Lemma 9.122 successively at the points of T. Each transform raises the Hirzebruch index by one and removes the unique negative-section contact exactly for the packet sections passing through the chosen point. Since every packet section meets the negative section in at most one point, after all t steps every final transform $S_i^\dagger$ is disjoint from the final negative section $S_0^\dagger$. This proves (i).

For (ii), let $\varepsilon_i := S_i \cdot S_0 \in \{0,1\}$. By Lemma 9.122, each of the t negative-section elementary transforms adds one to the fiber coefficient of S_i, and exactly one of those steps subtracts one again when $\varepsilon_i = 1$, namely the step at the unique point of T lying on S_i. Therefore the final class is

$$S_i^\dagger \equiv S_0^\dagger + (n+\varepsilon_i+t-\varepsilon_i)f^\dagger = S_0^\dagger + (n+t)f^\dagger.$$

So all prepared packet sections lie in the same ruling class.

Finally, (iii) follows by iterating the last statement of Lemma 9.122: a negative-section elementary transform never creates a new fiber carrying a pairwise packet collision. ☐

Corollary 9.124 (The non-scalar packet branch is finite after preparation)**.** *Assume the setup of Proposition 9.118, and let* $S_i^\dagger$ *be the prepared packet from Proposition 9.123. Assume that every pairwise intersection of the original packet sections* S_i *is supported on at most three fibers. Then exactly one of the following holds.*

(i) *the prepared packet is a scalar-multiple star;*

(ii) *the prepared packet is contained in a Frobenius tetrahedron*

$$0, \quad gx^{3^e}, \quad gy^{3^e}, \quad g(x-y)^{3^e},$$

and in particular contains at most four sections total.

Therefore, if there are at least five horizontal sections distinct from H_0*, then only the prepared scalar case can occur.*

Proof. By Proposition 9.123(iii), the prepared packet still has all pairwise intersections supported on at most three fibers. By Proposition 9.123(i)–(ii), all prepared packet sections are disjoint from the final negative section and lie in one ruling class. Hence Proposition 9.121 applies to the prepared packet and yields exactly the two displayed alternatives. The final assertion is immediate from the bound of at most four sections in the non-scalar branch. ☐

Proposition 9.125 (Local scalar-star model)**.** *Let* U *be a smooth surface with local coordinates* (t, z) *at a point* o*, and let*

$$A_0 = \{z = 0\}, \qquad A_i = \{z = \lambda_i t^u\} \quad (1 \le i \le m),$$

where

$$u \ge 1, \qquad \lambda_i \ne \lambda_j \text{ for } i \ne j.$$

Let $\tau_u \colon U_u \to U$ *be the composition of the* u *successive blowups at the unique common point of the strict transforms of* $A_0, A_1, \dots, A_m$*. Then:*

(1) *the exceptional locus is the chain* $G_1 - \cdots - G_u$*;*

(2) *one has*

$$G_i^2 = -2 \quad (1 \le i < u), \qquad G_u^2 = -1;$$

(3) *the strict transforms of* $A_0, \dots, A_m$ *meet* G_u *transversely at pairwise distinct points and are disjoint from* $G_1, \dots, G_{u-1}$*.*

Proof. We argue by induction on u. For $u = 1$, the first blowup separates the common point of the curves $A_0, \dots, A_m$ and creates a single exceptional curve G_1 with $G_1^2 = -1$. Since the tangent directions of the curves $A_i = \{z = \lambda_i t\}$ are distinct, their strict transforms meet G_1 transversely at pairwise distinct points.

Assume now that $u \ge 2$ and that the statement holds for $u - 1$. After the first blowup at o, in the standard chart with coordinates (t, w) given by $z = tw$, the strict transforms of $A_0, \dots, A_m$

are

$$A_0' = \{w = 0\}, \qquad A_i' = \{w = \lambda_i t^{u-1}\} \quad (1 \le i \le m),$$

and the new exceptional curve is $G_1 = \{t = 0\}$. The strict transforms still have a unique common point, namely

$$(t, w) = (0, 0) = A_0' \cap A_1' \cap \cdots \cap A_m' \cap G_1.$$

By the induction hypothesis, after the next $u - 1$ blowups at the successive unique common points, one obtains a chain $G_2 - \cdots - G_u$ with

$$G_i^2 = -2 \ \ (2 \le i < u), \qquad G_u^2 = -1,$$

and the strict transforms of $A_0', \dots, A_m'$ meet G_u transversely at distinct points and are disjoint from $G_2, \dots, G_{u-1}$. Since G_1 is created first and is blown up once more at the common point, its final self-intersection is -2, and it meets only G_2. This yields the full chain $G_1 - \cdots - G_u$ with the stated self-intersections and incidence relations. □

Corollary 9.126 (High-height section-only packets become scalar after preparation)**.** *Assume the setup of Proposition 9.118, and assume moreover that the witnessing fibration is section-only of height $h \ge 6$. Choose a clean section H_0 as in Corollary 9.116, let $S_i := \upsilon(H_i)$ for the horizontal sections distinct from H_0, and prepare the packet by elementary transforms at all packet–negative-section contact points as in Proposition 9.123. If every pairwise intersection of the original packet sections S_i is supported on at most three fibers, then the prepared packet on the final Hirzebruch surface is a scalar-multiple star.*

Proof. There are exactly $h-1 \ge 5$ horizontal sections distinct from H_0. So Corollary 9.124 rules out the non-scalar prepared branch. □

Remark 9.127 (Ramification of a multisection does not by itself produce an exceptional cluster)*.* A point of local degree at least 2 on a horizontal multisection does *not* by itself force any exceptional cluster on the minimal resolution.

Indeed, on the smooth ruled surface $Y = \mathbf{P}_t^1 \times \mathbf{P}_z^1$ with ruling $p = \mathrm{pr}_1$, the curve $S = \{t = z^3\}$ is a smooth horizontal 3-section in characteristic 3. Every point of $S \to \mathbf{P}_t^1$ has local degree 3, but the ruling on Y has no degenerate fibers and no exceptional locus. Thus ramification of a horizontal multisection is only tangency to a smooth fiber unless that point is an actual center of the birational map to a ruled model.

Proposition 9.128 (If a ramified point is actually blown up, the primitive local model is a fork)**.** *Let $p\colon \widetilde{X} \to \mathbf{P}^1$ be a $\mathbf{P}^1$-fibration on a smooth rational surface, let $H \subset D_{\mathrm{hor}}$ be a horizontal multisection, and let $\nu\colon \widetilde{X} \to Y$ be a birational morphism onto a smooth relatively minimal ruled surface, chosen so that ν is an isomorphism at the generic point of H. Put $S := \nu(H)$.*

Let $q \in S$ be a point such that, in local coordinates (t, z) centered at q and adapted to the ruling on Y, one has

$$f_0 = \{t = 0\}, \qquad S = \{t = z^a\}, \qquad a \ge 2.$$

Assume that over q the first a centers of ν^{-1} are exactly the forced tangency-resolution centers, and that no later center of ν^{-1} lies on the last exceptional curve created in that forced sequence. Then on $\widetilde{X}$ there exist curves

$$L \not\subset D, \qquad A, B_1, \dots, B_{a-1} \subset D$$

contained in the fiber over $p(q)$ such that

$$A^2 = -a, \qquad B_i^2 = -2 \ \ (1 \le i \le a-1), \qquad L^2 = -1,$$

and the local dual graph of $D + L$ near this cluster is

$$\begin{array}{ccccc} & & A & & \\ & & | & & \\ H & - & L & - & B_1 - \cdots - B_{a-1}. \end{array}$$

In particular:

(i) *the primitive ramification cluster is a fork, not a one-root bridge;*

(ii) *one has $L \cdot D = 3$;*

(iii) *the one-root staircase family $[b] \dashv [a, (2)^{a-1}]$ does* not *describe this primitive local model.*

Proof. Apply Proposition 9.120 to the tangency between

$$f_0 = \{t = 0\} \qquad \text{and} \qquad S = \{t = z^a\}.$$

Thus after the forced a blowups one obtains a local chain $F' - E_a - E_{a-1} - \cdots - E_1$, where

$$(F')^2 = -a, \qquad E_i^2 = -2 \ \ (1 \le i < a), \qquad E_a^2 = -1,$$

and the strict transforms of f_0 and S both meet E_a transversely at distinct points.

Let

$$L := E_a, \qquad A := F', \qquad B_i := E_{a-i} \ \ (1 \le i \le a-1).$$

By assumption the first a centers of ν^{-1} are exactly these forced tangency centers, and no later center lies on L. Hence the strict transform of L survives on $\widetilde{X}$ with self-intersection still equal to -1. Therefore $L \not\subset D$, because on the minimal resolution of a rank-one del Pezzo surface the components of a degenerate fiber outside D are exactly the vertical (-1)-curves.

On the other hand,

$$A^2 = -a \le -2, \qquad B_i^2 = -2,$$

so $A, B_1, \dots, B_{a-1} \subset D$. The displayed graph follows directly from the local tangency resolution. Since L meets exactly the three components $H,\ A,\ B_1$, one gets $L \cdot D = 3$. Thus the primitive local model is a fork, not a Route C bridge. □

Remark 9.129 (Why the ramification branch does not feed into the one-root staircase corollaries)*.* The primitive ramification fork from Proposition 9.128 is not the family $[b] \dashv [a, (2)^{a-1}]$ from Proposition 9.85. The crucial difference is that the last nonexceptional curve meets the strict transform of the ramification fiber itself, so its total contact with D is 3, not 2.

Moreover the primitive local fork is not, by itself, root-contractible: after contracting L, $B_1, \dots, B_{a-1}$, the image of A is the original smooth fiber of the ruled surface, hence a 0-curve rather than a point. So any root-contractibility statement for a lone multisection ramification point is necessarily global and must involve the whole vertical connected component of the fiber.

Remark 9.130 (What this sharpens)*.* The packet side and the multisection side now have precise local models.

(i) In the section-only high-height branch, after clearing all packet–negative-section contact points by the preparation process of Proposition 9.123, any packet whose pairwise collisions are supported on at most three fibers is forced to become a scalar-multiple star by Corollary 9.126. The local geometry of each marked scalar collision is then exactly the explicit star model of Proposition 9.125.

(ii) In the multisection branch, any ramification point of local degree at least 2 produces a primitive fork, and this fork survives on $\widetilde{X}$ whenever the tangency center is in general position for the birational map to the ruled model.

Thus the remaining high-height extraction problem is no longer the primitive local birational geometry of these two branches. It is the final global survival/exclusion step:

(a) exclude large scalar stars from the residual sector, or show that they force one of the already eliminated explicit models;

(a) show that a surviving primitive ramification fork is already covered by the existing fork analysis or by the one-root staircase elimination.

Remark 9.131 (What this pushes the high-height program to)*.* The packet theorem is now available in a prepared form on the section-only side. After choosing a clean section, one may first clear all packet–negative-section contact points by the preparation process of Proposition 9.123. The resulting packet is then a disjoint family of sections in one ruling class, so under the three-fiber support hypothesis it is reduced to two explicit regimes:

(i) a prepared scalar-multiple star; or

(ii) a finite Frobenius tetrahedron.

For height at least 6, Corollary 9.126 removes the second alternative. Together with Proposition 9.119, this means that every isolated collision of strong sections is already locally understood: transverse collisions give primitive contact-2 curves and higher-order collisions give primitive forks.

Thus the remaining high-height extraction problem is concentrated in two very narrow places:

(a) the *prepared scalar-star lemma*, excluding a large prepared scalar packet from the residual sector; and

(a) the *survival lemma*, upgrading a primitive contact-2 or fork core coming from a strong collision to an admissible Route C configuration on $\widetilde{X}$ or to one of the already eliminated explicit models.

The multisection branch is similarly narrowed by Proposition 9.120: a ramification point of local degree at least 2 produces a primitive fork model from the start.

9.29. A universal contact-defect sieve for surviving nonexceptional curves

The correct global replacement for the failed one-root staircase idea is not another bridge classification, but a defect inequality for any surviving nonexceptional vertical (-1)-curve. Any such curve can meet exceptional components only if the sum of their discrepancy defects is strictly less than 1. This immediately gives numerical restrictions for both surviving ramification forks and surviving scalar stars.

Lemma 9.132 (Universal discrepancy lower bound from self-intersection)**.** *Let* $\pi\colon(\widetilde{X},D)\to X$ *be the minimal resolution of a klt del Pezzo surface, and write*

$$\pi^*K_X = K_{\widetilde{X}} + \sum_{E\subset D} \mathrm{cf}_X(E)E.$$

If $E\subset D$ *is an irreducible component with*

$$E^2=-a, \qquad a\geq 2,$$

then

$$\mathrm{cf}_X(E)\geq 1-\frac{2}{a}.$$

Equivalently, if $\mu_X(E):=1-\mathrm{cf}_X(E)$*, then*

$$\mu_X(E)\leq\frac{2}{a}.$$

Proof. Intersect

$$\pi^*K_X = K_{\widetilde{X}} + \sum_{T\subset D} \mathrm{cf}_X(T)T$$

with E. Since E is exceptional, $\pi^*K_X\cdot E=0$. Because $E\cong\mathbf{P}^1$ and $E^2=-a$, adjunction gives $K_{\widetilde{X}}\cdot E=a-2$. Therefore

$$0=(a-2)-a\,\mathrm{cf}_X(E)+\sum_{T\neq E}(T\cdot E)\,\mathrm{cf}_X(T).$$

Since every coefficient $\mathrm{cf}_X(T)$ is nonnegative, we get $a\,\mathrm{cf}_X(E)\geq a-2$, hence

$$\mathrm{cf}_X(E)\geq 1-\frac{2}{a}.$$

The equivalent inequality for $\mu_X(E)$ is immediate. □

Proposition 9.133 (Universal contact-defect sieve)**.** *Let $\pi\colon (\widetilde{X}, D) \to X$ be the minimal resolution of a klt del Pezzo surface, and let $L \subset \widetilde{X}$ be a (-1)-curve such that $L \not\subset D$. For each irreducible component $E \subset D$ put $m_E := L \cdot E$. Then*

$$\sum_{E\subset D} m_E \,\mathrm{cf}_X(E) < 1.$$

Consequently,

$$\sum_{E\subset D} m_E\Big(1 - \frac{2}{-E^2}\Big) < 1$$

and equivalently

$$\sum_{E\subset D} \frac{2m_E}{-E^2} > L \cdot D - 1.$$

Proof. Put $C := \pi(L) \subset X$. Since $L \not\subset D$, the curve C is nonzero and effective. Because $-K_X$ is ample, $-K_X \cdot C > 0$. On the other hand,

$$\pi^* K_X = K_{\widetilde{X}} + \sum_{E\subset D} \mathrm{cf}_X(E) E,$$

so using $L^2 = -1$ and hence $K_{\widetilde{X}} \cdot L = -1$ by adjunction, we get

$$-K_X \cdot C = -\pi^* K_X \cdot L = 1 - \sum_{E\subset D} m_E \,\mathrm{cf}_X(E).$$

Thus

$$\sum_{E\subset D} m_E \,\mathrm{cf}_X(E) < 1.$$

Applying Lemma 9.132 termwise gives

$$\sum_{E\subset D} m_E\Big(1 - \frac{2}{-E^2}\Big) < 1.$$

Rearranging yields

$$\sum_{E\subset D} \frac{2m_E}{-E^2} > \sum_{E\subset D} m_E - 1 = L \cdot D - 1.$$

□

Corollary 9.134 (Surviving ramification forks satisfy a finite weight sieve)**.** *Keep the setup of Proposition 9.128, and write*

$$H^2 = -b, \qquad A^2 = -a, \qquad a \ge 2, \qquad b \ge 2.$$

Then

$$\frac{2}{a} + \frac{2}{b} > 1.$$

Equivalently, $(a-2)(b-2) < 4$. In particular:

(i) *if $b \ge 6$, then necessarily $a = 2$;*

(ii) *if $b \geq 4$, then necessarily $a \leq 3$;*

(iii) *if $a, b > 2$, then up to order $(a, b) \in \{(3,3), (3,4), (3,5)\}$.*

Proof. By Proposition 9.128, the surviving curve L meets the three components H, A, B_1 of D transversely once each. Hence Proposition 9.133 gives

$$\mathrm{cf}_X(H) + \mathrm{cf}_X(A) + \mathrm{cf}_X(B_1) < 1.$$

Since every discrepancy coefficient is nonnegative,

$$\mathrm{cf}_X(H) + \mathrm{cf}_X(A) < 1.$$

Now Lemma 9.132 gives

$$\mathrm{cf}_X(H) \geq 1 - \frac{2}{b}, \qquad \mathrm{cf}_X(A) \geq 1 - \frac{2}{a}.$$

Therefore

$$1 - \frac{2}{b} + 1 - \frac{2}{a} < 1,$$

which is equivalent to

$$\frac{2}{a} + \frac{2}{b} > 1.$$

The numerical consequences are immediate. ☐

Corollary 9.135 (Surviving scalar stars are A_1-heavy)**.** *Assume that above some marked scalar packet one has the local scalar-star configuration of Proposition 9.125, and suppose that on $\widetilde{X}$ the strict transform L of its last exceptional curve survives as a vertical (-1)-curve with $L \not\subset D$. Let $H_1, \ldots, H_r \subset D_{\mathrm{hor}}$ be the horizontal components passing through that scalar packet, and write*

$$H_i^2 = -b_i, \qquad b_i \geq 2.$$

Then

$$\sum_{i=1}^{r} \mathrm{cf}_X(H_i) < 1.$$

Hence

$$\sum_{i=1}^{r} \Bigl(1 - \frac{2}{b_i}\Bigr) < 1, \qquad \textit{equivalently} \qquad \sum_{i=1}^{r} \frac{2}{b_i} > r - 1.$$

In particular:

(i) *at most two of the integers b_i can exceed 2;*

(ii) *if two of them exceed 2, then after reordering one has $(b_1, b_2) \in \{(3,3), (3,4), (3,5)\}$;*

(iii) *at least $r - 2$ of the packet sections are (-2)-curves.*

Proof. In the local scalar-star model of Proposition 9.125, the last exceptional curve meets every packet section once. If the collision order is at least 2, it may also meet one vertical component of D, but all additional terms in Proposition 9.133 are nonnegative. Therefore

$$\sum_{i=1}^{r} \mathrm{cf}_X(H_i) < 1.$$

Now apply Lemma 9.132 to each H_i to obtain

$$\sum_{i=1}^{r} \Big(1 - \frac{2}{b_i}\Big) < 1.$$

If three of the b_i were at least 3, then the left-hand side would be at least

$$3 \cdot \frac{1}{3} = 1,$$

a contradiction. Thus at most two of them exceed 2, proving (i) and (iii). If exactly two exceed 2, say $b_1, b_2 > 2$, then all others are equal to 2, so the defect inequality becomes

$$1 - \frac{2}{b_1} + 1 - \frac{2}{b_2} < 1,$$

that is,

$$\frac{2}{b_1} + \frac{2}{b_2} > 1.$$

Up to order this leaves only $(b_1, b_2) \in \{(3,3), (3,4), (3,5)\}$. □

Remark 9.136 (What the new sieve buys)*.* The contact-defect sieve is the correct global replacement for the failed staircase idea.

(a) A surviving ramification fork is forced into a tiny numerical range: unless the horizontal side is a (-2)-curve, the ramification order is at most 5, and if both touched weights exceed 2 then only the finite pairs $(3,3),\ (3,4),\ (3,5)$ remain.

(b) A surviving scalar packet is automatically A_1-heavy: after discarding at most two packet sections, all remaining horizontal participants are (-2)-curves.

What still does *not* follow is existence of such a surviving curve: one still needs a global realization/survival statement ensuring that the residual sector produces either a surviving marked scalar packet or a surviving ramification fork to which the sieve can be applied.

9.30. The height-4 maximal-width corridor: exact excess and the remaining graph-theoretic gap

The conjectural three-section rigidity package is still unnecessary for the unconditional results proved in this paper. Once Corollary 9.13 applies, one has the exact excess identity

$$\Sigma = \sum_{F} (\sigma(F) - 1) = 3,$$

no fiber is contained in D, and one horizontal 1-section avoids all genuinely bad singly-hit multiplicity-one tips. What remains for a direct numerical closure is a fiberwise bound for the purely vertical connected components of the exceptional divisor.

The next proposition records the corrected graph-theoretic statement that follows from the dual-graph argument.

Proposition 9.137 (Corrected fiberwise graph bound in the maximal-width corridor)**.** *Assume the setup of Corollary 9.13. For a degenerate fiber F, let*

$$D_F := \sum_{E \subset D,\ E \subset F} E$$

be the vertical part of the exceptional divisor contained in F. Let $C_1, \dots, C_s$ be the connected components of D_F, and let $t(F)$ be the number of these connected components which meet D_{hor}. Put $v(F) := s - t(F)$, so $v(F)$ is the number of connected components of D_F which do not meet D_{hor}. Then

$$s \leq \sigma(F) + 1 \qquad \text{and} \qquad v(F) \leq \sigma(F) + 1 - t(F).$$

In particular, if $t(F) \geq 2$ for every degenerate fiber with $v(F) > 0$, then $v(F) \leq \sigma(F) - 1$.

Proof. Fix a degenerate fiber F. Let $N_1, \dots, N_r$ be the connected components of $F_{\mathrm{red}} - D$, and for each i let

$$m_i := \#\{\text{irreducible components of } N_i\}.$$

By the definition of $\sigma(F)$ used in Corollary 9.13,

$$\sigma(F) = \sum_{i=1}^{r} m_i,$$

so in particular $r \leq \sigma(F)$.

As in the previous dual-graph argument, contract each connected subgraph corresponding to some N_i and each connected subgraph corresponding to some C_j inside the dual graph of F_{red}. The resulting graph Γ_F is again a tree whose vertex set is $\{N_1, \dots, N_r\} \sqcup \{C_1, \dots, C_s\}$, and every edge joins some N_i to some C_j.

By [6, Lemma 2.6(a)], every irreducible component of $F_{\mathrm{red}} - D$ is a (-1)-curve. By [6, Lemma 2.4], the (-1)-curves in a degenerate fiber are non-branching. Therefore each N_i is a chain, so it meets the rest of the fiber in at most two components. Equivalently,

$$\deg_{\Gamma_F}(N_i) \leq 2 \qquad \text{for all } i.$$

Since Γ_F is a tree with $r + s$ vertices, $|E(\Gamma_F)| = r + s - 1$. Because Γ_F is bipartite with parts $\{N_i\}$ and $\{C_j\}$, every edge is incident with exactly one N_i-vertex, hence

$$|E(\Gamma_F)| = \sum_{i=1}^{r} \deg_{\Gamma_F}(N_i) \leq 2r.$$

Combining these equalities gives

$$r+s-1 \le 2r, \qquad \text{hence} \qquad s \le r+1 \le \sigma(F)+1.$$

By definition, $v(F) = s - t(F)$, so the first bound implies $v(F) \le \sigma(F) + 1 - t(F)$. If $t(F) \ge 2$ whenever $v(F) > 0$, then this becomes $v(F) \le \sigma(F) - 1$ for every degenerate fiber with $v(F) > 0$, and the same inequality is trivial when $v(F) = 0$. ☐

Corollary 9.138 (A sufficient missing input for the excess route)**.** *Assume the setup of Corollary 9.13, and assume that every degenerate fiber F with $v(F) > 0$ satisfies $t(F) \ge 2$. Then*

$$\sum_F v(F) \le \sum_F (\sigma(F) - 1) = \Sigma = 3.$$

Consequently, in this situation one has $\#\operatorname{Sing}(X) \le 7$ throughout the height-4 maximal-width corridor.

Proof. By Proposition 9.137, the hypothesis implies $v(F) \le \sigma(F) - 1$ for every degenerate fiber F. Summing over all degenerate fibers and using Corollary 9.13 gives

$$\sum_F v(F) \le \sum_F (\sigma(F) - 1) = \Sigma = 3.$$

Let c_{hor} be the number of connected components of D which contain at least one horizontal component, and let c_{vert} be the number of connected components of D which are purely vertical. Then

$$\#\pi_0(D) = c_{\mathrm{hor}} + c_{\mathrm{vert}}.$$

Since $D_{\mathrm{hor}} = H_0 + H_1 + H_2 + H_3$, one has $c_{\mathrm{hor}} \le 4$. Every purely vertical connected component of D is contained in a unique degenerate fiber of p, so

$$c_{\mathrm{vert}} = \sum_F v(F) \le 3.$$

Therefore $\#\pi_0(D) \le 4+3 = 7$. Finally, the connected components of the exceptional divisor are exactly the exceptional divisors over the singular points of X, so $\#\operatorname{Sing}(X) = \#\pi_0(D) \le 7$. ☐

Remark 9.139 (Why the leaf argument does not close the count)*.* The tempting stronger statement $v(F) \le \sigma(F) - 1$ does *not* follow from the sole fact that a degenerate fiber contains a (-1)-curve. The obstruction is that a leaf of the dual graph of a degenerate fiber need not itself be a (-1)-curve. A standard example is the fiber of type $[2, 1, 2]$, obtained by blowing up a smooth fiber once and then blowing up the intersection of the two resulting (-1)-curves. Its two leaves are (-2)-curves, while the unique (-1)-curve lies in the middle. Therefore a purely vertical connected component of D_F can be attached to the rest of the fiber by a single edge without contradicting minimality of the resolution.

Remark 9.140 (Why this subsection is not a global seven-point theorem)*.* Every statement in this subsection concerns a *fixed* witnessing $\mathbf{P}^1$-fibration of height 4 and maximal width. By definition, the inequality $\mathrm{ht}(X) \geq 4$ says only that the minimal possible height of a witnessing fibration is at least 4; it does not itself produce a witnessing fibration of height exactly 4. Likewise, maximal width is a separate existence problem. Thus the excess package above is a corridor analysis, not a global selection theorem for arbitrary residual surfaces.

Remark 9.141 (Status of the endgame). The clean-section Hirzebruch model remains useful for organizing the residual geometry, but by itself it is still only a corridor analysis on a fixed witnessing fibration. The later sections convert this corridor analysis into the global theorem by proving the remaining height-selection statement and thereby removing the need for any additional boxed Route C inputs.

10. A two-part endpoint formulation for the excess route

The previous subsection isolates two genuinely global issues behind the excess endgame: one must choose a witnessing $\mathbf{P}^1$-fibration in the height-4, maximal-width corridor, and one must rule out purely vertical exceptional clusters which touch the horizontal divisor only once. Historically, these two tasks were packaged into a single endpoint statement.

Remark 10.1 (Two-part endpoint formulation for the excess route). Let X be a characteristic-3 klt del Pezzo surface of Picard rank 1 such that $\mathrm{ht}(X) \geq 4$ and X has no descendant with elliptic boundary. Let $\pi\colon \widetilde{X} \to X$ be the minimal resolution and let $D = \mathrm{Exc}(\pi)_{\mathrm{red}}$. Then there exists a witnessing $\mathbf{P}^1$-fibration $p\colon \widetilde{X} \to \mathbf{P}^1$ such that:

(1) p has height 4 and maximal width, so $D_{\mathrm{hor}} = H_0 + H_1 + H_2 + H_3$ is the union of four horizontal 1-sections;

(2) for every degenerate fiber F, if the vertical divisor

$$D_F := \sum_{E \subset D,\ E \subset F} E$$

has a connected component disjoint from D_{hor}, equivalently if $v(F) > 0$, then at least two connected components of D_F meet D_{hor}, i.e. $t(F) \geq 2$.

Corollary 10.2 (This endpoint formulation implies the seven-point theorem). *Assume the two-part endpoint formulation of Remark 10.1. Then every characteristic-*3 *klt del Pezzo surface of Picard rank* 1 *satisfies* $\#\,\mathrm{Sing}(X) \leq 7$.

Proof. Suppose for contradiction that there exists a characteristic-3 klt del Pezzo surface X of Picard rank 1 with $\#\,\mathrm{Sing}(X) \geq 8$. By Theorem 9.1, the surface X already lies in the residual height-≥ 4, no-descendant sector. Therefore the endpoint formulation of Remark 10.1 applies to X, and its minimal resolution admits a witnessing $\mathbf{P}^1$-fibration of height 4 and maximal width satisfying the two-touch condition $v(F) > 0 \Longrightarrow t(F) \geq 2$ for every degenerate fiber F.

Now Corollary 9.13 applies to this fibration and gives the exact excess identity

$$\Sigma = \sum_F (\sigma(F) - 1) = 3.$$

The two-touch condition is precisely the hypothesis of Corollary 9.138. Hence $\#\,\mathrm{Sing}(X) \le 7$, contradicting the assumption. □

Remark 10.3 (Why this was the right endpoint formulation)*.* Remark 10.1 bypasses the earlier boxed Route C entry and frontier-completeness inputs. It does not ask for a surviving contact-2 bridge, a frontier classification beyond what had already been proved, or a surviving scalar packet / ramification fork to which the contact-defect sieve applies. It asks only for a global choice of witnessing fibration in the exact corridor where the excess count is sharp, together with the fiberwise two-touch property that converts Proposition 9.137 into the numerical bound

$$\sum_F v(F) \le \Sigma = 3.$$

This was the cleanest two-part formulation of the last missing step before the final closure.

11. Reduction of the endpoint formulation

The endpoint formulation of Remark 10.1 has two logically separate parts: a *height-selection* problem (producing a witnessing fibration of height 4 and maximal width) and the *two-touch* problem inside that maximal-width corridor. The next results record the reductions that isolate the height-selection side before the final closure in Section 12.

Corollary 11.1 (First sharp reduction of the height-selection problem)**.** *Let X lie in the residual sector of Theorem 9.1. To prove the height-selection statement* $\mathrm{ht}(X) \ge 4$*, no elliptic descendant* $\implies$ *existence of a witnessing fibration of height* 4 *and maximal width, it is enough to rule out the following two global branches.*

(i) *the* multi-special-horizontal branch*: some witnessing fibration carries several special horizontal components, already reduced on the section-only side to packet / isolated-collision geometry by Remarks 9.130 and 9.131;*

(ii) *the* special-multisection branch*: some witnessing fibration carries a special horizontal multisection of degree at least* 2*. On the purely inseparable side, Corollary 11.8 yields a surviving primitive ramification fork, while Corollary 11.9 and Corollary 11.14 force the ordinary singleton* $[2]$*-root branch into the cubic/two-fiber corridor. On the separable one-section side, Proposition 11.16 and Corollary 11.73 show that no separate low-degree mixed local corridor remains.*

At this stage of the argument, the unresolved part was therefore concentrated in the global survival / realization of several special horizontals or of a special multisection.

Proof. If a residual witness has height $h \ge 5$, then Corollary 11.5 reduces it to either several special horizontal components or a special multisection.

Assume next that $h = 4$ and let $p\colon \widetilde{X} \to \mathbf{P}^1$ be a witnessing fibration of height 4. If p has maximal width, then Theorem 11.110 already removes this case. Assume therefore that p is not of maximal width. By Corollary 11.6, some horizontal component is special.

If every horizontal component is a multisection, then one is immediately on the special-multisection branch. If the degree partition is $3+1$ or $2+1+1$, then Proposition 11.16 shows that the multisection image on the ruled model is singular, so this case lies in the low-degree mixed corridor. Corollary 11.73 then shows that this corridor contributes no separate local branch beyond the several-special-horizontal and special-multisection alternatives already listed. □

Definition 11.2 (Special horizontal components for arbitrary witnessing fibrations)**.** Let $p\colon \widetilde{X} \to \mathbf{P}^1$ be a witnessing $\mathbf{P}^1$-fibration on the minimal resolution of a singular klt del Pezzo surface, and let $H \subset D_{\mathrm{hor}}$ be a horizontal component.

We call H *ordinary* if the connected component of D containing H is a chain and, after applying the 2-tail reduction of Definition 9.92 to the touched pair relative to H, the reduced touched core is neither a touched tip of weight 2 nor the singleton chain [3]. Otherwise we call H *special.*

Lemma 11.3 (Ordinary horizontal components contribute at least one half per horizontal degree)**.** *Assume the setup of Definition 11.2. If $H \subset D_{\mathrm{hor}}$ is ordinary, then*

$$\mu_X(H) \le \frac{1}{2} \qquad \textit{and hence} \qquad \mathrm{cf}_X(H) = 1 - \mu_X(H) \ge \frac{1}{2}.$$

Proof. Because H is ordinary, the connected component of D containing H is a chain $A = [A_1, \ldots, A_r]$, and $H = A_i$ for some index i. Let $(A^\sharp, H^\sharp)$ be the corresponding 2-tail reduction.

If $H^\sharp$ is an interior vertex of $A^\sharp$, then Proposition 9.94 gives

$$\mu(A^\sharp, H^\sharp) \le \frac{1}{2}.$$

If $H^\sharp$ is a tip, then ordinariness means precisely that the reduced touched core is neither a touched 2-tip nor the singleton chain [3]. Hence Proposition 9.96 again gives

$$\mu(A^\sharp, H^\sharp) \le \frac{1}{2}.$$

By Lemma 9.93, removing terminal 2-tails away from the touched component can only increase the local weight, so

$$\mu(A, H) \le \mu(A^\sharp, H^\sharp) \le \frac{1}{2}.$$

Finally, Lemma 9.106 identifies $\mu_X(H) = \mu(A, H)$, which proves the claim. □

Proposition 11.4 (Global special-horizontal multiplicity bound)**.** *Let $p\colon \widetilde{X} \to \mathbf{P}^1$ be a witnessing $\mathbf{P}^1$-fibration of height $h := F \cdot D$, where F is a general fiber. Let*

$$S := \sum_{H \text{ special}} (F \cdot H)$$

be the total horizontal multiplicity carried by special horizontal components in the sense of Definition 11.2. Then $S \ge h - 3$.

Proof. By Lemma 9.108,

$$\sum_{H\subset D_{\mathrm{hor}}} \mathrm{cf}_X(H)\,(F\cdot H) < 2.$$

Split the horizontal components into special and ordinary ones. For special components we only use the trivial bound $\mathrm{cf}_X(H) \geq 0$. For ordinary components, Lemma 11.3 gives

$$\mathrm{cf}_X(H) \geq \frac{1}{2}.$$

Therefore

$$2 > \sum_{H \text{ special}} \mathrm{cf}_X(H)\,(F\cdot H) + \sum_{H \text{ ordinary}} \mathrm{cf}_X(H)\,(F\cdot H) \geq 0 + \frac{1}{2}(h-S).$$

Hence $S > h-4$. Since S and h are integers, this implies $S \geq h-3$. □

Corollary 11.5 (Residual high-height dichotomy for the endpoint formulation)**.** *Let X lie in the residual sector of Theorem 9.1, and let $p\colon \widetilde{X} \to \mathbf{P}^1$ be a witnessing $\mathbf{P}^1$-fibration of height $h \geq 5$. Then at least one of the following holds:*

- (i) *there are at least two distinct special horizontal components;*
- (ii) *there exists a special horizontal multisection of degree at least* 2*.*

In particular, the high-height part of the endpoint formulation of Remark 10.1 is already reduced to a special-horizontal problem.

Proof. By Proposition 11.4, the total special horizontal multiplicity satisfies $S \geq h-3 \geq 2$. If all special horizontal components were 1-sections and there were at most one of them, then one would have $S \leq 1$, a contradiction. Thus either there are at least two distinct special horizontal components, or some special horizontal component has degree at least 2 over the base. □

Corollary 11.6 (Height-4 non-maximal-width witnesses already contain a special horizontal component)**.** *Assume that* $\mathrm{ht}(X) = 4$ *and let $p\colon \widetilde{X} \to \mathbf{P}^1$ be a witnessing $\mathbf{P}^1$-fibration of height* 4*. If p is not of maximal width, then at least one horizontal component is special in the sense of Definition 11.2.*

*In particular, the only height-*4 *non-maximal-width partitions in which the special component need not itself be a multisection are*

$$3+1 \qquad \textit{and} \qquad 2+1+1.$$

Proof. If p is not of maximal width, then some horizontal component has degree greater than 1, but in every case the height is still $h=4$. Applying Proposition 11.4 gives $S \geq 1$, so at least one horizontal component is special.

If the degree partition is 4 or $2+2$, then every horizontal component is a multisection, so any special horizontal component is automatically a special multisection. The only remaining non-maximal-width partitions are therefore

$$3+1 \qquad \text{and} \qquad 2+1+1,$$

in which the special component could in principle be the section side. □

The multisection branch can also be pushed further on the purely inseparable side.

Corollary 11.7 (Ramification forks survive away from finitely many bad points)**.** *Let* $p\colon \widetilde{X} \to \mathbf{P}^1$ *be a* $\mathbf{P}^1$*-fibration on a smooth rational surface, let* $H \subset D_{\mathrm{hor}}$ *be a horizontal multisection, and let* $\nu\colon \widetilde{X} \to Y$ *be a birational morphism onto a smooth relatively minimal ruled surface, chosen so that* ν *is an isomorphism at the generic point of* H*. Put* $S := \nu(H)$*. Fix a factorization of* ν^{-1} *into finitely many blowups. Then there exists a finite subset* $B_\nu \subset S$ *with the following property: if* $q \in S \setminus B_\nu$ *is a ramification point of local degree at least* 2 *for* $S \to \mathbf{P}^1$*, then the primitive ramification fork of Proposition 9.128 survives on* $\widetilde{X}$ *as a vertical curve*

$$L \not\subset D, \qquad L^2 = -1.$$

Proof. For each ramification point $q \in S$ of local degree at least 2, let $a(q)$ be that local degree. Proposition 9.128 identifies the forced initial order-$a(q)$ tangency-resolution subcluster over q and its last exceptional curve. Declare q to lie in B_ν if, after that forced tangency-resolution stage, some later blowup center in the chosen factorization lies on the then- current transform of the last exceptional curve.

Because the factorization uses only finitely many blowups, only finitely many points q can be bad in this sense. Hence B_ν is finite. For $q \notin B_\nu$, all the hypotheses of Proposition 9.128 are satisfied by construction, so the primitive ramification fork survives on $\widetilde{X}$. □

Corollary 11.8 (Purely inseparable multisections automatically yield surviving primitive ramification forks)**.** *Keep the setup of Corollary 11.7. Assume in addition that* $S \to \mathbf{P}^1$ *is purely inseparable of degree greater than* 1*. Then there exists a point* $q \in S$ *and a vertical curve*

$$L \subset \widetilde{X}, \qquad L \not\subset D,$$

such that the primitive ramification fork over q *survives on* $\widetilde{X}$*.*

Proof. For a purely inseparable map of degree greater than 1, every point of $S \cong \mathbf{P}^1$ has local degree at least 2. Since the bad set B_ν from Corollary 11.7 is finite, one can choose $q \in S \setminus B_\nu$. Applying that corollary to q gives the required surviving primitive ramification fork. □

Corollary 11.9 (Ordinary surviving ramification forks are cubic in characteristic 3)**.** *Keep the setup of Proposition 9.128, and let* a *be the local ramification degree, so that* $A^2 = -a$*. Assume that the horizontal multisection* H *is ordinary in the sense of Definition 11.2. Then* $a \le 3$*. If moreover* $\operatorname{char} k = 3$ *and the map* $H \to \mathbf{P}^1$ *is purely inseparable of degree greater than* 1*, then* $a = 3$*.*

If in addition the connected component of D containing H is the one-component chain $[b]$, then

$$\frac{1}{a} + \frac{1}{b} > \frac{1}{2}.$$

Proof. By Proposition 9.128, the surviving curve L meets $H,\ A,\ B_1$ transversely once each. Proposition 9.133 therefore gives

$$\mathrm{cf}_X(H) + \mathrm{cf}_X(A) + \mathrm{cf}_X(B_1) < 1.$$

Since H is ordinary, Lemma 11.3 yields

$$\mathrm{cf}_X(H) \geq \frac{1}{2}.$$

Hence

$$\mathrm{cf}_X(A) < \frac{1}{2}.$$

By Lemma 9.132,

$$\mathrm{cf}_X(A) \geq 1 - \frac{2}{a}.$$

Therefore

$$1 - \frac{2}{a} < \frac{1}{2},$$

so $a < 4$. Because $a \geq 2$ is an integer, it follows that $a \leq 3$.

If the map $H \to \mathbf{P}^1$ is purely inseparable in characteristic 3, then its local degree is a positive power of 3. Since

$$a > 1 \qquad \text{and} \qquad a \leq 3,$$

one gets $a = 3$.

Finally, when the connected component containing H is the singleton chain $[b]$, the last inequality is exactly the reformulation of Corollary 9.134. □

Proposition 11.10 (Smooth rational cubic multisections on Hirzebruch surfaces)**.** *Let $Y = \mathbb{F}_n$ with negative section S_0 and fiber class f. If $M \equiv 3S_0 + mf$ is a smooth rational multisection not containing S_0, then*

$$m = \frac{3n+2}{2}, \qquad M^2 = 6.$$

In particular, n is even.

Proof. Using

$$S_0^2 = -n, \qquad S_0 \cdot f = 1, \qquad f^2 = 0, \qquad K_{\mathbb{F}_n} = -2S_0 - (n+2)f,$$

one computes

$$M^2 = -9n + 6m, \qquad M \cdot K_{\mathbb{F}_n} = 3(n-2) - 2m.$$

Since $M \cong \mathbf{P}^1$, adjunction gives

$$\frac{M^2 + M \cdot K_{\mathbb{F}_n}}{2} + 1 = 0.$$

Substituting the above formulas yields $-9n + 6m + 3(n-2) - 2m = -2$, hence

$$4m = 6n + 4, \qquad m = \frac{3n+2}{2}.$$

Then

$$M^2 = -9n + 6 \cdot \frac{3n+2}{2} = 6.$$

Thus $3n + 2$ is even, so n is even. □

Remark 11.11 (The cubic backbone and a supplementary search reduction). Corollary 11.9 and Proposition 11.10 show that the purely inseparable ordinary-root branch is governed by a rigid ruled-model backbone: after passing to a relatively minimal ruled surface, the horizontal image is a smooth rational cubic multisection of self-intersection 6. A supplementary finite ordinary-fiber search on that backbone shows that over an ordinary fiber no valid interesting local module of Picard defect 0 appears through eight additional blowups, and defects add across disjoint ordinary fibers.

Proposition 11.12 (Blowup budget in the clean threaded [2]-root model). *Let $Y \to \mathbf{P}^1$ be a relatively minimal ruled surface, and let $S \subset Y$ be a smooth rational multisection of degree $a \geq 2$. Then $S^2 = 2a$. Consequently, if a birational morphism $\nu^{-1} \colon Y \dashrightarrow \widetilde{X}$ is obtained by blowing up finitely many points on successive strict transforms of S, and the final strict transform $H \subset \widetilde{X}$ satisfies $H^2 = -2$, then exactly $2a + 2$ of the blowup centers lie on successive strict transforms of S.*

Assume now that

$$\operatorname{char} k = 3, \qquad a = 3^e \geq 3,$$

and that every blowup centered on a transform of S is either:

(i) *one of the a forced centers resolving a primitive order-a tangency over one of r distinct fibers of the ruling; or*

(ii) *a* thread blowup *at the unique point where the current strict transform of H meets the current unique nonexceptional (-1)-curve over one of those same r fibers.*

Then $r \leq 2$. Moreover every extra thread blowup merely lengthens the detached chain over that fiber by one terminal (-2)-curve and does not create any new connected component of D. In particular, this clean threaded branch contributes at most three connected components to D : the root component containing H, and at most two detached ramification chains.

Proof. Write

$$Y = \mathbb{F}_n, \qquad S \sim aS_0 + bf.$$

Since $p_a(S) = 0$, adjunction gives $(a-1)(2b - na - 2) = 0$. Because $a \geq 2$, one gets $2b = na + 2$.

Therefore $S^2 = -na^2 + 2ab = 2a$. Each blowup centered on the current strict transform of S lowers its self-intersection by 1. So changing $2a$ to -2 requires exactly $2a + 2$ such centers.

Now consider one primitive order-a tangency $t = z^a$. Its full forced tangency resolution consists of a successive blowups at the unique current point where the strict transforms of the multisection and the fiber meet. The resulting local configuration is the primitive fork

$$H - L - F, \qquad L - E_{a-1} - \cdots - E_1,$$

where

$$L^2 = -1, \qquad F^2 = -a, \qquad E_i^2 = -2 \quad (1 \le i \le a-1).$$

Thus each primitive ramification fiber consumes exactly a of the $2a + 2$ available centers on successive transforms of S. Hence $ra \le 2a + 2$, which implies $r \le 2$ for every $a \ge 3$.

Finally, a thread blowup is centered at the unique point where the current transform of H meets the current nonexceptional (-1)-curve over one of those fibers. The old (-1)-curve becomes a new (-2)-curve in D, while a new (-1)-curve appears at the H -end of the same local branch. So the detached chain over that fiber simply gains one more terminal (-2)-curve and no new connected component of D is created. Since each primitive ramification fiber contributes one detached chain, the total number of connected components of D coming from this branch is at most $1 + r \le 3$. ☐

Remark 11.13 (Meaning for the remaining special singleton [2] branch). Proposition 11.12 shows that the clean threaded purely inseparable [2] -root model is too small to serve as a stand-alone source of a Picard-rank-one surface with at least eight singular points. Any remaining candidate in that branch must therefore use non-threaded service patterns or interact with other residual geometry.

Corollary 11.14 (Non-clean purely inseparable service patterns still live on at most two fibers)**.** *Keep the setup of the first paragraph of Proposition 11.12. Assume only that*

$$\operatorname{char} k = 3, \qquad a = 3^e \ge 3,$$

and that the final strict transform $H \subset \widetilde{X}$ satisfies $H^2 = -2$. Do not *assume the clean threaded hypotheses* (i) *and* (ii) *of that proposition. Let $\mathcal{T}$ be the set of fibers of $Y \to \mathbf{P}^1$ whose total transforms contain at least one blowup center lying on a successive strict transform of S. Then $\#\mathcal{T} \le 2$. In particular, even a non-clean purely inseparable ordinary singleton* [2] *-root service pattern can involve the multisection side over at most two fibers of the ruled model.*

Proof. The first part of Proposition 11.12 is independent of the clean threaded assumptions and shows that exactly $2a + 2$ of the blowup centers lie on successive strict transforms of S. Now fix a fiber in $\mathcal{T}$. Because the map $S \to \mathbf{P}^1$ is purely inseparable of degree a, every point of S has local degree a. Therefore, once a blowup center lies on the current transform of S over that fiber, the first a such centers over that fiber are the forced order-a tangency-resolution centers. Distinct fibers of the ruling are disjoint on the ruled surface, so the centers lying on transforms of S over different fibers are disjoint groups. Hence $a\,\#\mathcal{T} \leq 2a+2$. Since $a \geq 3$, this implies $\#\mathcal{T} \leq 2$. □

Remark 11.15 (What was genuinely left at this stage)*.* At this stage, the proved reductions above sharpened the endpoint formulation of Remark 10.1 in four ways.

(a) In the high-height regime $h \geq 5$, one is no longer facing an arbitrary witnessing fibration. Corollary 11.5 reduces the problem to either several special horizontal components or a special multisection.

(b) On the multisection side, the purely inseparable branch is now split into explicit subcorridors. Corollary 11.8 produces a surviving primitive ramification fork; Corollary 11.9 and Proposition 11.10 show that every ordinary surviving fork is cubic in characteristic 3 and comes from a rigid smooth rational cubic backbone of self-intersection 6; and Proposition 11.12 shows that the clean threaded special singleton [2] branch has at most two primitive ramification fibers and at most three connected components of D coming from that branch.

(c) A supplementary finite ordinary-fiber search on the cubic backbone shows that over an ordinary fiber no valid interesting local module of Picard defect 0 appears through eight additional blowups, and defects add across disjoint ordinary fibers. So coupled ordinary-fiber constructions in that searched range cannot preserve Picard rank 1.

(d) In height 4, a non-maximal-width witness already contains a special horizontal component by Corollary 11.6. Thus the height-selection part of the conjecture is reduced to understanding how special horizontal components survive in the residual sector, while the two-touch part is concentrated entirely in the height-4, maximal-width corridor of Section 10.

So at this stage the remaining proof was no longer a single amorphous global selection problem. It was concentrated in the following explicit residual branches:

(i) separable special multisections of degree at least 2, together with the residual global survival / exclusion problem for the non-clean purely inseparable service patterns not covered by Proposition 11.12 and the supplementary cubic-backbone search described above;

(ii) the section-only high-height packet / isolated-collision branch already reduced in Remarks 9.130 and 9.131;

(iii) the height-4, maximal-width single-attached-fiber problem $v(F) > 0 \Longrightarrow t(F) \geq 2$.

This was the sharpest honest reduction available at this stage of the argument.

11.1. The one-section separable branch: every higher-degree image is singular

A further unconditional reduction becomes available as soon as one horizontal 1-section is present. After contracting the vertical curves disjoint from that section, every other horizontal component of degree greater than 1 acquires positive arithmetic genus on the ruled model. Since its normalization is still $\mathbf{P}^1$, such an image is automatically singular. This applies in particular to the height-4 partitions

$$3+1 \qquad \text{and} \qquad 2+1+1$$

isolated in Corollary 11.6.

Proposition 11.16 (One-section ruled model: class and arithmetic genus of every other horizontal image)**.** *Let $p\colon \widetilde{X} \to \mathbf{P}^1$ be a witnessing $\mathbf{P}^1$-fibration, and assume that $H_0 \subset D_{\mathrm{hor}}$ is a horizontal 1-section. Let $\nu\colon \widetilde{X} \to \mathbb{F}_n$ be the contraction of all vertical curves disjoint from H_0, and put $S_0 := \nu(H_0)$. Then $S_0^2 = -n, \qquad n = -H_0^2 \geq 2$, so S_0 is the negative section of $\mathbb{F}_n$.*

Let $H \subset D_{\mathrm{hor}} - H_0$ be any other horizontal component, and put

$$d := F \cdot H, \qquad C := \nu(H), \qquad s := H \cdot H_0.$$

Then

$$s \in \{0,1\}, \qquad C \equiv dS_0 + (nd+s)f,$$

where f denotes a fiber of the ruling on $\mathbb{F}_n$, and

$$p_a(C) = \frac{(d-1)(nd+2s-2)}{2}.$$

Proof. Because ν contracts only vertical curves, it is a birational morphism over the same base $\mathbf{P}^1$, and its target is a relatively minimal ruled surface. Since the contracted curves are disjoint from H_0, the morphism ν is an isomorphism in a neighborhood of H_0. Hence $S_0^2 = H_0^2 < 0$. Therefore S_0 is the negative section of some Hirzebruch surface $\mathbb{F}_n$, and $n = -H_0^2 \geq 2$ because $H_0 \subset D$ is not a (-1)-curve.

Now let $H \subset D_{\mathrm{hor}} - H_0$ be another horizontal component. Since ν is over the base, $C \cdot f = d$. Write $C \equiv dS_0 + bf$ for some integer b. Because ν is an isomorphism near H_0, one has $C \cdot S_0 = H \cdot H_0 = s$. Distinct irreducible components of the reduced exceptional divisor meet at most once, so $s \in \{0,1\}$. On the other hand, $(dS_0 + bf) \cdot S_0 = -nd + b$, hence $b = nd + s$. This proves the class formula $C \equiv dS_0 + (nd+s)f$.

Finally, on $\mathbb{F}_n$ one has $K_{\mathbb{F}_n} = -2S_0 - (n+2)f$. Therefore

$$C^2 = (dS_0 + (nd+s)f)^2 = -nd^2 + 2d(nd+s) = nd^2 + 2ds,$$

and

$$K_{\mathbb{F}_n} \cdot C = (-2S_0 - (n+2)f) \cdot (dS_0 + (nd+s)f) = -dn - 2d - 2s.$$

Adjunction for the arithmetic genus gives

$$p_a(C) = 1 + \frac{C^2 + K_{\mathbb{F}_n} \cdot C}{2} = 1 + \frac{nd^2 + 2ds - dn - 2d - 2s}{2} = \frac{(d-1)(nd+2s-2)}{2}.$$

$\square$

Corollary 11.17 (With one section fixed, every genuine multisection image is singular)**.** *Keep the setup of Proposition 11.16. If $d \geq 2$, then the image $C = \nu(H)$ is singular.*

Equivalently, once a horizontal section is fixed, every smooth horizontal image on the ruled model is again a section.

Proof. By Proposition 11.16,

$$p_a(C) = \frac{(d-1)(nd + 2s - 2)}{2}.$$

Since

$$n \geq 2, \qquad d \geq 2, \qquad s \geq 0,$$

one has $nd + 2s - 2 \geq 2d - 2 > 0$, and therefore $p_a(C) > 0$. On the other hand, $H \cong \mathbf{P}^1$ and $\nu|_H \colon H \to C$ is birational, so the normalization of C is rational. Hence C cannot be smooth. □

Lemma 11.18 (A singular horizontal point is either a branch-collision point or a ramification point)**.** *Let $q \in C \subset Y$ be a singular point of an irreducible horizontal curve on a smooth ruled surface $Y \to \mathbf{P}^1$, and let $\eta \colon \overline{C} \to C$ be the normalization. Then exactly one of the following holds.*

- (i) *the fiber $\eta^{-1}(q)$ contains at least two points, so q is a collision point of at least two local branches of C;*
- (ii) *$\eta^{-1}(q) = \{x\}$ and the map $\overline{C} \to \mathbf{P}^1$ has local degree at least 2 at x.*

Proof. If $\eta^{-1}(q)$ contains at least two points, then q has at least two branches by definition.

Assume now that $\eta^{-1}(q) = \{x\}$. If the induced map $\overline{C} \to \mathbf{P}^1$ had local degree 1 at x, then in local coordinates (t, z) on the ruled surface adapted to the ruling and a local parameter u on $\overline{C}$ at x, one could write

$$t = u, \qquad z = \varphi(u)$$

for some convergent power series φ. The image would then be the graph $z = \varphi(t)$, which is smooth at q, contradicting the assumption that q is singular. Therefore the local degree at x must be at least 2. □

Corollary 11.19 (The one-section height-4 partitions are singular-image cases)**.** *Assume that* $\operatorname{ht}(X) = 4$ *and that a witnessing fibration has degree partition either*

$$3 + 1 \qquad \textit{or} \qquad 2 + 1 + 1.$$

Choose one of the horizontal sections $H_0 \subset D_{\mathrm{hor}}$ and form $\nu \colon \widetilde{X} \to \mathbb{F}_n$ as in Proposition 11.16. Let $H \subset D_{\mathrm{hor}} - H_0$ be the unique multisection, and put

$$C := \nu(H), \qquad d := F \cdot H \in \{2, 3\}, \qquad s := H \cdot H_0 \in \{0, 1\}.$$

Then C is a singular rational curve. More precisely:

(i) *if* $d = 2$*, then*

$$C \equiv 2S_0 + (2n+s)f \qquad \textit{and} \qquad \delta(C) = p_a(C) = n + s - 1 \geq 1;$$

(ii) *if* $d = 3$*, then*

$$C \equiv 3S_0 + (3n+s)f \qquad \textit{and} \qquad \delta(C) = p_a(C) = 3n + 2s - 2 \geq 4.$$

In particular, every singular point of C *is locally either a collision of horizontal branches or a unibranch ramification point of local degree at least* 2*.*

Proof. The singularity statement is Corollary 11.17. Since $H \cong \mathbf{P}^1$ and $\nu|_H$ is birational, the normalization of C is rational, so $\delta(C) = p_a(C)$. Applying Proposition 11.16 with

$$d = 2 \qquad \text{or} \qquad d = 3$$

gives the displayed formulas.

The final local dichotomy is exactly Lemma 11.18. □

Remark 11.20 (What this does to the separable special-multisection branch)*.* The one-section part of the separable special-multisection branch is no longer a smooth-multisection problem.

Indeed, once one section is fixed, every horizontal component of degree greater than 1 is forced by Corollary 11.17 to have a singular ruled-model image. So in the partitions

$$3 + 1 \qquad \text{and} \qquad 2 + 1 + 1,$$

the residual geometry is concentrated entirely in singular rational 3-sections and 2-sections on $\mathbb{F}_n, \qquad n \geq 2$. By Lemma 11.18, every singular point of such a curve is locally of one of two broad types:

(a) a collision of at least two horizontal branches, with the same forced initial blowup pattern as the local two-branch collision model of Proposition 9.119;

(b) a unibranch point of local degree at least 2.

The next subsection pushes the degree-two case further by a ruled-surface obstruction and a marked plane conic/cubic reduction. The following subsection shows that in the low-degree partitions

$$3 + 1 \qquad \text{and} \qquad 2 + 1 + 1$$

the remaining unibranch side is still a finite low-order problem: after a finite preliminary Euclidean singular phase, the only remaining local steps are corner blowups and order-2 or order-3 tangency blocks.

So the one-section separable branch is reduced to a genuinely global survival / root-contractibility problem for these explicit singular-image models. In particular, item (i) of Remark 11.15 can now be sharpened: in the one-section partitions, only *singular-image* separable multisections remain.

11.2. The bisection case: a ruled-surface obstruction and a plane reduction

The degree-two part of the one-section branch can be pushed further. The first observation is a general ruled-surface fact: an irreducible multisection cannot avoid a section of nonnegative self-intersection.

Lemma 11.21 (Generalized Hirzebruch obstruction)**.** *Let $C \subset \mathbb{F}_n$ be a section with $C^2 \geq 0$, and let $B \subset \mathbb{F}_n$ be an irreducible d-section with $d \geq 2$. Then $B \cdot C > 0$. In particular, $B \cap C \neq \varnothing$.*

Proof. Write

$$C \equiv S_0 + af, \qquad B \equiv dS_0 + bf,$$

where

$$S_0^2 = -n, \qquad S_0 \cdot f = 1, \qquad f^2 = 0.$$

Since $C^2 = -n + 2a \geq 0$, one has

$$a > 0 \qquad \text{if } n > 0,$$

and

$$a \geq 0 \qquad \text{if } n = 0.$$

Because B is irreducible and $d \geq 2$, it is not equal to the negative section. Hence $B \cdot S_0 = b - dn \geq 0$, so $b \geq dn$. Therefore

$$B \cdot C = (dS_0 + bf) \cdot (S_0 + af) = d(-n + a) + b \geq da.$$

If $n > 0$, then $a > 0$, so $B \cdot C > 0$. If $n = 0$ and $a > 0$, the same inequality gives $B \cdot C > 0$.

It remains to consider the case $n = 0, \quad a = 0$. Then $\mathbb{F}_0 \cong \mathbf{P}^1 \times \mathbf{P}^1$ and C is a fiber of one ruling. In this case $B \cdot C = b$. If $b = 0$, then $B \equiv dS_0$, but every effective divisor of class $(d, 0)$ on $\mathbf{P}^1 \times \mathbf{P}^1$ is a sum of d fibers and is reducible for $d \geq 2$, contradicting the irreducibility of B. Hence $b > 0$, so again $B \cdot C > 0$. □

Corollary 11.22 (In the $2+1+1$ partition, the bisection meets the second section on the ruled model)**.** *Assume the setup of Corollary 11.19, and assume $d = 2$. Let $H_1 \subset D_{\text{hor}} - H_0 - H$ be the remaining horizontal section, and put $S_1 := \nu(H_1)$. Then $C \cdot S_1 > 0$. In particular, the ruled-model bisection is never disjoint from the second section.*

Proof. By Proposition 11.16, the curve S_1 is a section on $\mathbb{F}_n$, and $S_1^2 = n + 2(H_0 \cdot H_1) \geq n \geq 2$. Therefore S_1 is a section of nonnegative self-intersection. Since C is an irreducible 2-section, Lemma 11.21 gives $C \cdot S_1 > 0$. □

Lemma 11.23 (Local normal forms for singular points of a bisection)**.** *Let $Y \to \mathbf{P}^1$ be a smooth ruled surface, let $C \subset Y$ be an irreducible bisection, and let $p \in C$ be a singular point. Choose* *formal local coordinates* (t, z) *at* p *adapted to the ruling, so that* $t = 0$ *is the local fiber through* p*. Then exactly one of the following holds.*

(i) ***Two-branch collision.*** *The completed local equation of* C *at* p *is* $z(z - t^m) = 0$ *for some integer* $m \geq 1$*. Equivalently,* C *has two smooth branches through* p *whose local intersection multiplicity is* m*.*

(ii) ***Unibranch ramification type.*** *The normalization of* C *at the unique preimage of* p *admits a formal parameter* u *for which*

$$t = u^2, \qquad z = u^{2m+1}$$

for some integer $m \geq 1$*. Equivalently, the completed local equation of* C *at* p *is* $z^2 = t^{2m+1}$*.*

In case (ii)*, the map from the normalization of* C *to the base has ramification index* 2 *at the unique preimage of* p*.*

Proof. Because $C \cdot F_p = 2$ for the fiber F_p through p, the singularity multiplicity satisfies $\mathrm{mult}_p(C) \leq 2$. Since p is singular, one has $\mathrm{mult}_p(C) = 2$.

Assume first that p has two branches. Since the total degree over the base is 2, each branch has local degree 1 over the base. After a formal fiberwise coordinate change, the two branches can therefore be written as

$$z = \varphi_1(t), \qquad z = \varphi_2(t).$$

Replacing z by $z - \varphi_1(t)$ makes the first branch equal to $z = 0$. Then the second branch is $z = \psi(t)$ for some nonzero series $\psi(t)$. Writing $\psi(t) = u(t)t^m$ with $u(0) \neq 0$ and replacing z by $u(t)^{-1}z$ gives the local equation $z(z - t^m) = 0$. This is case (i).

Assume now that p is unibranch. Let u be a formal parameter on the normalization of C at the unique preimage of p. Because the local degree to the base is 2, and $\mathrm{char}\, k = 3 \neq 2$, one may choose the base coordinate so that $t = u^2$. Write $z = \phi(u)$. Subtracting the even part of $\phi(u)$ by a formal fiberwise coordinate change $z \mapsto z - g(t)$ leaves only odd powers of u. Since C is singular, the first nonzero odd term has order at least 3, say $2m + 1$. Thus one can write $z = u^{2m+1}h(u^2)$ with $h(0) \neq 0$. Replacing z by $h(t)^{-1}z$ gives

$$t = u^2, \qquad z = u^{2m+1}.$$

Eliminating u gives the equation $z^2 = t^{2m+1}$. This is case (ii).

In case (ii), the parameterization $t = u^2$ shows directly that the map from the normalization to the base has ramification index 2 at the corresponding point. ☐

Lemma 11.24 (A singular-point elementary transform peels one layer from a bisection singularity)**.** *Keep the setup of Lemma 11.23, and perform the elementary transform at* p*. Then the image of* C *has the following local form.*

(i) *If* C *is of collision type* $z(z - t^m) = 0$*, then the image is smooth if* $m = 1$*, and otherwise* *it is again of collision type* $z(z - t^{m-1}) = 0$.

(ii) *If* C *is of unibranch type* $t = u^2$, $z = u^{2m+1}$, *then the image is smooth if* $m = 1$, *and otherwise it is again of unibranch type* $t = u^2$, $z = u^{2m-1}$.

Proof. In formal coordinates adapted to the ruling, the elementary transform is given by

$$(t, z) \longmapsto (t, w), \qquad w = \frac{z}{t},$$

which is the blowup of p followed by the contraction of the strict transform of the fiber $t = 0$.

For collision type $z(z - t^m) = 0$, the two branches become

$$w = 0 \qquad \text{and} \qquad w = t^{m-1}.$$

If $m = 1$, these are two distinct smooth branches through different points, so the image is smooth. If $m \geq 2$, the image is again a collision of order $m - 1$.

For unibranch type

$$t = u^2, \qquad z = u^{2m+1},$$

one gets

$$w = \frac{z}{t} = u^{2m-1}.$$

If $m = 1$, then $t = u^2$, $\quad w = u$, so the image is smooth. If $m \geq 2$, the image is again of unibranch type with parameterization

$$t = u^2, \qquad w = u^{2m-1}.$$

□

Corollary 11.25 (A bisection has at most two unibranch singular points on every ruled model)**.** *Let* C *be an irreducible rational bisection on a smooth ruled surface. Then* C *has at most two unibranch singular points. More generally, every curve obtained from* C *by a sequence of singular-point elementary transforms has at most two unibranch singular points.*

Proof. The normalization of C is $\mathbf{P}^1$, and the induced map to the base has degree 2. Because $\operatorname{char} k = 3 \neq 2$, this map is separable. By Riemann–Hurwitz, $-2 = 2(-2) + \deg R$, so its ramification divisor has degree 2. Hence there are exactly two ramification points on the normalization.

By Lemma 11.23, a unibranch singular point on any ruled model of the bisection comes from a ramification point of that normalization map. Therefore at most two such points can occur on any model in a singular-point elementary-transform sequence. □

Lemma 11.26 (Elementary transform class formula away from the negative section)**.** *Let*

$$\epsilon_p \colon \mathbb{F}_n \dashrightarrow \mathbb{F}_{n-1}$$

be the elementary transform at a point $p \in \mathbb{F}_n - S_0$*. Let* $C \subset \mathbb{F}_n$ *be an irreducible curve of class* $C \equiv aS_0 + bf$*, and write* $m := \mathrm{mult}_p(C)$*. If* $C' \subset \mathbb{F}_{n-1}$ *is the image of* C*, then* $C' \equiv aS_0' + (b-m)f'$*, where* $S_0'^2 = -(n-1)$ *is the negative section of* $\mathbb{F}_{n-1}$ *and* f' *is a fiber of its ruling.*

Proof. Let

$$\sigma\colon Z := \mathrm{Bl}_p \mathbb{F}_n \to \mathbb{F}_n$$

be the blowup of p, let $E \subset Z$ be the exceptional curve, let F be the fiber of $\mathbb{F}_n$ through p, and let $F' \subset Z$ be the strict transform of F. Then $F'^2 = -1$, so contracting F' gives a morphism $\tau\colon Z \to \mathbb{F}_{n-1}$ realizing the elementary transform.

Because $p \notin S_0$, one has $\tau^* f' = \sigma^* f$ and

$$\tau^* S_0' = \sigma^* S_0 + F' = \sigma^* S_0 + \sigma^* f - E.$$

The strict transform of C on Z is $\widetilde{C} = \sigma^* C - mE$. Since $\widetilde{C} \cdot F' = (aS_0 + bf) \cdot f - m = a - m$, one gets

$$\tau^* C' = \widetilde{C} + (a-m)F'.$$

Expanding and using the displayed formulas for $\tau^* S_0'$ and $\tau^* f'$ gives

$$\tau^* C' = a\,\tau^* S_0' + (b-m)\,\tau^* f'.$$

Hence $C' \equiv aS_0' + (b-m)f'$, as claimed. ☐

Proposition 11.27 (Rational bisections reduce to plane conics or cubics)**.** *Let* $C \subset \mathbb{F}_n$ *be an irreducible rational curve with*

$$C \equiv 2S_0 + (2n+s)f, \qquad s \in \{0,1\}.$$

Then there exists a sequence of $n-1$ *elementary transforms*

$$\mathbb{F}_n \dashrightarrow \mathbb{F}_{n-1} \dashrightarrow \cdots \dashrightarrow \mathbb{F}_1$$

such that each center is a singular point of the current curve and lies off the current negative section. The final image $C_1 \subset \mathbb{F}_1$ *satisfies* $C_1 \equiv 2S_1 + (2+s)f$*. After contracting the negative section* S_1*, the image of* C_1 *is an irreducible rational plane curve of degree* $2+s$*. Equivalently:*

(i) *if* $s = 0$*, the branch reduces birationally to a plane conic;*

(ii) *if* $s = 1$*, the branch reduces birationally to a singular plane cubic.*

Proof. We argue by induction on n. If $n = 1$, there is nothing to prove.

Assume now that $n \geq 2$. By Proposition 11.16, one has $p_a(C) = n + s - 1 > 0$, so C is singular. Let $p \in C$ be a singular point. We claim that $p \notin S_0$. Indeed, if $p \in S_0$, then because S_0 is smooth and p is singular on C, one would have

$$(C \cdot S_0)_p \geq \operatorname{mult}_p(C)\operatorname{mult}_p(S_0) \geq 2,$$

contradicting the total intersection number $C \cdot S_0 = s \leq 1$. So every singular point of C lies off the negative section.

Let F_p be the fiber through p. Since C is a bisection, $C \cdot F_p = 2$. Hence $\operatorname{mult}_p(C) \leq 2$. Because p is singular, one also has $\operatorname{mult}_p(C) \geq 2$, so in fact $\operatorname{mult}_p(C) = 2$. Perform the elementary transform at p. By Lemma 11.26 with $a = 2, \quad m = 2$, the image $C' \subset \mathbb{F}_{n-1}$ has class $C' \equiv 2S_0' + (2(n-1)+s)f'$. It is again irreducible and rational, being birational to C. Applying the induction hypothesis to C' finishes the construction of the sequence.

For the final plane statement, let $\kappa\colon \mathbb{F}_1 \to \mathbf{P}^2$ be the contraction of S_1, and let $H := \kappa^*(\text{line}) = S_1 + f$. Then

$$C_1 \equiv 2S_1 + (2+s)f = (2+s)H - sS_1,$$

so the image $\Gamma := \kappa(C_1)$ has plane degree $\deg \Gamma = 2+s$. It is irreducible and rational because C_1 is. If $s = 0$, then Γ is an irreducible plane conic, hence smooth. If $s = 1$, then Γ is a rational plane cubic and is therefore singular. □

Proposition 11.28 (Marked-point plane models for rational bisections)**.** *Keep the setup of Proposition 11.27, and let $\kappa\colon \mathbb{F}_1 \to \mathbf{P}^2$ be the contraction of the negative section $E \subset \mathbb{F}_1$ to a point $q \in \mathbf{P}^2$. Let $\Gamma := \kappa(C_1)$. Then Γ is an irreducible rational plane curve of degree $2+s$, and the ruling on $\mathbb{F}_1$ restricts to the normalization of the projection of Γ from q. More precisely:*

(i) *if $s = 0$, then $q \notin \Gamma$ and Γ is a smooth plane conic;*

(ii) *if $s = 1$, then $q \in \Gamma_{\mathrm{reg}}$, the curve Γ is a singular plane cubic, and projection from q realizes the original degree-2 map from the normalization of the bisection to the base.*

In case $s = 1$, that projection is separable of degree 2 and has exactly two ramification points.

Proof. The final class computation in the proof of Proposition 11.27 gives $C_1 \equiv 2E + (2+s)f = (2+s)H - sE$, where $H = \kappa^*(\text{line})$. Hence $\deg \Gamma = 2+s$ and the multiplicity of Γ at q is exactly s. If $s = 0$, then $q \notin \Gamma$, and Proposition 11.27 already shows that Γ is a smooth conic. If $s = 1$, then $q \in \Gamma$ with multiplicity 1, so $q \in \Gamma_{\mathrm{reg}}$. In that case Proposition 11.27 shows that Γ is a singular irreducible cubic.

The fibers of the ruling on $\mathbb{F}_1$ are the strict transforms of the lines through q. Therefore the restriction of the ruling to C_1 is exactly the normalization of the projection of Γ from q. Since all elementary transforms in Proposition 11.27 are performed over the same base, this is the same degree-2 map as the original map from the normalization of the bisection to $\mathbf{P}^1$.

When $s = 1$, the source and target are both $\mathbf{P}^1$, and in characteristic 3 a degree-2 morphism is separable. Riemann–Hurwitz therefore gives $-2 = 2(-2) + \deg R$, so $\deg R = 2$. Since every ramification point of a degree-2 separable morphism contributes 1 to $\deg R$, there are exactly two ramification points. □

Corollary 11.29 (The touched bisection subcase is a marked singular plane cubic)**.** *Assume the setup of Corollary 11.19, and assume* $d = 2, \qquad s = 1$. *Then, after successive singular-point elementary transforms, the ruled-model bisection reduces birationally to an irreducible singular plane cubic* $\Gamma \subset \mathbf{P}^2$ *together with a smooth marked point* $q \in \Gamma$. *Projection from* q *realizes the original degree-*2 *map from the normalization of the bisection to the base. In particular, every ruled model obtained along the singular-point elementary-transform sequence has at most two unibranch singular points.*

Proof. Combine Proposition 11.27, Proposition 11.28, and Corollary 11.25. □

Remark 11.30 (Relation with the unexpected-cubic route)*.* Corollary 11.29 places the touched bisection subcase squarely in marked plane-cubic geometry: one has a singular cubic $\Gamma \subset \mathbf{P}^2$ together with a smooth marked point $q \in \Gamma$, and projection from q recovers the original degree-2 map to the base. By Proposition 11.28, this projection has exactly two ramification points, so the residual local freedom is concentrated in collision strings and at most two unibranch ramification strings. This is exactly the range where Kettinger proves that characteristic 3 has no unexpected plane cubics [9]. The marked-cubic model therefore remains a useful global compression of the touched degree-two branch. However, the later Corollary 11.72 shows that for the Route C singularity-count problem one does not need a separate unexpected-cubic closure on the local side: every surviving singularity of the ordinary bisection is either impossible by the contact-defect sieve or already absorbed by the old touched-tip / branching geometry.

Remark 11.31 (What this adds to the one-section separable branch)*.* The degree-two part of the one-section separable branch is no longer an arbitrary ruled-surface singularity problem. In the partition $2 + 1 + 1$, the bisection is forced by Corollary 11.22 to meet the second section on the ruled model; Lemma 11.23 shows that every singular point is an explicit plane double point; Lemma 11.24 shows that singular-point elementary transforms shorten those local strings one step at a time; and Proposition 11.27 reduces the whole branch to a marked plane conic or cubic. In the touched subcase $s = 1$, Corollary 11.29 gives a marked singular plane cubic with exactly two ramification directions for the projection from the marked point. At this stage of the paper, the only one-section separable cases still not reduced to such explicit plane models are the residual conic bookkeeping problem in degree 2 and the singular trisection branch in degree 3. The later Subsections 11.11 and 11.12 show that neither creates a genuinely new local corridor.

11.3. Low-degree unibranch singularities: only corner and order-$2/3$ terminal phases remain

The previous remark reduces the one-section separable branch to singular ruled-model images. On the multibranch side, Proposition 9.119 already gives the primitive local collision model. The unibranch side still needs a local endgame analysis. In the one-section partitions

$$2 + 1 + 1 \qquad \text{and} \qquad 3 + 1,$$

that endgame is much smaller than an arbitrary plane-branch resolution, simply because the local degree over the base is at most 3.

Lemma 11.32 (Order pairs follow the Euclidean algorithm under blowup)**.** *Let*

$$\gamma\colon \operatorname{Spec} k[[u]] \to (\mathbb{A}^2_{x,y}, 0)$$

be a primitive formal branch, and write

$$\operatorname{ord}_u x(\gamma) = r, \qquad \operatorname{ord}_u y(\gamma) = s, \qquad \gcd(r,s) = 1.$$

Blow up the origin and choose the unique affine chart containing the strict transform of γ*. Then the new vanishing-order pair is*

$$(r, s-r) \qquad \textit{if } s > r,$$

and

$$(r-s, s) \qquad \textit{if } r > s.$$

Consequently, repeated blowups at the unique current point of the strict transform realize the Euclidean algorithm on the pair (r,s)*. In particular, if* $r \le 3$*, then after finitely many such blowups one reaches a smooth strict transform with local order pair*

$$(1, a) \qquad \textit{or} \qquad (a, 1)$$

for some integer $1 \le a \le 3$*.*

Proof. Assume first that $s > r$. In the blowup chart

$$x = x_1, \qquad y = x_1 y_1,$$

which is the unique chart containing the strict transform of γ, one has

$$\operatorname{ord}_u x_1 = r, \qquad \operatorname{ord}_u y_1 = s - r.$$

This gives the pair $(r, s-r)$. The case $r > s$ is symmetric, using the chart

$$x = y_1 x_1, \qquad y = y_1.$$

Thus each blowup replaces the current order pair by one subtraction step of the Euclidean algorithm.

Because $\gcd(r,s) = 1$, iterating this process eventually reaches a pair of the form

$$(1, a) \qquad \text{or} \qquad (a, 1)$$

for some $a \ge 1$. At that stage the strict transform is smooth. If initially $r \le 3$, then every remainder in the Euclidean algorithm is at most 3, so $a \le 3$. ☐

Corollary 11.33 (First smooth stage for local degree 2 and 3)**.** *Let $q \in C \subset Y$ be a unibranch singular point of an irreducible horizontal curve on a smooth ruled surface $Y \to \mathbf{P}^1$, and let $\eta\colon \overline{C} \to C$ be the normalization. Assume that the induced map $\overline{C} \to \mathbf{P}^1$ has local degree $e \in \{2, 3\}$ at the unique point of $\eta^{-1}(q)$. Then after finitely many blowups at the unique current point of the strict transform over q, one reaches a stage where the strict transform Γ is smooth. At that stage the reduced exceptional divisor over q is a simple normal-crossings divisor, and the unique point of Γ on it is of one of the following two kinds:*

(i) *a smooth point of one exceptional component, with local contact order $a \le e$; if the resolution is not yet finished at that stage, then necessarily $a \in \{2, 3\}$; more precisely,*

$$a = 2 \text{ if } e = 2, \qquad a \in \{2, 3\} \text{ if } e = 3;$$

(ii) *a corner point of two exceptional components.*

Proof. Choose local coordinates (t, z) at q adapted to the ruling, so that the local degree over the base is $\operatorname{ord}_u t = e \in \{2, 3\}$ on the normalization parameter u. Applying Lemma 11.32 to the successive strict transforms gives a finite blowup sequence after which the branch becomes smooth. At that first smooth stage, the total exceptional divisor created by the previous blowups is a normal-crossings divisor, so the point of the smooth strict transform on that divisor is either a smooth point of one exceptional component or a corner of two exceptional components.

In the first case, if the local contact order were $a = 1$, then the branch would already meet the exceptional divisor transversely and the local embedded resolution would be finished. So whenever the resolution is not yet finished, one must have $a \ge 2$. The bound $a \le e$ comes from the final order pair

$$(1, a) \qquad \text{or} \qquad (a, 1)$$

of Lemma 11.32. Since $e \in \{2, 3\}$, this leaves only the displayed possibilities. □

Proposition 11.34 (Low-degree unibranch points have only corner and order-2/3 terminal steps)**.** *Keep the setup of Corollary 11.33. Consider the remaining blowups after the first stage where the strict transform becomes smooth.*

Then every such remaining blowup is of one of the following two kinds:

(i) *a* corner blowup*: the smooth strict transform passes through a corner of the current reduced exceptional divisor;*

(ii) *a* tangency blowup block of order *a: the smooth strict transform is tangent to a single exceptional component with contact order $a \in \{2, 3\}$. In this case the next a forced blowups are exactly the order-a two-branch tangency resolution of Proposition 9.119, applied to the smooth strict transform and the touched exceptional component.*

In particular, after the branch first becomes smooth, no higher-order terminal behavior can occur in the one-section partitions

$$2 + 1 + 1 \qquad \text{and} \qquad 3 + 1.$$

Proof. By Corollary 11.33, once the strict transform is smooth the unresolved contact with the reduced exceptional divisor is supported either at a corner point or at a smooth point of one exceptional component with contact order $a \in \{2, 3\}$. This gives the stated dichotomy for every remaining blowup.

In case (ii), both the smooth strict transform and the touched exceptional component are smooth branches through one point, with intersection multiplicity exactly a. So the local analytic model is the same as for two smooth branches with order-a tangency, and Proposition 9.119 gives the forced order-a resolution block. No order larger than 3 can occur because Corollary 11.33 already bounds $a \leq 3$. □

Corollary 11.35 (Finite low-order endgame for the one-section unibranch branch)**.** *Assume the setup of Corollary 11.19. Let $q \in C$ be a unibranch singular point. Then, after a finite preliminary singular phase, the remaining local embedded resolution over q is built only from:*

(a) *corner blowups of a smooth branch through a corner of the reduced exceptional divisor;*

(b) *order-2 tangency blocks;*

(c) *order-3 tangency blocks, and these can occur only in the degree-3 case.*

Equivalently, the unibranch part of the one-section separable branch is not an arbitrary high-order ramification problem: in the partitions

$$2 + 1 + 1 \qquad \textit{and} \qquad 3 + 1$$

it is a finite low-order local endgame.

Proof. At a unibranch singular point of the image C, Lemma 11.18 says that the local degree over the base is at least 2. Since $C \cdot f = d \in \{2, 3\}$, that local degree is also at most d. Thus the only possibilities are

$$e = 2 \qquad \text{or} \qquad (e, d) = (3, 3).$$

Applying Proposition 11.34 gives exactly the three displayed terminal step types, with order-3 occurring only when $d = 3$. □

Remark 11.36 (What has now been isolated on the one-section separable side)*.* Combined with Proposition 9.119, Corollary 11.35 shows that the one-section separable branch has only two remaining local mechanisms:

(a) multibranch collisions of local branches of the multisection image;

(b) unibranch singularities whose post-smoothing endgame uses only corner blowups and order-2/3 tangency blocks.

So the unibranch side is no longer a vague "ramification" corridor. It is a finite low-order local problem. What is still missing is the genuinely global step: turning one of these low-order local blocks into a surviving admissible Route C configuration on $\widetilde{X}$ or else showing that the branch cannot occur in the residual sector at all.

11.4. Terminal corner and order-2/3 blocks are low-order forks, not chain bridges

The natural hope would be that the terminal corner and order-2 blocks from Corollary 11.35 already fall into the existing contact-2 chain corridors. The local calculation shows that this is not quite what happens. The exceptional components touched by those terminal blocks are already exceptional before the block begins, so they have self-intersection at most -1. After the terminal block they therefore remain inside D, and the last exceptional curve has total contact $L \cdot D = 3$, not 2. Thus the correct local output is a small fork corridor rather than a new chain bridge.

Proposition 11.37 (A surviving terminal corner blowup is a primitive fork)**.** *Let $\nu\colon \widetilde{X} \to Y$ be a birational morphism from a smooth surface to a smooth ruled surface, and let $H \subset \widetilde{X}$ be a smooth horizontal curve such that $\nu|_H\colon H \to C := \nu(H)$ is birational. Fix a point $q \in C$. Assume that in a factorization of ν^{-1} over q one reaches a smooth intermediate surface on which the strict transform Γ of C is smooth and passes through a corner $p = A_0 \cap B_0$ of the reduced exceptional divisor over q, where A_0, B_0 are smooth exceptional curves meeting transversely. Assume that the next blowup is exactly the blowup of p, and that no later center of ν^{-1} lies on the exceptional curve L created by that blowup.*

Then on $\widetilde{X}$ there are vertical curves

$$L \not\subset D, \qquad A, B \subset D$$

contained in the fiber through q with local dual graph

$$\begin{array}{ccccc} & & A & & \\ & & | & & \\ H & - & L & - & B \end{array}$$

and

$$L^2 = -1, \qquad A^2 \leq -2, \qquad B^2 \leq -2.$$

In particular, $L \cdot D = 3$.

Proof. Choose local coordinates (x, y) at p so that

$$A_0 = \{x = 0\}, \qquad B_0 = \{y = 0\}, \qquad \Gamma = \{y = x\}.$$

Blowing up p creates one exceptional curve L with $L^2 = -1$, and the strict transforms of

$$\Gamma, \qquad A_0, \qquad B_0$$

meet L transversely at three distinct points. So after that blowup the local graph is exactly the displayed fork.

Because

$$A_0 \quad \text{and} \quad B_0$$

are components of the exceptional divisor already present before the terminal blowup, one has

$$A_0^2 \leq -1, \qquad B_0^2 \leq -1.$$

The corner blowup lowers each self-intersection by 1, so immediately after that blowup,

$$(A_0')^2 \leq -2, \qquad (B_0')^2 \leq -2.$$

Any later blowup can only decrease those numbers further. Hence the final strict transforms A, B on $\widetilde{X}$ still satisfy

$$A^2 \leq -2, \qquad B^2 \leq -2.$$

By [7, Lemma 2.8(b)], every vertical component of a degenerate fiber outside D is a (-1)-curve. Therefore $A, B \subset D$. By assumption no later center lies on L, so its strict transform on $\widetilde{X}$ still has self-intersection -1. Hence $L \not\subset D$. Since L meets exactly the three displayed components of D, one gets $L \cdot D = 3$. □

Proposition 11.38 (Surviving terminal order-a tangency blocks are primitive forks)**.** *Keep the setup of Proposition 11.37. Assume instead that in a factorization of ν^{-1} over q one reaches a smooth intermediate surface on which the strict transform Γ of C is smooth and tangent of order $a \in \{2,3\}$ to a smooth exceptional curve A_0 at a smooth point p of the reduced exceptional divisor over q. Assume that the next a blowups are exactly the forced tangency-resolution centers, and that no later center of ν^{-1} lies on the last exceptional curve L created in that forced block. Then on $\widetilde{X}$ there exist vertical curves*

$$L \not\subset D, \qquad A, B_1, \dots, B_{a-1} \subset D$$

contained in the fiber through q with local dual graph

$$\begin{array}{ccccccc} & & A & & & & \\ & & | & & & & \\ H & - & L & - & B_{a-1} & - \cdots - & B_1 \end{array}$$

and

$$L^2 = -1, \qquad B_i^2 \leq -2 \;\; (1 \leq i \leq a-1), \qquad A^2 \leq -(a+1).$$

In particular, $L \cdot D = 3$. For $a = 2$ one gets the fork $H - L - A - B_1$, while for $a = 3$ one gets the fork $H - L - A - B_2 - B_1$.

Proof. Choose local coordinates (t, z) at p so that

$$A_0 = \{z = 0\}, \qquad \Gamma = \{z = t^a\}.$$

Applying Lemma 9.20 to the pair A_0, Γ shows that after the forced a blowups one obtains a chain $F_1 - \cdots - F_a$ with

$$F_i^2 = -2 \;\; (1 \leq i < a), \qquad F_a^2 = -1,$$

and the strict transforms of

$$A_0 \qquad \text{and} \qquad \Gamma$$

both meeting F_a transversely at distinct points. Set

$$L := F_a, \qquad B_i := F_{a-i} \ \ (1 \le i \le a-1),$$

and let A denote the final strict transform of A_0.

Since A_0 is already exceptional before this terminal block, one has $A_0^2 \le -1$. Each of the a forced blowups is centered on the current strict transform of A_0, so immediately after the block, $A^2 \le -1 - a = -(a+1)$. Any later blowup can only decrease that self-intersection further. Likewise the curves $B_1, \dots, B_{a-1}$ have self-intersection -2 immediately after the forced block, and later centers can only decrease those values. Therefore $A, B_1, \dots, B_{a-1} \subset D$ by [7, Lemma 2.8(b)].

By assumption no later center lies on L, so its strict transform on $\widetilde{X}$ still satisfies $L^2 = -1$. Hence $L \not\subset D$. The displayed local graph is exactly the one given by Lemma 9.20. Since L meets precisely the three components

$$H, \qquad A, \qquad B_{a-1}$$

of D, one gets $L \cdot D = 3$. □

Corollary 11.39 (Finite defect sieve for the terminal low-order forks)**.** *The surviving forks from Propositions 11.37 and 11.38 satisfy the following numerical restrictions.*

(i) *In the tangency case, write*

$$H^2 = -\beta, \qquad A^2 = -\alpha, \qquad \alpha, \beta \ge 2.$$

Then

$$\frac{2}{\alpha} + \frac{2}{\beta} > 1.$$

Equivalently, $(\alpha - 2)(\beta - 2) < 4$. *In particular, if* $\alpha, \beta > 2$*, then up to order* $(\alpha, \beta) \in \{(3,3), (3,4), (3,5)\}$. *Since Proposition 11.38 also gives* $\alpha \ge a+1$*, the order-*3 *case forces either* $\beta = 2$*, or else* $(\alpha, \beta) \in \{(4,3), (5,3)\}$.

(ii) *In the corner case, write*

$$H^2 = -\beta, \qquad A^2 = -\alpha, \qquad B^2 = -\gamma, \qquad \alpha, \beta, \gamma \ge 2.$$

Then

$$\frac{2}{\alpha} + \frac{2}{\beta} + \frac{2}{\gamma} > 2, \qquad \text{equivalently} \qquad \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} > 1.$$

After reordering so that $\alpha \le \beta \le \gamma$*, this leaves only the spherical triples*

$$(\alpha, \beta, \gamma) = (2,2,n) \ \ (n \ge 2), \qquad (2,3,3), \qquad (2,3,4), \qquad (2,3,5).$$

Proof. In the tangency case, Proposition 11.38 shows that the surviving curve L meets the three components

$$H, \qquad A, \qquad B_{a-1}$$

of D transversely once each. Thus Proposition 9.133 gives

$$\mathrm{cf}_X(H) + \mathrm{cf}_X(A) + \mathrm{cf}_X(B_{a-1}) < 1.$$

Discarding the nonnegative last term and using Lemma 9.132, we obtain

$$1 - \frac{2}{\beta} + 1 - \frac{2}{\alpha} < 1,$$

which is the displayed inequality. The finite list for $\alpha, \beta > 2$ is exactly the same elementary consequence as in Corollary 9.134. Since Proposition 11.38 also gives $\alpha \geq a + 1$, in the order-3 case one has $\alpha \geq 4$. So if $\beta > 2$, then the only possibilities are $(\alpha, \beta) \in \{(4, 3), (5, 3)\}$.

In the corner case, Proposition 11.37 gives $L \cdot H = L \cdot A = L \cdot B = 1$. Hence Proposition 9.133 gives

$$\mathrm{cf}_X(H) + \mathrm{cf}_X(A) + \mathrm{cf}_X(B) < 1.$$

Applying Lemma 9.132 to all three terms yields

$$1 - \frac{2}{\beta} + 1 - \frac{2}{\alpha} + 1 - \frac{2}{\gamma} < 1,$$

which is equivalent to

$$\frac{2}{\alpha} + \frac{2}{\beta} + \frac{2}{\gamma} > 2.$$

Now reorder so that $\alpha \leq \beta \leq \gamma$. If $\alpha \geq 3$, then

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} \leq 1,$$

so necessarily $\alpha = 2$. If also $\beta \geq 4$, then

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} \leq \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1,$$

again impossible. Thus $\beta \in \{2, 3\}$. If $\beta = 2$, then any $\gamma \geq 2$ works. If $\beta = 3$, then

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{\gamma} > 1$$

forces $\gamma < 6$, so $\gamma \in \{3, 4, 5\}$. This gives exactly the displayed list. □

Corollary 11.40 (The surviving one-section unibranch endgame is a finite fork corridor)**.** *Assume the setup of Corollary 11.19, and let $q \in C$ be a unibranch singular point. Assume that, in the factorization of ν^{-1} over q, the last exceptional curve created in the terminal block survives on $\widetilde{X}$. Then the local configuration on $\widetilde{X}$ is one of exactly the following three types:*

(i) *a corner fork*

$$\begin{array}{ccccc} & & A & & \\ & & | & & \\ H & - & L & - & B; \end{array}$$

(ii) *an order-*2 *tangency fork* $H - L - A - B_1$;

(iii) *an order-*3 *tangency fork* $H - L - A - B_2 - B_1$.

In every case $L \cdot D = 3$, *so the terminal one-section unibranch mechanisms do* not *directly enter the contact-*2 *chain-bridge corridor. Instead they lie in the explicit low-order fork corridor of Corollary 11.39.*

Proof. By Corollary 11.35, after the preliminary singular phase the terminal part of the local embedded resolution is either a corner blowup, an order-2 tangency block, or an order-3 tangency block. If the last exceptional curve of that terminal block survives on $\widetilde{X}$, then Proposition 11.37 handles the corner case and Proposition 11.38 handles the order-2 and order-3 cases. The final statement follows from those propositions together with Corollary 11.39. ☐

Remark 11.41 (What this honestly leaves). The attempted direct reduction of the one-section unibranch branch to the existing short-bridge / special-tip *chain* corridors is therefore too optimistic. The terminal corner and order-2/3 blocks do not create a contact-2 bridge: they create a surviving contact-3 fork, with only three possible low-order combinatorial shapes.

So the one-section separable branch is now narrowed as follows.

(a) multibranch collisions are governed by Proposition 9.119;

(b) unibranch singularities, if their terminal block survives, land in the finite low-order fork corridor of Corollary 11.40.

What remains is the genuinely global step: either show that these low-order forks are already absorbed by the existing fork/frontier analysis, or prove that they cannot survive in the residual one-section separable sector.

11.5. With ordinary section side, the one-section unibranch branch is horizontally low-weight

The terminal-fork reduction of the previous subsection still left an *a priori* arbitrary horizontal weight $\beta = -H^2$. In the genuinely one-section branch, this can be sharpened further. If every horizontal section distinct from the multisection is ordinary, then the fiber anticanonical inequality forces the multisection itself to carry discrepancy coefficient strictly smaller than

$$\frac{1}{2},$$

so its self-intersection can only be

$$-2 \quad \text{or} \quad -3.$$

Thus the horizontal arm of every surviving terminal fork is already numerically of the same low-weight type as the two special tip-visible cores isolated earlier on the chain side.

Proposition 11.42 (Ordinary sections force the multisection arm to have weight 2 or 3)**.** *Keep the setup of Corollary 11.19. Let $H \subset D_{\mathrm{hor}}$ be the unique horizontal multisection, and write $d := F \cdot H \in \{2, 3\}$. Let $S_1, \ldots, S_{4-d} \subset D_{\mathrm{hor}} - H$ be the horizontal sections. Assume that every S_i is ordinary in the sense of Definition 11.2. Assume moreover that some unibranch singular point of $C = \nu(H)$ has a surviving terminal local block on $\widetilde{X}$ as in Corollary 11.40, and write $H^2 = -\beta$. Then $\beta \in \{2, 3\}$. Equivalently,*

$$\mathrm{cf}_X(H) < \frac{1}{2}.$$

Proof. By Lemma 9.108, a witnessing general fiber satisfies

$$\sum_{T \subset D_{\mathrm{hor}}} \mathrm{cf}_X(T)\,(F \cdot T) < 2.$$

Since the horizontal degree partition is

$$d + \underbrace{1 + \cdots + 1}_{4-d \text{ sections}},$$

this becomes

$$d\ \mathrm{cf}_X(H) + \sum_{i=1}^{4-d} \mathrm{cf}_X(S_i) < 2.$$

Each section S_i is ordinary, so Lemma 11.3 gives

$$\mathrm{cf}_X(S_i) \geq \frac{1}{2}.$$

Therefore

$$d\ \mathrm{cf}_X(H) < 2 - \frac{4-d}{2} = \frac{d}{2},$$

hence

$$\mathrm{cf}_X(H) < \frac{1}{2}.$$

Now Lemma 9.132 applied to $H^2 = -\beta$ gives

$$\mathrm{cf}_X(H) \geq 1 - \frac{2}{\beta}.$$

Combining the two inequalities yields

$$1 - \frac{2}{\beta} < \frac{1}{2},$$

so $\beta < 4$. Since $\beta \geq 2$, we conclude that $\beta \in \{2, 3\}$. □

Corollary 11.43 (The pure one-section unibranch branch splits into a weight-2 corridor and a finite weight-3 list)**.** *Keep the assumptions of Proposition 11.42. Let the surviving terminal* *block over the chosen unibranch singular point be one of the low-order forks from Corollary 11.40. Then exactly one of the following holds.*

(i) $H^2 = -2$. *So the horizontal side already has the same numerical weight as the touched* (-2)*-tip geometry isolated earlier in Corollary 9.97.*

(ii) $H^2 = -3$. *Then the remaining touched vertical weights are finite. More precisely:*

  (a) *in the tangency case of Proposition 11.38, writing* $A^2 = -\alpha$, *one has* $\alpha \in \{3, 4, 5\}$ *for an order-*2 *terminal block, and* $\alpha \in \{4, 5\}$ *for an order-*3 *terminal block;*

  (b) *in the corner case of Proposition 11.37, writing*

$$A^2 = -\alpha, \qquad B^2 = -\gamma,$$

  one has, up to order,

$$\{\alpha, \gamma\} \in \{\{2, 2\}, \{2, 3\}, \{2, 4\}, \{2, 5\}\}.$$

In particular, once all section sides are ordinary, the only genuinely new one-section unibranch survivors beyond the horizontal [2]*-case form a finite horizontal-*[3] *list.*

Proof. By Proposition 11.42, one has $\beta \in \{2, 3\}$. If $\beta = 2$, then alternative (i) holds.

Assume now that $\beta = 3$. In the tangency case, Corollary 11.39(i) gives $(\alpha - 2)(\beta - 2) < 4$. Substituting $\beta = 3$ yields $\alpha < 6$. Since Proposition 11.38 gives

$$\alpha \geq a + 1 \qquad \text{for } a \in \{2, 3\},$$

we get $\alpha \in \{3, 4, 5\}$ for $a = 2$, and $\alpha \in \{4, 5\}$ for $a = 3$. This proves (ii)(a).

In the corner case, Corollary 11.39(ii) gives

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} > 1.$$

With $\beta = 3$ this becomes

$$\frac{1}{\alpha} + \frac{1}{\gamma} > \frac{2}{3}.$$

After reordering so that $\alpha \leq \gamma$, if $\alpha \geq 3$, then

$$\frac{1}{\alpha} + \frac{1}{\gamma} \leq \frac{1}{3} + \frac{1}{3} = \frac{2}{3},$$

a contradiction. Hence $\alpha = 2$. The inequality then reads

$$\frac{1}{2} + \frac{1}{\gamma} > \frac{2}{3},$$

so $\gamma < 6$. Therefore $\gamma \in \{2, 3, 4, 5\}$, which proves (ii)(b). □

Remark 11.44 (What this buys for the endpoint reduction). The genuinely one-section separable branch is now smaller than the previous terminal-fork statement suggested. If the multisection is the only special horizontal component and all section sides are ordinary, then every surviving unibranch terminal block has horizontal weight 2 or 3. So the residual one-section unibranch branch splits cleanly into two pieces:

(a) a horizontal [2]-side, which is numerically parallel to the touched (-2)-tip obstruction already isolated on the chain side;

(b) a finite horizontal [3]-list, namely the cases enumerated in Corollary 11.43.

Thus the unibranch part of the pure one-section branch is no longer an infinite weighted fork corridor. Only the horizontal [2]-case and a finite horizontal [3] list remain. If instead one of the section sides is itself special, then the witness already contains at least two special horizontal components, so one leaves the pure one-section branch and returns to the multi-special problem isolated in Remark 11.15.

11.6. The horizontal-[3] branch already enters the old special-tip / branching corridors

The weight-3 alternative from Corollary 11.43 still looked like a new finite fork problem. In fact it is not. The contact-defect inequality forces one of the vertical components adjacent to the surviving last exceptional curve to have discrepancy coefficient strictly smaller than

$$\frac{1}{3}.$$

On a chain side, that is already strong enough to force the old touched-(-2)-tip geometry. So the horizontal-[3] branch does not create a genuinely new local side type: it is absorbed by the previously isolated special-tip / branching corridors.

Lemma 11.45 (On a chain side, discrepancy coefficient below one third forces a touched (-2)-tip)**.** *Let (A, E) be a touched chain pair, and let $(A^\sharp, E^\sharp)$ be its 2-tail core from Definition 9.92. If*

$$\mathrm{cf}_X(E) < \frac{1}{3},$$

then $E^\sharp$ is a tip of $A^\sharp$ of weight 2.

Proof. By Lemma 9.106, $\mathrm{cf}_X(E) = 1 - \mu(A, E)$. So the hypothesis is equivalent to

$$\mu(A, E) > \frac{2}{3}.$$

By Lemma 9.93, $\mu(A, E) \leq \mu(A^\sharp, E^\sharp)$, hence

$$\mu(A^\sharp, E^\sharp) > \frac{2}{3}.$$

If $E^\sharp$ were an interior vertex of $A^\sharp$, then Proposition 9.94 would give

$$\mu(A^\sharp, E^\sharp) \leq \frac{1}{2},$$

a contradiction. So $E^\sharp$ is a tip.

Now Proposition 9.96 shows that a touched tip with

$$\mu > \frac{1}{2}$$

has reduced touched core either a touched tip of weight 2 or the singleton chain [3]. But for the singleton chain [3] one has

$$\mu = \frac{2}{3},$$

whereas here

$$\mu(A^\sharp, E^\sharp) > \frac{2}{3}.$$

Therefore the reduced touched core cannot be [3]. Hence $E^\sharp$ must be a touched tip of weight 2. □

Proposition 11.46 (Every surviving horizontal-[3] low-order fork already has a special vertical side)**.** *Keep the assumptions of Proposition 11.42, and assume moreover that $H^2 = -3$. Let the surviving terminal local block over the chosen unibranch singular point be one of the low-order forks from Corollary 11.40. Then there exists a vertical component $E \subset D$ adjacent to the surviving curve L such that*

$$\mathrm{cf}_X(E) < \frac{1}{3}.$$

Consequently, if the connected component of D containing E is a chain, then the 2-tail core of the touched pair relative to E is a touched tip of weight 2. If that connected component is not a chain, then one is already on the branching side of the residual Route C gap.

Proof. Since $H^2 = -3$, Lemma 9.132 gives

$$\mathrm{cf}_X(H) \geq 1 - \frac{2}{3} = \frac{1}{3}.$$

Assume first that the terminal block is a tangency fork as in Proposition 11.38. Let $E := B_{a-1}$, so $L \cdot H = L \cdot A = L \cdot E = 1$. By Proposition 9.133,

$$\mathrm{cf}_X(H) + \mathrm{cf}_X(A) + \mathrm{cf}_X(E) < 1.$$

Also Proposition 11.38 gives $A^2 \leq -(a+1) \leq -3$, hence

$$\mathrm{cf}_X(A) \geq 1 - \frac{2}{3} = \frac{1}{3}$$

by Lemma 9.132. Therefore

$$\mathrm{cf}_X(E) < 1 - \frac{1}{3} - \frac{1}{3} = \frac{1}{3}.$$

Assume next that the terminal block is a corner fork as in Proposition 11.37. Then $L \cdot H = L \cdot A = L \cdot B = 1$, so Proposition 9.133 gives

$$\mathrm{cf}_X(H) + \mathrm{cf}_X(A) + \mathrm{cf}_X(B) < 1.$$

Using

$$\mathrm{cf}_X(H) \geq \frac{1}{3},$$

we obtain

$$\mathrm{cf}_X(A) + \mathrm{cf}_X(B) < \frac{2}{3}.$$

Hence at least one of $A,\ B$ has discrepancy coefficient strictly smaller than

$$\frac{1}{3}.$$

Take E to be such a component.

The final statement is exactly Lemma 11.45 when the component of D containing E is a chain. If that component is not a chain, then by definition one is already on the branching side. □

Corollary 11.47 (The horizontal-[3] one-section branch is not a new corridor)**.** *Keep the setup of Proposition 11.46. Then every surviving one-section unibranch terminal block with* $H^2 = -3$ *already falls into one of the two side types that were previously isolated elsewhere in the note:*

(i) *a chain side whose* 2*-tail core is a touched tip of weight* 2*;*

(ii) *a branching side.*

In particular, the horizontal-[3] *branch does not create a genuinely new residual one-section mechanism.*

Proof. This is exactly Proposition 11.46. □

Remark 11.48 (What this changes in the one-section endgame)*.* The previous low-weight split left two pieces: a horizontal [2] corridor and a finite horizontal [3] list. The new proposition shows that the second piece is not genuinely new. Whenever $H^2 = -3$, one of the vertical sides adjacent to the surviving terminal fork already lands in the old special-tip / branching geometry.

So, inside the genuinely one-section separable branch with ordinary section sides, the only still-distinctive new unibranch mechanism is the horizontal [2] corridor. The horizontal [3] cases have already been funneled back into the previously isolated Route C side types.

11.7. A global proof-first trichotomy at height at least four

The local reductions on the multisection and one-section sides can be organized into a single global statement. The key observation is that ordinary multisections are already expensive in the fiber anticanonical inequality: each unit of horizontal degree carried by an ordinary multisection contributes at least one half.

Lemma 11.49 (Ordinary multisections carry total horizontal degree at most three)**.** *Let* $p\colon \widetilde{X} \to \mathbf{P}^1$ *be a witnessing* $\mathbf{P}^1$*-fibration on the minimal resolution of a singular klt del Pezzo surface, let* F *be a general fiber, and let*

$$\mathcal{M}_{\mathrm{ord}} := \{H \subset D_{\mathrm{hor}}\,;\, F \cdot H \geq 2, \quad H \text{ is ordinary in the sense of Definition 11.2}\}.$$

Then

$$\sum_{H \in \mathcal{M}_{\mathrm{ord}}} (F \cdot H) \leq 3.$$

Proof. By Lemma 11.3, every $H \in \mathcal{M}_{\mathrm{ord}}$ satisfies

$$\mathrm{cf}_X(H) \geq \frac{1}{2}.$$

Applying Lemma 9.108 to the witnessing fiber gives

$$\sum_{T \subset D_{\mathrm{hor}}} \mathrm{cf}_X(T)\,(F \cdot T) < 2.$$

Discarding all horizontal components outside $\mathcal{M}_{\mathrm{ord}}$ and using the preceding lower bound yields

$$2 > \sum_{T \subset D_{\mathrm{hor}}} \mathrm{cf}_X(T)\,(F \cdot T) \geq \sum_{H \in \mathcal{M}_{\mathrm{ord}}} \mathrm{cf}_X(H)\,(F \cdot H) \geq \frac{1}{2} \sum_{H \in \mathcal{M}_{\mathrm{ord}}} (F \cdot H).$$

Therefore

$$\sum_{H \in \mathcal{M}_{\mathrm{ord}}} (F \cdot H) < 4.$$

Since the left-hand side is an integer, it is at most 3. ☐

Proposition 11.50 (Global trichotomy for witnessing fibrations of height at least four)**.** *Let* $p\colon \widetilde{X} \to \mathbf{P}^1$ *be a witnessing* $\mathbf{P}^1$*-fibration of height* $h := F \cdot D \geq 4$ *on the minimal resolution of a singular klt del Pezzo surface. Then exactly one of the following holds.*

(i) *every horizontal component is a* 1*-section;*

(ii) *there exists a special horizontal multisection*

$$H \subset D_{\mathrm{hor}}, \qquad F \cdot H \geq 2;$$

(iii) *there exists a unique horizontal multisection* $H \subset D_{\mathrm{hor}}$*, this component is ordinary, one has* $F \cdot H \in \{2, 3\}$*, and every other horizontal component is a* 1*-section.*

In case (iii)*, after choosing any horizontal section* $H_0 \subset D_{\mathrm{hor}} - H$ *and contracting all vertical curves disjoint from* H_0 *to a ruled surface* $\nu\colon \widetilde{X} \to \mathbb{F}_n$*, the image* $C := \nu(H)$ *is singular, and every singular point of* C *is either:*

(a) *a collision point of at least two local branches of C;*

(b) *a unibranch point whose local degree over the base is* $2 \quad$ *or* $\quad 3$.

Proof. If every horizontal component is a 1-section, we are in case (i). So assume that some horizontal component has degree at least 2.

If one such multisection is special, then case (ii) holds. Assume from now on that no horizontal multisection is special. Then every horizontal multisection is ordinary, so Lemma 11.49 gives

$$\sum_{H \subset D_{\mathrm{hor}},\ F \cdot H \geq 2} (F \cdot H) \leq 3.$$

Because $h = F \cdot D \geq 4$, not all horizontal components can be multisections. Hence there is at least one horizontal 1-section.

On the other hand, every horizontal multisection has degree at least 2. Since their total degree is at most 3, there can be only one such component. Call it H. Its degree must then be $F \cdot H \in \{2, 3\}$, and every other horizontal component is a 1-section. This proves case (iii) and the trichotomy.

Now assume case (iii), and choose a horizontal section $H_0 \subset D_{\mathrm{hor}} - H$. Let $\nu \colon \widetilde{X} \to \mathbb{F}_n$ be the contraction of all vertical curves disjoint from H_0. Because $F \cdot H \geq 2$, Corollary 11.17 shows that the ruled-model image $C := \nu(H)$ is singular.

Let $q \in C$ be a singular point. By Lemma 11.18, either q is a collision point of at least two local branches of C, or the normalization map has a unique preimage x above q and the induced map to the base has local degree at least 2 at x. Since the global degree of $H \to \mathbf{P}^1$ is $F \cdot H \in \{2, 3\}$, that local degree is automatically at most 3. This proves the final assertion. □

Corollary 11.51 (Residual witnesses reduce to section-only, special-multisection, or low-degree mixed type)**.** *Let X be a characteristic-3 klt del Pezzo surface of Picard rank 1 with $\# \operatorname{Sing}(X) \geq 8$. Let $p \colon \widetilde{X} \to \mathbf{P}^1$ be a witnessing $\mathbf{P}^1$-fibration on the minimal resolution. Then $\operatorname{ht}(X) \geq 4$ by Theorem 9.1, and exactly one of the following three global branches occurs.*

(i) ***Section-only branch.*** *Every horizontal component is a 1-section. If $\operatorname{ht}(X) = 4$, then $D_{\mathrm{hor}} = H_0 + H_1 + H_2 + H_3$ is already a maximal-width witness, so one is in the final two-touch corridor of Section 10. If $\operatorname{ht}(X) \geq 5$, then the remaining global problem is the section-only packet / isolated-collision branch of Remarks 9.130 and 9.131.*

(ii) ***Special-multisection branch.*** *There exists a special horizontal multisection of degree at least 2. This contains the residual separable special-multisection branch and the purely inseparable branch isolated in Corollaries 11.8 and 11.9.*

(iii) ***Low-degree mixed branch.*** *There exists a unique ordinary horizontal multisection $H \subset D_{\mathrm{hor}}$ of degree* $2 \quad$ *or* $\quad 3$, *and every other horizontal component is a 1-section. After fixing any section and passing to the ruled model, the image of H is singular, and every singular point is either a branch collision or a unibranch point of local degree* $2 \quad$ *or* $\quad 3$.

Proof. By Theorem 9.1, $\mathrm{ht}(X) \geq 4$. So Proposition 11.50 applies to any witnessing fibration.

If case (i) of that proposition holds, then we are in the section-only branch. When $\mathrm{ht}(X) = 4$, all four units of horizontal degree are carried by sections, so the witness already has maximal width. When $\mathrm{ht}(X) \geq 5$, the later packet / scalar-star / isolated-collision reductions apply by construction.

If case (ii) of Proposition 11.50 holds, then we are in the special-multisection branch.

If case (iii) holds, then we are in the low-degree mixed branch with the stated singular-image description. □

Corollary 11.52 (Height-4 non-maximal-width witnesses are either special or low-degree mixed)**.** *Assume that* $\mathrm{ht}(X) = 4$ *and let* $p\colon \widetilde{X} \to \mathbf{P}^1$ *be a witnessing* $\mathbf{P}^1$*-fibration which is not of maximal width. Then exactly one of the following holds.*

- (i) *there exists a special horizontal multisection of degree at least* 2*;*
- (ii) *there exists a unique ordinary horizontal multisection* $H \subset D_{\mathrm{hor}}$ *with* $F \cdot H \in \{2, 3\}$*, and the horizontal degree partition is therefore*

$$2+1+1 \qquad \textit{or} \qquad 3+1.$$

*In particular, every height-*4 *non-maximal-width witness outside the special-multisection branch already lies in the low-degree one-section singular-image corridor.*

Proof. Because the witness is not of maximal width, not every horizontal component is a 1-section. Apply Proposition 11.50. Case (i) is impossible, so only cases (ii) and (iii) remain. In case (iii), the unique multisection has degree $2 \quad \text{or} \quad 3$, and the remaining horizontal degree is $4 - (F \cdot H)$, carried entirely by sections. Thus the degree partition is

$$2+1+1 \qquad \text{or} \qquad 3+1.$$

□

Remark 11.53 (What this adds globally)*.* The new trichotomy removes the possibility of a genuinely high-degree *ordinary* mixed branch. Once special multisections are set aside, every residual witnessing fibration of height at least 4 is forced into one of only two remaining horizontal types:

- (a) the section-only branch; or
- (b) a unique ordinary bisection / trisection plus sections.

So the global height-selection problem is now sharper than before: the unresolved global branches are precisely the section-only packet / isolated-collision branch, the special-multisection branch, the low-degree mixed singular-image branch, and the height-4, maximal-width two-touch problem.

11.8. The low-degree mixed branch is either multi-special or height four with one old special section

The proof-first trichotomy leaves a unique ordinary bisection or trisection together with horizontal sections. The same fiber anticanonical inequality still forces almost all of those section sides to be special. As a result, above height four the low-degree mixed branch is already absorbed by the several-special-horizontal problem, while in height four the only residual case with no such several-special phenomenon has a unique special section. That unique special section already lies on one of the old special-tip / branching side types, and the ruled model relative to it lives on $\mathbb{F}_2$ or $\mathbb{F}_3$.

Proposition 11.54 (Low-degree mixed witnesses have at most $3-d$ ordinary sections)**.** *Assume that $p\colon \widetilde{X} \to \mathbf{P}^1$ lies in case* (iii) *of Proposition 11.50. Let $H \subset D_{\mathrm{hor}}$ be the unique ordinary horizontal multisection, put $d := F \cdot H \in \{2,3\}$, and let o be the number of ordinary horizontal sections contained in $D_{\mathrm{hor}} - H$. Then $o \leq 3-d$.*

Equivalently, if r denotes the number of special horizontal sections contained in $D_{\mathrm{hor}} - H$, then

$$r \geq h-3, \qquad h := F \cdot D.$$

In particular:

(i) *if $d=3$, then every horizontal section is special;*

(ii) *if $d=2$, then at most one horizontal section is ordinary.*

Proof. By case (iii) of Proposition 11.50, the component H is ordinary, so Lemma 11.3 gives

$$\mathrm{cf}_X(H) \geq \frac{1}{2}.$$

Every ordinary horizontal section $S \subset D_{\mathrm{hor}} - H$ also satisfies

$$\mathrm{cf}_X(S) \geq \frac{1}{2}$$

by the same lemma.

Applying Lemma 9.108 to a witnessing general fiber gives

$$\sum_{T \subset D_{\mathrm{hor}}} \mathrm{cf}_X(T)\,(F \cdot T) < 2.$$

Among the horizontal components, the ordinary multisection H contributes degree d, the o ordinary sections contribute degree o, and the remaining special sections contribute only nonnegative terms. Hence

$$2 > d\ \mathrm{cf}_X(H) + \sum_{i=1}^{o} \mathrm{cf}_X(S_i) \geq \frac{d}{2} + \frac{o}{2}.$$

Therefore $d+o<4$. Since $d+o$ is an integer and $d \in \{2,3\}$, we conclude that $o \leq 3-d$.

For the equivalent form, note that every horizontal component distinct from H is a section, so the total number of horizontal sections is $h-d$. Hence $r=(h-d)-o \geq (h-d)-(3-d)=h-3$. The final two statements are immediate. ☐

Corollary 11.55 (Outside the several-special corridor, the low-degree mixed branch is height four with a unique special section)**.** *Assume the setup of Corollary 11.51, and suppose that we are in the low-degree mixed branch. If the chosen witnessing fibration does* not *have at least two special horizontal sections, then* $\mathrm{ht}(X)=4$ *and there is a unique special horizontal section* $H_0 \subset D_{\mathrm{hor}}$*. Consequently the horizontal degree partition is*

$$2+1+1 \qquad \textit{or} \qquad 3+1,$$

the unique multisection is ordinary, and every horizontal component distinct from

$$H \quad \textit{and} \quad H_0$$

is an ordinary section.

Proof. Let r be the number of special horizontal sections. By Proposition 11.54,

$$r \geq h-3, \qquad h=\mathrm{ht}(X).$$

If $r \leq 1$, then $h-3 \leq 1$, so $h \leq 4$. On the other hand, Corollary 11.51 already gives $h \geq 4$. Hence $h=4$. Then $r \geq h-3=1$, and together with $r \leq 1$ this yields $r=1$. So there is a unique special horizontal section H_0. Since we are in the low-degree mixed branch, Proposition 11.50 shows that the unique multisection H is ordinary of degree

$$2 \quad \text{or} \quad 3,$$

and every other horizontal component is a section. Because $h=4$, the degree partition is

$$2+1+1 \qquad \text{or} \qquad 3+1.$$

Every section distinct from H_0 is then ordinary by uniqueness of the special section. ☐

Proposition 11.56 (The unique special section is an old special side and forces $\mathbb{F}_2$ or $\mathbb{F}_3$)**.** *Assume the setup of Corollary 11.55. Let* $H_0 \subset D_{\mathrm{hor}}$ *be the unique special horizontal section, let* $H \subset D_{\mathrm{hor}}$ *be the unique ordinary multisection, and put* $d := F \cdot H \in \{2,3\}$*. Then*

$$\mathrm{cf}_X(H_0) < \frac{1}{2}.$$

In particular, $H_0^2 \in \{-2,-3\}$*.*

If the connected component of D *containing* H_0 *is a chain, then the reduced touched core relative to* H_0 *is either a touched tip of weight* 2 *or the singleton chain* [3]*. If that connected component is not a chain, then one is already on the branching side.*

Let $\nu\colon \widetilde{X} \to \mathbb{F}_n$ *be the contraction of all vertical curves disjoint from* H_0*, put*

$$S_0 := \nu(H_0), \qquad C := \nu(H), \qquad s := H \cdot H_0 \in \{0,1\},$$

and let f *denote a fiber of the ruling on* $\mathbb{F}_n$*. Then* $n = -H_0^2 \in \{2,3\}$*. Moreover:*

(i) *if* $d = 2$*, then*

$$C \equiv 2S_0 + (2n+s)f, \qquad \delta(C) = n + s - 1 \in \{1,2,3\},$$

and the elementary-transform sequence from Proposition 11.27 has length $n - 1 \leq 2$*;*

(ii) *if* $d = 3$*, then*

$$C \equiv 3S_0 + (3n+s)f, \qquad \delta(C) = 3n + 2s - 2 \in \{4,6,7,9\}.$$

Finally, the ordinary multisection H *has bounded self-intersection:*

(a) *if* $d = 2$*, then*

$$\mathrm{cf}_X(H) < \frac{3}{4} \qquad \textit{and hence} \qquad H^2 \in \{-2,-3,-4,-5,-6,-7\};$$

(b) *if* $d = 3$*, then*

$$\mathrm{cf}_X(H) < \frac{2}{3} \qquad \textit{and hence} \qquad H^2 \in \{-2,-3,-4,-5\}.$$

Proof. Because H is ordinary, Lemma 11.3 gives

$$\mathrm{cf}_X(H) \geq \frac{1}{2}.$$

By Corollary 11.55, every horizontal section distinct from H_0 is ordinary. Since the height is 4, there are exactly $3 - d$ such ordinary sections. Therefore Lemma 9.108 yields

$$2 > d\ \mathrm{cf}_X(H) + \mathrm{cf}_X(H_0) + \sum_{i=1}^{3-d} \mathrm{cf}_X(S_i) \geq \frac{d}{2} + \mathrm{cf}_X(H_0) + \frac{3-d}{2} = \frac{3}{2} + \mathrm{cf}_X(H_0).$$

Hence

$$\mathrm{cf}_X(H_0) < \frac{1}{2}.$$

Applying Lemma 9.132 to $H_0^2 = -a$ gives

$$1 - \frac{2}{a} \leq \mathrm{cf}_X(H_0) < \frac{1}{2},$$

so $a < 4$. Since $a \geq 2$, we conclude that $a \in \{2,3\}$, equivalently $H_0^2 \in \{-2,-3\}$.

The chain / branching dichotomy is now immediate from the fact that H_0 is special in the sense of Definition 11.2: if the connected component of D containing H_0 is a chain, then its reduced touched core relative to H_0 is by definition either a touched tip of weight 2 or the singleton chain [3]; if not, then one is on the branching side.

The formula $n = -H_0^2$ comes from Proposition 11.16. Since $H_0^2 \in \{-2, -3\}$, we get $n \in \{2, 3\}$. The class and arithmetic-genus formulas for C are exactly those of Corollary 11.19. Substituting

$$n \in \{2,3\} \qquad \text{and} \qquad s \in \{0,1\}$$

gives the displayed finite sets of values for $\delta(C)$. When $d = 2$, Proposition 11.27 uses precisely $n-1$ singular-point elementary transforms, so here that sequence has length at most 2.

For the bounds on H, argue again from Lemma 9.108. If $d = 2$, there is exactly one ordinary horizontal section distinct from H_0, so

$$2 > 2\ \mathrm{cf}_X(H) + \mathrm{cf}_X(H_0) + \mathrm{cf}_X(S_1) \geq 2\ \mathrm{cf}_X(H) + 0 + \frac{1}{2}.$$

Hence

$$\mathrm{cf}_X(H) < \frac{3}{4}.$$

Writing $H^2 = -\beta$, Lemma 9.132 gives

$$1 - \frac{2}{\beta} < \frac{3}{4},$$

so $\beta < 8$. Since $\beta \geq 2$, we obtain $\beta \in \{2,3,4,5,6,7\}$.

If $d = 3$, then H_0 is the only horizontal section, so

$$2 > 3\ \mathrm{cf}_X(H) + \mathrm{cf}_X(H_0) \geq 3\ \mathrm{cf}_X(H) + 0.$$

Therefore

$$\mathrm{cf}_X(H) < \frac{2}{3}.$$

Again writing $H^2 = -\beta$, Lemma 9.132 yields

$$1 - \frac{2}{\beta} < \frac{2}{3},$$

hence $\beta < 6$. Since $\beta \geq 2$, this gives $\beta \in \{2,3,4,5\}$. ☐

Corollary 11.57 (Global form of the low-degree mixed branch)**.** *Assume the setup of Corollary 11.51, and suppose that we are in the low-degree mixed branch. Then exactly one of the following holds.*

(i) *the chosen witnessing fibration already has at least two special horizontal sections, so it lies in the several-special-horizontal corridor from Remark 11.15;*

(ii) $\mathrm{ht}(X) = 4$, *there is a unique special horizontal section* H_0, *and the unique ordinary multisection has degree* 2. *Relative to* H_0, *the ruled model lies on*

$$\mathbb{F}_2 \quad \textit{or} \quad \mathbb{F}_3,$$

the bisection has total $\delta \leq 3$, and Proposition 11.27 reduces the whole branch by at most two singular-point elementary transforms to either a smooth plane conic or a marked singular plane cubic;

(iii) $\operatorname{ht}(X) = 4$, *there is a unique special horizontal section H_0, and the unique ordinary multisection has degree 3. Relative to H_0, the ruled model lies on*

$$\mathbb{F}_2 \quad \textit{or} \quad \mathbb{F}_3,$$

the trisection class is one of the four explicit classes from Proposition 11.56, and the section side is already one of the old special-tip / branching geometries.

In particular, the low-degree mixed branch does not create a new high-height problem: above height 4 it is already absorbed by the several-special-horizontal corridor, and in height 4 the only genuinely new mixed cases are the finite bisection plane-model package and the singular trisection attached to a single old special section.

Proof. If the witnessing fibration has at least two special horizontal sections, then we are in case (i). Otherwise Corollary 11.55 applies, giving height 4 and a unique special horizontal section H_0. Then Proposition 11.56 gives $n \in \{2, 3\}$ and the stated finite ruled-model packages. When the multisection degree is 2, the final plane reduction is exactly Proposition 11.27 together with Proposition 11.28. When the multisection degree is 3, the same proposition shows that the whole branch is concentrated on the four explicit trisection classes listed there, with the unique section already lying in the old special-tip / branching corridor. □

Remark 11.58 (What this adds to the global proof-first picture)*.* The low-degree mixed branch is now much smaller than the trichotomy alone suggests.

First, it is not a new high-height branch at all: Proposition 11.54 shows that above height 4 the branch already belongs to the several-special-horizontal problem.

Second, in height 4, the only way to stay outside that several-special corridor is to have a *unique* special horizontal section. Proposition 11.56 shows that this section already lies on one of the old special-tip / branching side types and has self-intersection

$$-2 \quad \text{or} \quad -3.$$

So the ruled model relative to that section lives only on

$$\mathbb{F}_2 \quad \text{or} \quad \mathbb{F}_3.$$

Thus the residual mixed geometry is now completely explicit: the degree-two branch is a one- or two-step reverse elementary-transform problem from a smooth plane conic or a marked singular cubic, while the degree-three branch is a finite trisection-class problem on

$$\mathbb{F}_2 \quad \text{or} \quad \mathbb{F}_3$$

attached to a single old special section.

11.9. The singular trisection branch: one-step reduction and absorption of surviving unibranch blocks

The residual degree-three mixed branch from Corollary 11.57 is now a singular rational trisection

$$C \subset \mathbb{F}_n, \qquad C \equiv 3S_0 + (3n+s)f, \qquad n \in \{2,3\}, \qquad s \in \{0,1\},$$

attached to a unique special section H_0. There are two further unconditional reductions. First, one singular-point elementary transform already moves the branch to an explicit finite package. Second, any surviving unibranch terminal block already has an old special vertical side, so the trisection branch does not create a new unibranch fork corridor.

Proposition 11.59 (One singular-point transform reduces the trisection to a finite package)**.** *Assume the setup of Proposition 11.56, and suppose that $d = 3$. Thus*

$$C \equiv 3S_0 + (3n+s)f, \qquad n \in \{2,3\}, \qquad s \in \{0,1\}.$$

Let $p \in C$ be a singular point. Then:

(i) $p \notin S_0$;

(ii) $\mathrm{mult}_p(C) \in \{2,3\}$.

Let

$$\epsilon_p \colon \mathbb{F}_n \dashrightarrow \mathbb{F}_{n-1}$$

be the elementary transform at p, and let $C' \subset \mathbb{F}_{n-1}$ be the image of C. Writing

$$m := \mathrm{mult}_p(C) \in \{2,3\}, \qquad s' := s + 3 - m \in \{s, s+1\},$$

one has $C' \equiv 3S_0' + (3(n-1)+s')f'$. Consequently:

(a) *if $n = 2$, then after this single singular-point elementary transform and the contraction $\kappa \colon \mathbb{F}_1 \to \mathbf{P}^2$ of the negative section, the branch reduces birationally to an irreducible rational plane curve $\Gamma \subset \mathbf{P}^2$ of degree $3 + s' \in \{3,4,5\}$, with marked point $q := \kappa(S_1)$ of multiplicity $s' \in \{0,1,2\}$;*

(b) *if $n = 3$, then after one singular-point elementary transform the branch reduces to one of the three explicit classes on*

$$\mathbb{F}_2 : \qquad 3S_0 + 6f, \qquad 3S_0 + 7f, \qquad 3S_0 + 8f.$$

The first two already belong to the $n = 2$ package from (a)*. Hence after at most two singular-point elementary transforms every singular trisection branch reduces either to a marked plane cubic / quartic / quintic package or to the single exceptional $\mathbb{F}_2$ class $3S_0 + 8f$.*

Proof. Since $C \cdot S_0 = s \leq 1$, a singular point of C cannot lie on S_0. Indeed, if $p \in C \cap S_0$ were singular on C, then

$$(C \cdot S_0)_p \geq \operatorname{mult}_p(C) \geq 2,$$

contradicting the total intersection number $C \cdot S_0 = s \leq 1$. This proves (i).

Let F_p be the fiber through p. Because $C \cdot F_p = 3$, one has $\operatorname{mult}_p(C) \leq 3$. Since p is singular on C, one also has $\operatorname{mult}_p(C) \geq 2$, so $\operatorname{mult}_p(C) \in \{2, 3\}$. This proves (ii).

Now apply Lemma 11.26 with

$$a = 3, \qquad b = 3n + s, \qquad m = \operatorname{mult}_p(C).$$

It yields

$$C' \equiv 3S_0' + (3n + s - m)f' = 3S_0' + (3(n-1) + (s + 3 - m))f'.$$

So with $s' := s + 3 - m$, one gets $C' \equiv 3S_0' + (3(n-1) + s')f'$. Because $m \in \{2, 3\}$, one has $s' \in \{s, s+1\}$.

Assume first that $n = 2$. Then $C' \subset \mathbb{F}_1$ and $C' \equiv 3S_1 + (3 + s')f$. Let $\kappa \colon \mathbb{F}_1 \to \mathbf{P}^2$ be the contraction of

$$S_1, \qquad q := \kappa(S_1), \qquad H := \kappa^*(\text{line}) = S_1 + f.$$

Then

$$C' \equiv 3S_1 + (3 + s')f = (3 + s')H - s'S_1.$$

Hence the plane image $\Gamma := \kappa(C')$ has degree $\deg \Gamma = 3 + s'$ and multiplicity $\operatorname{mult}_q(\Gamma) = s'$. Since C' is birational to the original rational curve, it is irreducible and rational, and so is Γ. Because

$$s \in \{0, 1\} \qquad \text{and} \qquad s' \in \{s, s+1\},$$

one has

$$s' \in \{0, 1, 2\}, \qquad 3 + s' \in \{3, 4, 5\}.$$

This proves (a).

Assume next that $n = 3$. Then $C' \subset \mathbb{F}_2$ and

$$C' \equiv 3S_0' + (6 + s')f', \qquad s' \in \{0, 1, 2\}.$$

So the only possibilities are

$$3S_0' + 6f', \qquad 3S_0' + 7f', \qquad 3S_0' + 8f'.$$

The first two correspond exactly to the

$$n = 2, \qquad s' \in \{0, 1\}$$

package from (a), so one more singular-point elementary transform reduces them to marked plane cubic / quartic / quintic models. The third is the single residual $\mathbb{F}_2$ class $3S_0 + 8f$. This proves (b). ☐

Proposition 11.60 (Every surviving unibranch trisection block already has an old special vertical side)**.** *Assume the setup of Proposition 11.56, and suppose that $d = 3$. Let $q \in C$ be a unibranch singular point. Assume that, in the factorization of ν^{-1} over q, the last exceptional curve created in the terminal block survives on $\widetilde{X}$. Then there exists a vertical component $E \subset D$ adjacent to that surviving last exceptional curve such that*

$$\mathrm{cf}_X(E) < \frac{1}{3}.$$

Consequently, if the connected component of D containing E is a chain, then the corresponding 2-tail core is a touched tip of weight 2; if that connected component is not a chain, then one is already on the branching side.

Proof. Let $H \subset D_{\mathrm{hor}}$ be the unique trisection. In the low-degree mixed branch it is ordinary, so Lemma 11.3 gives

$$\mathrm{cf}_X(H) \geq \frac{1}{2}.$$

By Corollary 11.40, the terminal surviving local block over q is one of the following:

(i) a corner fork

$$\begin{array}{ccccc} & & A & & \\ & & | & & \\ H & - & L & - & B; \end{array}$$

(ii) an order-2 tangency fork $H - L - A - B_1$;

(iii) an order-3 tangency fork $H - L - A - B_2 - B_1$.

In every case

$$L \not\subset D \qquad \text{and} \qquad L \cdot D = 3.$$

Assume first that the terminal block is a tangency fork as in Proposition 11.38. Let

$$E := B_{a-1}, \qquad a \in \{2, 3\}.$$

Then $L \cdot H = L \cdot A = L \cdot E = 1$. By Proposition 9.133,

$$\mathrm{cf}_X(H) + \mathrm{cf}_X(A) + \mathrm{cf}_X(E) < 1.$$

Also Proposition 11.38 gives $A^2 \leq -(a + 1) \leq -3$, so Lemma 9.132 yields

$$\mathrm{cf}_X(A) \geq \frac{1}{3}.$$

Together with

$$\mathrm{cf}_X(H) \geq \frac{1}{2},$$

this gives

$$\mathrm{cf}_X(E) < 1 - \frac{1}{2} - \frac{1}{3} = \frac{1}{6} < \frac{1}{3}.$$

Assume next that the terminal block is a corner fork as in Proposition 11.37. Then $L \cdot H = L \cdot A = L \cdot B = 1$, so Proposition 9.133 gives

$$\mathrm{cf}_X(H) + \mathrm{cf}_X(A) + \mathrm{cf}_X(B) < 1.$$

Using again

$$\mathrm{cf}_X(H) \geq \frac{1}{2},$$

one gets

$$\mathrm{cf}_X(A) + \mathrm{cf}_X(B) < \frac{1}{2}.$$

Hence at least one of $A,\ B$ satisfies

$$\mathrm{cf}_X(E) < \frac{1}{4} < \frac{1}{3}.$$

Take such a component as E.

The final conclusion is exactly Lemma 11.45 when the connected component of D containing E is a chain. If that connected component is not a chain, then by definition one is already on the branching side. ☐

Corollary 11.61 (The singular trisection branch is not a new unibranch corridor)**.** *Assume the setup of Corollary 11.57, and suppose that we are in case* (iii)*. Then every surviving unibranch terminal block already falls into one of the old side types isolated earlier in the note:*

(i) *a chain side whose* 2*-tail core is a touched tip of weight* 2*;*

(ii) *a branching side.*

In particular, the singular trisection branch does not create a genuinely new unibranch fork corridor. Any genuinely new trisection obstruction must come from collision points of C, or from the single exceptional one-step global package $C \equiv 3S_0 + 8f \subset \mathbb{F}_2$ from Proposition 11.59.

Proof. This is Proposition 11.60 together with Proposition 11.59. ☐

Remark 11.62 (What now remains in the trisection target)*.* The residual trisection branch is now much smaller than the finite-class statement of Corollary 11.57 alone suggested.

First, one singular-point elementary transform already reduces every trisection to an explicit finite global package: either a marked plane cubic / quartic / quintic package, or the single exceptional class $3S_0 + 8f \subset \mathbb{F}_2$. So the trisection branch is no longer an arbitrary problem on

$$\mathbb{F}_2 \quad \text{or} \quad \mathbb{F}_3.$$

Second, Proposition 11.60 shows that the local unibranch side is not genuinely new: whenever the terminal block survives, one already lands in the old touched-(-2)-tip / branching geometry.

So the honest remaining trisection gap is now concentrated in two places only:

(a) collision singularities of the trisection image;

(b) the global bookkeeping needed either to force one more off-negative-section singular point in the exceptional $\mathbb{F}_2$ class $3S_0 + 8f$, or to exclude that class directly in the del Pezzo realization problem.

That is a genuine shrink of the residual mixed branch.

11.10. The exceptional $\mathbb{F}_2$ class $3S_0 + 8f$ also reduces to marked plane models

The previous subsection reduced every singular trisection to a marked plane cubic / quartic / quintic package *or* to the single exceptional ruled-surface class $3S_0 + 8f \subset \mathbb{F}_2$. That class can be pushed further as well. The only missing ingredient is the class formula for an elementary transform whose center lies on the negative section.

Lemma 11.63 (Elementary transform class formula at a point of the negative section)**.** *Let*

$$\epsilon_p \colon \mathbb{F}_n \dashrightarrow \mathbb{F}_{n+1}$$

be the elementary transform at a point $p \in S_0$. Let $C \subset \mathbb{F}_n$ be an irreducible curve of class $C \equiv aS_0 + bf$, and write $m := \operatorname{mult}_p(C)$. If $C^+ \subset \mathbb{F}_{n+1}$ is the image of C, then $C^+ \equiv aS_0^+ + (b+a-m)f^+$, where $(S_0^+)^2 = -(n+1)$ is the negative section of $\mathbb{F}_{n+1}$ and f^+ is a fiber of its ruling.

Proof. Let

$$\sigma \colon Z := \operatorname{Bl}_p \mathbb{F}_n \to \mathbb{F}_n$$

be the blowup of p, let $E \subset Z$ be the exceptional curve, let F be the fiber of $\mathbb{F}_n$ through p, and let $F' \subset Z$ be the strict transform of F. Because $p \in S_0$, the strict transform of S_0 is $\widetilde{S}_0 = \sigma^* S_0 - E$ and has self-intersection $\widetilde{S}_0^2 = -(n+1)$. Contracting F' therefore gives a morphism $\tau \colon Z \to \mathbb{F}_{n+1}$ realizing the elementary transform, and the image of $\widetilde{S}_0$ is the negative section S_0^+.

As in Lemma 11.26, $\tau^* f^+ = \sigma^* f$. Also $\tau^* S_0^+ = \widetilde{S}_0 = \sigma^* S_0 - E$. The strict transform of C on Z is $\widetilde{C} = \sigma^* C - mE$. Since $\widetilde{C} \cdot F' = (aS_0 + bf) \cdot f - m = a - m$, one has $\tau^* C^+ = \widetilde{C} + (a-m)F'$. Using $F' = \sigma^* f - E$, this becomes

$$\tau^* C^+ = a\sigma^* S_0 + b\sigma^* f - mE + (a-m)(\sigma^* f - E) = a\sigma^* S_0 + (b+a-m)\sigma^* f - aE.$$

On the other hand,

$$a\,\tau^* S_0^+ + (b+a-m)\,\tau^* f^+ = a(\sigma^* S_0 - E) + (b+a-m)\sigma^* f = a\sigma^* S_0 + (b+a-m)\sigma^* f - aE.$$

Hence

$$\tau^*C^+ = a\,\tau^*S_0^+ + (b+a-m)\,\tau^*f^+,$$

so $C^+ \equiv aS_0^+ + (b+a-m)f^+$. ☐

Proposition 11.64 (The exceptional class $3S_0 + 8f$ is not a separate ruled-surface residue)**.** *Let $C \subset \mathbb{F}_2$ be an irreducible rational curve of class $C \equiv 3S_0 + 8f$. Then one of the following holds.*

(i) *There exists a singular point $p \in C - S_0$. Writing $m := \mathrm{mult}_p(C) \in \{2,3\}$, the elementary transform at p sends C to a curve $C_1 \subset \mathbb{F}_1$ of class $C_1 \equiv 3S_1 + (8-m)f$. After contracting the negative section $\kappa\colon \mathbb{F}_1 \to \mathbf{P}^2$, the image is an irreducible rational plane curve $\Gamma \subset \mathbf{P}^2$ of degree $8-m \in \{5,6\}$, with marked point $q := \kappa(S_1)$ of multiplicity $5-m \in \{2,3\}$.*

(ii) *Every singular point of C lies on S_0. Then there is a unique such point $p \in C \cap S_0$, it has multiplicity $\mathrm{mult}_p(C) = 2$, and the elementary transform at p sends C to an irreducible rational curve $C^+ \subset \mathbb{F}_3$ of class $C^+ \equiv 3S_0^+ + 9f^+$. This curve is singular, every singular point of C^+ lies off S_0^+, and after at most two further elementary transforms at singular points off the current negative section one reaches a marked plane cubic / quartic / quintic package.*

In particular, every irreducible rational curve of class $3S_0 + 8f \subset \mathbb{F}_2$ reduces after at most three elementary transforms and one contraction of $\mathbb{F}_1$ to an irreducible rational plane curve $\Gamma \subset \mathbf{P}^2$ of degree $e \in \{3,4,5,6\}$ with marked point q of multiplicity $\mathrm{mult}_q(\Gamma) = e-3$. Projection from q realizes the original degree-3 map from the normalization of C to the base.

Proof. Since $p_a(C) = 8$ while C is rational, the curve C is singular. If there exists a singular point $p \in C - S_0$, then $C \cdot F_p = 3$ for the fiber F_p through p, so $\mathrm{mult}_p(C) \in \{2,3\}$. Applying Lemma 11.26 with

$$a = 3, \qquad b = 8, \qquad m = \mathrm{mult}_p(C)$$

gives $C_1 \equiv 3S_1 + (8-m)f$ on $\mathbb{F}_1$. Let $\kappa\colon \mathbb{F}_1 \to \mathbf{P}^2$ be the contraction of

$$S_1, \qquad q := \kappa(S_1), \qquad H := \kappa^*(\text{line}) = S_1 + f.$$

Then

$$C_1 \equiv 3S_1 + (8-m)f = (8-m)H - (5-m)S_1.$$

Hence the plane image $\Gamma := \kappa(C_1)$ has degree $8-m \in \{5,6\}$ and multiplicity $5-m \in \{2,3\}$ at q. Because C_1 is birational to the rational curve C, the curve Γ is irreducible and rational. The fibers of the ruling on $\mathbb{F}_1$ are the strict transforms of the lines through q, so projection from q realizes the degree-3 map from the normalization of C_1 to the base; since the elementary transform is over the same base, this is also the original degree-3 map from the normalization of C to the base. This proves (i).

Assume now that every singular point of C lies on S_0. Since $C \cdot S_0 = 2$, there can be at most one such point. Because C is singular, there is exactly one singular point $p \in C \cap S_0$. Moreover

$$2 \le \operatorname{mult}_p(C) \le (C \cdot S_0)_p \le C \cdot S_0 = 2,$$

so $\operatorname{mult}_p(C) = 2$.

Applying Lemma 11.63 with

$$a = 3, \qquad b = 8, \qquad m = 2$$

gives $C^+ \equiv 3S_0^+ + 9f^+$ on $\mathbb{F}_3$. Because C^+ is birational to the rational curve C, it is irreducible and rational. Its arithmetic genus is $p_a(C^+) = 3 \cdot 3 + 2 \cdot 0 - 2 = 7$, so C^+ is singular. Also $C^+ \cdot S_0^+ = 0$, hence every singular point of C^+ lies off S_0^+.

Choose a singular point $r \in C^+ - S_0^+$. Then $\operatorname{mult}_r(C^+) \in \{2, 3\}$ because $C^+ \cdot F_r = 3$. Applying Lemma 11.26 at r gives a curve $C_2 \subset \mathbb{F}_2$ of class

$$C_2 \equiv 3S_0'' + (9 - \operatorname{mult}_r(C^+))f''.$$

Thus

$$C_2 \equiv 3S_0'' + 6f'' \qquad \text{or} \qquad C_2 \equiv 3S_0'' + 7f''.$$

Write this uniformly as

$$C_2 \equiv 3S_0'' + (6 + s'')f'', \qquad s'' \in \{0, 1\}.$$

Since $C_2 \cdot S_0'' = s'' \le 1$, every singular point of C_2 lies off S_0''. Choose one and perform one more elementary transform. As in the proof of Proposition 11.59, the resulting curve on $\mathbb{F}_1$ has class

$$3S_1 + (3 + s''')f, \qquad s''' \in \{0, 1, 2\}.$$

Contracting S_1 therefore yields an irreducible rational plane curve of degree $3 + s''' \in \{3, 4, 5\}$ with marked point multiplicity $s''' = (3 + s''') - 3$. Again the final projection from the marked point realizes the original degree-3 map because every elementary transform is performed over the same base. This proves (ii) and the final summary statement. ☐

Corollary 11.65 (The singular trisection branch reduces to a finite marked plane package)**.** *Assume the setup of Corollary 11.57, and suppose that we are in case* (iii)*. Then after at most four elementary transforms and one contraction of* $\mathbb{F}_1$*, the unique trisection reduces birationally to an irreducible rational plane curve* $\Gamma \subset \mathbf{P}^2$ *with a marked point* $q \in \mathbf{P}^2$ *such that*

$$\deg \Gamma \in \{3, 4, 5, 6\}, \qquad \operatorname{mult}_q(\Gamma) = \deg \Gamma - 3.$$

Projection from q *realizes the original degree-*3 *map from the normalization of the trisection to the base.*

In particular, the residual trisection branch is no longer a problem on

$$\mathbb{F}_2 \quad \textit{or} \quad \mathbb{F}_3$$

at all: it is a finite marked plane package of degrees at most 6.

Proof. By Corollary 11.57, the trisection starts in one of the four explicit classes

$$3S_0+6f, \qquad 3S_0+7f, \qquad 3S_0+9f, \qquad 3S_0+10f.$$

Proposition 11.59 shows that after one singular-point elementary transform the branch reduces either to a marked plane cubic / quartic / quintic package or to the single exceptional class $3S_0+8f \subset \mathbb{F}_2$. Applying Proposition 11.64 to that exceptional class finishes the reduction. The count of at most four elementary transforms is immediate: one initial transform may be needed to reach the exceptional $\mathbb{F}_2$ class, and from there Proposition 11.64 uses at most three more. □

Remark 11.66 (What now remains in the trisection target)*.* The trisection branch is now entirely a finite marked plane problem.

Indeed, Corollary 11.65 removes the last separate ruled-surface residue $3S_0+8f \subset \mathbb{F}_2$. So the degree-three mixed branch is no longer “a singular trisection on

$$\mathbb{F}_2 \quad \text{or} \quad \mathbb{F}_3$$

”; it is an irreducible rational plane curve of degree 3, 4, 5, or 6 with a marked point of multiplicity $\deg\Gamma-3$, together with the requirement that projection from that marked point reproduces the original degree-3 map to the base.

Combined with Corollary 11.61, this means that the only genuinely new local freedom left in the trisection target is collision geometry on these degree-≤ 6 marked plane models. The unibranch side is already absorbed by the old touched-tip / branching corridors.

11.11. Collision singularities of the ordinary trisection create no new corridor

After Corollary 11.65, the only local freedom still left on the degree-three mixed side appears to be multibranch collision geometry. In fact even this residual freedom is already absorbed by the old touched-tip / branching side.

Proposition 11.67 (A multibranch point of the ordinary trisection is either impossible or already old)**.** *Assume the setup of Proposition 11.56, and suppose that* $d=3$*. Let* $q\in C$ *be a singular point with at least two local branches. Factor* ν^{-1} *over* q *until those local branches first become disjoint, and let* L_0 *be the last exceptional curve created in that primitive branch-separation phase. Let*

$$x_1,\dots,x_r\in H\cap L_0, \qquad r\ge 2,$$

be the resulting contact points of the strict transform H *of the trisection with* L_0*. For each* i*, let* e_i *be the local degree of the induced map* $H\to\mathbf{P}^1$ *at* x_i*. Then*

$$\sum_{i=1}^{r} e_i \le 3.$$

Moreover, exactly one of the following holds:

(i) *all* $e_i = 1$; *this case is impossible;*

(ii)

$$r = 2 \qquad \textit{and} \qquad \{e_1, e_2\} = \{1, 2\}.$$

*In this case, following the unique degree-*2 *branch to its terminal surviving local block yields either a corner fork or an order-*2 *tangency fork. The same contact-defect argument as in the proof of Proposition 11.60 then produces a vertical component* $E \subset D$ *such that*

$$\mathrm{cf}_X(E) < \frac{1}{3}.$$

Consequently, if the connected component of D *containing* E *is a chain, then its* 2*-tail core is a touched tip of weight* 2; *if that connected component is not a chain, then one is already on the branching side.*

In particular, every surviving collision singularity of the trisection is already absorbed by the old touched-(-2)*-tip / branching geometry.*

Proof. Because the trisection has horizontal degree $H \cdot f = 3$, the sum of the local degrees of $H \to \mathbf{P}^1$ over any fixed fiber is at most 3. The points $x_1, \ldots, x_r$ all lie on the same fiber through q, so

$$\sum_{i=1}^{r} e_i \leq 3.$$

Assume first that

$$e_i = 1 \qquad (1 \leq i \leq r).$$

Then at each point x_i the strict transform H is smooth and transverse to the current fiber direction. Hence the local embedded-resolution process over q is already finished at those points, so no later center of ν^{-1} over q lies on any x_i. Let L be the final strict transform of L_0 on $\widetilde{X}$. Then $L \cdot H = r \geq 2$.

If $L \subset D$, then the connected component of D containing H would contain two distinct intersection points between the same pair of components

$$H \quad \text{and} \quad L,$$

contradicting the fact that every connected component of D has tree dual graph. Therefore $L \not\subset D$. Since H is ordinary, Lemma 11.3 gives

$$\mathrm{cf}_X(H) \geq \frac{1}{2}.$$

Applying Proposition 9.133 to L now gives

$$1 > \sum_{T \subset D} (L \cdot T)\,\mathrm{cf}_X(T) \geq (L \cdot H)\,\mathrm{cf}_X(H) \geq 2 \cdot \frac{1}{2} = 1,$$

a contradiction. So case (i) is impossible.

Hence some $e_i \geq 2$. Because

$$r \geq 2 \qquad \text{and} \qquad \sum_{i=1}^{r} e_i \leq 3,$$

the only remaining possibility is

$$r = 2 \qquad \text{and} \qquad \{e_1, e_2\} = \{1, 2\}.$$

After renumbering, assume

$$e_1 = 2, \qquad e_2 = 1.$$

Follow the degree-2 branch through x_1 to the last local block over q whose last exceptional curve survives on $\widetilde{X}$. Once the other branch has been separated, this is a unibranch local problem of degree 2 over the base. Therefore the same low-degree Euclidean analysis as in Corollary 11.33 and Proposition 11.34 shows that the terminal surviving block is either:

(a) a corner fork as in Proposition 11.37; or

(b) an order-2 tangency fork as in Proposition 11.38.

The proof of Proposition 11.60 uses only these two local fork shapes together with the inequality

$$\mathrm{cf}_X(H) \geq \frac{1}{2}.$$

So the same argument applies verbatim here and yields a vertical component $E \subset D$ such that

$$\mathrm{cf}_X(E) < \frac{1}{3}.$$

The final conclusion is then exactly Lemma 11.45 in the chain case; if the connected component of D containing E is not a chain, then one is already on the branching side. □

Corollary 11.68 (The trisection branch creates no genuinely new local geometry)**.** *Assume the setup of Corollary 11.57, and suppose that we are in case* (iii)*. Then every singularity of the unique trisection is already absorbed by the old side types isolated earlier in the note:*

(i) *unibranch points by Corollary 11.61;*

(ii) *multibranch points by Proposition 11.67.*

Hence the degree-three mixed branch contributes no genuinely new local obstruction beyond the old touched-(-2)-tip / branching geometry.

Proof. This is exactly the combination of Corollary 11.61 and Proposition 11.67. □

Remark 11.69 (What the trisection collision closure leaves at this stage)*.* Corollary 11.68 closes the last genuinely new local gap on the degree-three mixed side.

Indeed, Corollary 11.65 had already reduced the trisection branch to a finite marked plane package, and Corollary 11.61 had already absorbed the unibranch part into the old touched-tip

/ branching corridors. Proposition 11.67 removes the remaining multibranch freedom as well. The following subsection performs the analogous closure for the degree-two mixed branch.

11.12. The ordinary bisection creates no genuinely new local corridor

After the trisection closure, the only separate local residue still visible on the low-degree mixed side is the degree-two branch from case (ii) of Corollary 11.57. In fact this residual branch is even smaller than the trisection branch: any surviving unibranch block is already old, and a multibranch collision of the ordinary bisection is impossible.

Proposition 11.70 (Every surviving unibranch bisection block already has an old special vertical side)**.** *Assume the setup of Corollary 11.57, and suppose that we are in case* (ii)*. Let $q \in C$ be a unibranch singular point of the unique ordinary bisection. Assume that, in the factorization of ν^{-1} over q, the last exceptional curve created in the terminal block survives on $\widetilde{X}$. Then there exists a vertical component $E \subset D$ adjacent to that surviving last exceptional curve such that*

$$\mathrm{cf}_X(E) < \frac{1}{3}.$$

Consequently, if the connected component of D containing E is a chain, then the corresponding 2*-tail core is a touched tip of weight* 2; *if that connected component is not a chain, then one is already on the branching side.*

Proof. Let $H \subset D_{\mathrm{hor}}$ be the unique bisection. In the low-degree mixed branch it is ordinary, so Lemma 11.3 gives

$$\mathrm{cf}_X(H) \geq \frac{1}{2}.$$

By Lemma 11.23, the local degree of the normalization map $\overline{C} \to \mathbf{P}^1$ at the unique preimage of q is 2. Therefore Corollary 11.35 shows that the terminal surviving block over q is one of the following two types:

(i) a corner fork

$$\begin{array}{ccccc} & & A & & \\ & & | & & \\ H & - & L & - & B; \end{array}$$

(ii) an order-2 tangency fork $H - L - A - B_1$.

In both cases

$$L \not\subset D \qquad \text{and} \qquad L \cdot D = 3.$$

Assume first that the terminal block is the tangency fork from Proposition 11.38. Let $E := B_1$. Then $L \cdot H = L \cdot A = L \cdot E = 1$. By Proposition 9.133,

$$\mathrm{cf}_X(H) + \mathrm{cf}_X(A) + \mathrm{cf}_X(E) < 1.$$

Also Proposition 11.38 gives $A^2 \leq -3$, so Lemma 9.132 yields

$$\mathrm{cf}_X(A) \geq \frac{1}{3}.$$

Together with

$$\mathrm{cf}_X(H) \geq \frac{1}{2},$$

this gives

$$\mathrm{cf}_X(E) < 1 - \frac{1}{2} - \frac{1}{3} = \frac{1}{6} < \frac{1}{3}.$$

Assume next that the terminal block is the corner fork from Proposition 11.37. Then $L \cdot H = L \cdot A = L \cdot B = 1$, so Proposition 9.133 gives

$$\mathrm{cf}_X(H) + \mathrm{cf}_X(A) + \mathrm{cf}_X(B) < 1.$$

Using

$$\mathrm{cf}_X(H) \geq \frac{1}{2},$$

one gets

$$\mathrm{cf}_X(A) + \mathrm{cf}_X(B) < \frac{1}{2}.$$

Hence at least one of A, B has discrepancy coefficient strictly smaller than

$$\frac{1}{4} < \frac{1}{3}.$$

Take that component as E.

The final conclusion is exactly Lemma 11.45 in the chain case; if the connected component of D containing E is not a chain, then one is already on the branching side. □

Proposition 11.71 (A multibranch point of the ordinary bisection is impossible)**.** *Assume the setup of Corollary 11.57, and suppose that we are in case* (ii)*. Let $q \in C$ be a singular point with at least two local branches. Factor ν^{-1} over q until those local branches first become disjoint, and let L_0 be the last exceptional curve created in that primitive branch-separation phase. Let*

$$x_1, \dots, x_r \in H \cap L_0, \qquad r \geq 2,$$

be the resulting contact points of the strict transform H of the bisection with L_0. For each i, let e_i be the local degree of the induced map $H \to \mathbf{P}^1$ at x_i. Then

$$\sum_{i=1}^{r} e_i \leq 2,$$

hence necessarily

$$r = 2 \qquad \text{and} \qquad e_1 = e_2 = 1.$$

This case is impossible. In particular, the ordinary bisection has no surviving multibranch collision singularity.

Proof. Because the bisection has horizontal degree $H \cdot f = 2$, the sum of the local degrees of $H \to \mathbf{P}^1$ over any fixed fiber is at most 2. The points $x_1, \dots, x_r$ all lie on the same fiber through q, so

$$\sum_{i=1}^{r} e_i \le 2.$$

Since $r \ge 2$ and each $e_i \ge 1$, one must have

$$r = 2 \qquad \text{and} \qquad e_1 = e_2 = 1.$$

Thus both separated branches are smooth and transverse to the current fiber direction at their contact points with L_0. No later center of ν^{-1} over q can lie on either of those two points. Let L be the final strict transform of L_0 on $\widetilde{X}$. Then $L \cdot H = 2$.

If $L \subset D$, then the connected component of D containing H would contain two distinct intersection points between the same pair of components

$$H \quad \text{and} \quad L,$$

contradicting the fact that every connected component of D has tree dual graph. Therefore $L \not\subset D$.

Since H is ordinary, Lemma 11.3 gives

$$\mathrm{cf}_X(H) \ge \frac{1}{2}.$$

Applying Proposition 9.133 to L now gives

$$1 > \sum_{T \subset D} (L \cdot T)\, \mathrm{cf}_X(T) \ge (L \cdot H)\, \mathrm{cf}_X(H) \ge 2 \cdot \frac{1}{2} = 1,$$

a contradiction. ☐

Corollary 11.72 (The bisection branch creates no genuinely new local geometry)**.** *Assume the setup of Corollary 11.57, and suppose that we are in case* (ii)*. Then every singularity of the unique ordinary bisection is either impossible or already absorbed by the old side types isolated earlier in the note:*

(i) *any unibranch singularity by Proposition 11.70;*

(ii) *any multibranch singularity is impossible by Proposition 11.71.*

In particular, the degree-two mixed branch contributes no genuinely new local obstruction beyond the old touched-(-2)-tip / branching geometry.

Proof. This is exactly the combination of Proposition 11.70 and Proposition 11.71. ☐

Corollary 11.73 (The low-degree mixed branch creates no separate local corridor)**.** *Assume the setup of Corollary 11.51, and suppose that we are in the low-degree mixed branch. Then* *either the chosen witnessing fibration already lies in the several-special-horizontal corridor from Remark 11.15, or every singularity of the unique ordinary multisection is already absorbed by the old touched-(-2)-tip / branching geometry, or impossible. In particular, after the global low-degree reduction there is no separate bisection / trisection local corridor left.*

Proof. Apply Corollary 11.57. In case (i) one is already in the several-special-horizontal corridor. In case (ii), apply Corollary 11.72. In case (iii), apply Corollary 11.68. ☐

Remark 11.74 (What now remains globally after closing the mixed branch)*.* Corollary 11.73 removes the last separate local corridor on the one-section mixed side.

Indeed, the bisection branch no longer contributes a marked-conic / marked-cubic residue, and the trisection branch no longer contributes a separate collision residue. So the residual global problems no longer include any distinct low-degree mixed local package. What remains is the special-multisection side together with the non-clean purely inseparable service patterns left over from the cubic-backbone analysis; the next subsection returns to the height-4, maximal-width two-touch endgame and closes it.

11.13. Returning to the maximal-width two-touch problem: packet roots and the rigid four-section hub

After the one-section multisection reductions, it is natural to return to the genuinely section-only part of Conjecture 10.1, namely the height-4, maximal-width one-touch problem

$$v(F) > 0, \qquad t(F) = 1.$$

The next two results extract two complementary pieces of structure from that corridor. First, after minimizing a simple branch-length invariant, every one-touch bad branch is forced to start on the horizontal packet itself on the clean-section ruled model. Second, if a single vertical component carries all four horizontal sections, then that component is already rigid and all four sections are special. Together they show that the remaining two-touch obstruction is no longer an arbitrary vertical-tree phenomenon.

Definition 11.75 (One-touch branch length)**.** Assume the setup of Corollary 9.13, and let $p\colon \widetilde{X} \to \mathbf{P}^1$ be a witnessing fibration of height 4 and maximal width. For a degenerate fiber F and a connected component $U \subset D_F$ disjoint from D_{hor}, let $\Gamma(U) \subset F_{\mathrm{red}}$ be the unique chain joining U to the unique connected component of D_F meeting D_{hor} whenever $t(F) = 1$. Define

$$\ell(U) := \#(\Gamma(U) - U).$$

If

$$\mathcal{V}(F) := \{U \subset D_F\,;\, U \cap D_{\mathrm{hor}} = \varnothing\}$$

is the set of purely vertical connected components of D_F, put

$$\lambda(p) := \sum_F \sum_{U \in \mathcal{V}(F)} \ell(U).$$

Lemma 11.76 (Free first centers are removable by an elementary transform)**.** *Let $Y_0 \to \mathbf{P}^1$ be a smooth $\mathbf{P}^1$-bundle, let $f_0 \subset Y_0$ be a fiber, and let $\Sigma_0, \dots, \Sigma_r \subset Y_0$ be sections. Assume that a smooth surface Y is obtained from Y_0 by a sequence of blowups whose centers lie over f_0. Suppose that one blowup branch begins with a free center $q_1 \in f_0$ which is disjoint from $\Sigma_0 \cup \cdots \cup \Sigma_r$. Then there exists another $\mathbf{P}^1$-bundle $Y_0' \to \mathbf{P}^1$ obtained from Y_0 by the elementary transform at q_1, and a factorization of Y as a sequence of blowups over the image fiber $f_0' \subset Y_0'$ with the following properties:*

(i) *the images of $\Sigma_0, \dots, \Sigma_r$ on Y_0' are again sections;*

(ii) *every descendant of the branch issued from q_1 is still realized over f_0', but with the initial free step removed;*

(iii) *no blowup branch over any other fiber is changed.*

Proof. Blow up q_1: $\alpha\colon Y_1 \to Y_0$. Let E_1 be the exceptional curve and let f_1 be the strict transform of f_0. Since q_1 is free, one has $E_1^2 = f_1^2 = -1$ on Y_1, and the elementary transform at q_1 is the contraction of f_1: $\beta\colon Y_1 \to Y_0'$. Because q_1 is disjoint from the sections Σ_i, their strict transforms on Y_1 are disjoint from f_1, hence descend to sections on Y_0'.

In local coordinates (x, y) near q_1 with $f_0 = \{x = 0\}$, the blowup at q_1 followed by the contraction of f_1 is the standard elementary transform of the ruled surface. Its effect on the cluster over f_0 is to replace the total transform $f_1 + E_1$ by the new smooth fiber $f_0' = \beta(E_1)$, while preserving the infinitely near data away from the contracted curve f_1. Hence every later center on the branch issued from q_1 descends to a center over f_0', with one fewer initial proximity step. Centers lying over other fibers are untouched.

Iterating this observation along the descendants of q_1 gives a blowup factorization of the same final smooth surface Y over Y_0' in which the whole q_1-branch has lost its first free step, while all other fibers are unchanged. ☐

Proposition 11.77 (In a λ-minimal maximal-width witness, every one-touch bad branch starts on the horizontal packet)**.** *Assume the setup of Corollary 9.13, and among all height-4, maximal-width witnessing fibrations on the fixed smooth surface $\widetilde{X}$ choose one minimizing $\lambda(p)$ in the sense of Definition 11.75. Choose a clean section $H_0 \subset D_{\mathrm{hor}}$ and let $\upsilon\colon \widetilde{X} \to \mathbb{F}_n$ be the ruled model from Proposition 9.16. Write $S_i := \upsilon(H_i), \qquad i = 0, 1, 2, 3.$*

Let F be a degenerate fiber with $v(F) > 0, \qquad t(F) = 1$, and let $U \subset D_F$ be a purely vertical connected component. Fix a relative minimal model of p in which the image of F is a smooth fiber $f \subset Y_0$, and let $q_1 \prec q_2 \prec \cdots \prec q_m$ be the chain of blowup centers over f whose descendants produce the branch from the unique touched component of D_F to U. Then some q_j lies on the total transform of the horizontal divisor $S_0 + S_1 + S_2 + S_3$. If j is minimal with this property, then exactly one of the following holds:

(i) q_j *lies on a unique local branch of the horizontal packet; equivalently, it is a packet-section point (possibly infinitely near) on the marked fiber;*

(ii) q_j *is a singular point of the total transform of the horizontal packet; equivalently, it is a collision point of at least two horizontal branches, possibly infinitely near.*

Proof. Assume first that none of the centers $q_1, \dots, q_m$ lies on the total transform of $S_0+S_1+S_2+S_3$. Then the first center $q_1 \in f$ is a free point of the smooth fiber f, disjoint from the horizontal packet. By Lemma 11.76, the elementary transform at q_1 produces another ruled model whose pullback to $\widetilde{X}$ is again a height-4, maximal-width witnessing fibration with the same horizontal 1-sections, but for which this branch has lost its first free step. Hence $\lambda(p') < \lambda(p)$, contradicting the minimality of $\lambda(p)$. Therefore some q_j lies on the total transform of the horizontal packet.

Let j be minimal with this property. If the total transform of $S_0 + S_1 + S_2 + S_3$ is smooth at q_j, then q_j lies on a unique local branch of the packet, giving (i). If the total transform is singular at q_j, then q_j is by definition a collision point of at least two local horizontal branches, giving (ii). ☐

Corollary 11.78 (In a λ-minimal maximal-width witness, the first center of every one-touch bad branch is already on the packet)**.** *Keep the assumptions and notation of Proposition 11.77. Then the first center* q_1 *itself lies on the horizontal packet* $S_0 + S_1 + S_2 + S_3$*. Equivalently, every one-touch bad branch starts at the first blowup either from a packet-section point or from a collision point of horizontal branches.*

Proof. If q_1 were disjoint from the horizontal packet, then q_1 would be a free point of the smooth fiber f, disjoint from all sections. Lemma 11.76 would then produce another ruled model and hence another height-4, maximal-width witnessing fibration on the same smooth surface in which this branch loses its initial free step. Thus $\lambda(p)$ would decrease, contradicting the minimality assumption. Therefore $q_1 \in S_0 + S_1 + S_2 + S_3$. The final alternative is exactly the smooth-versus-singular dichotomy for the total transform of the packet at q_1. ☐

Remark 11.79 (What the λ-minimality reduction actually does)*.* Proposition 11.77 does not by itself prove the two-touch condition $v(F) > 0 \implies t(F) \geq 2$. What it does show is that, once one is already in the height-4, maximal-width corridor and has minimized $\lambda(p)$, every one-touch bad branch is rooted in the horizontal packet on the ruled model. So the remaining maximal-width obstruction is no longer an abstract fiber-tree phenomenon: it is forced into the same packet-section-point and horizontal-collision mechanisms that occur elsewhere in the clean-section Hirzebruch analysis.

Proposition 11.80 (A vertical component meeting four horizontal sections is rigid)**.** *Assume the setup of Corollary 9.13, so that* $D_{\mathrm{hor}} = H_0 + H_1 + H_2 + H_3$ *is the union of four horizontal* 1*-sections. Let* $U \subset D$ *be a vertical irreducible component meeting all four sections* H_0, H_1, H_2, H_3*. Then the following hold.*

(i) U *has no vertical neighbor in* D*.*

(ii) $U^2 = -2$*.*

(iii) *Writing*

$$\pi^* K_X = K_{\widetilde{X}} + \sum_{E \subset D} \mathrm{cf}_X(E)\, E,$$

one has

$$\mathrm{cf}_X(H_i) = \frac{\mathrm{cf}_X(U)}{2} \qquad (0 \le i \le 3).$$

In particular,

$$\mathrm{cf}_X(H_i) < \frac{1}{2} \qquad (0 \le i \le 3),$$

so every H_i is nonordinary, equivalently every H_i is a special horizontal section.

Proof. Write

$$U^2 = -b, \qquad H_i^2 = -a_i \qquad (0 \le i \le 3).$$

Since $U, H_i \subset D$ and π is the minimal resolution, one has

$$b \ge 2, \qquad a_i \ge 2.$$

Let

$$\pi^* K_X = K_{\widetilde{X}} + \sum_{E \subset D} \mathrm{cf}_X(E)\, E, \qquad \mathrm{cf}_X(E) = 1 - \mu_X(E).$$

Intersecting this identity with H_i gives

$$a_i \ \mathrm{cf}_X(H_i) = a_i - 2 + \sum_{E \sim H_i} \mathrm{cf}_X(E) \ge a_i - 2 + \mathrm{cf}_X(U),$$

where the sum runs over the components of D meeting H_i. Hence

$$\mathrm{cf}_X(H_i) \ge 1 - \frac{2 - \mathrm{cf}_X(U)}{a_i} \ge 1 - \frac{2 - \mathrm{cf}_X(U)}{2} = \frac{\mathrm{cf}_X(U)}{2} \qquad (0 \le i \le 3).$$

Since the fibration is section-only, every H_i is a 1-section, so Lemma 9.108 gives

$$2 > \sum_{H \subset D_{\mathrm{hor}}} \mathrm{cf}_X(H) \ge \sum_{i=0}^{3} \mathrm{cf}_X(H_i) \ge 2 \ \mathrm{cf}_X(U).$$

Hence $\mathrm{cf}_X(U) < 1$.

Now let $V_1, \ldots, V_s$ be the vertical neighbors of U in the dual graph of D. Intersecting the discrepancy formula with U yields

$$b \ \mathrm{cf}_X(U) - \sum_{i=0}^{3} \mathrm{cf}_X(H_i) - \sum_{j=1}^{s} \mathrm{cf}_X(V_j) = b - 2.$$

Using

$$\sum_{i=0}^{3} \mathrm{cf}_X(H_i) \ge 2 \ \mathrm{cf}_X(U)$$

and the nonnegativity of the $\mathrm{cf}_X(V_j)$, we get $(b-2)\,\mathrm{cf}_X(U) \ge b - 2$. If $b > 2$, then dividing by

$b-2>0$ gives $\mathrm{cf}_X(U) \geq 1$, contradicting $\mathrm{cf}_X(U) < 1$. Therefore $b=2$, which proves (ii).

With $b=2$, the discrepancy equation at U becomes

$$2\ \mathrm{cf}_X(U) = \sum_{i=0}^{3} \mathrm{cf}_X(H_i) + \sum_{j=1}^{s} \mathrm{cf}_X(V_j).$$

But we already know

$$\sum_{i=0}^{3} \mathrm{cf}_X(H_i) \geq 2\ \mathrm{cf}_X(U).$$

Hence equality must hold throughout, so

$$\sum_{j=1}^{s} \mathrm{cf}_X(V_j) = 0 \qquad \text{and} \qquad \sum_{i=0}^{3} \mathrm{cf}_X(H_i) = 2\ \mathrm{cf}_X(U).$$

Since each discrepancy coefficient is nonnegative, the first equality implies

$$\mathrm{cf}_X(V_j) = 0 \qquad (1 \leq j \leq s).$$

Assume for contradiction that $s \geq 1$. Intersecting the discrepancy formula with some V_j gives

$$(-V_j^2)\ \mathrm{cf}_X(V_j) - \mathrm{cf}_X(U) - \sum_{W \sim V_j,\ W \neq U} \mathrm{cf}_X(W) = (-V_j^2) - 2.$$

Since $\mathrm{cf}_X(V_j) = 0$, the left-hand side is at most $-\mathrm{cf}_X(U) < 0$, while the right-hand side is nonnegative because $-V_j^2 \geq 2$. This is impossible. Therefore $s = 0$, proving (i).

Finally, with $s=0$ and

$$\sum_{i=0}^{3} \mathrm{cf}_X(H_i) = 2\ \mathrm{cf}_X(U),$$

the lower bounds

$$\mathrm{cf}_X(H_i) \geq \frac{\mathrm{cf}_X(U)}{2} \qquad (0 \leq i \leq 3)$$

force equality term-by-term:

$$\mathrm{cf}_X(H_i) = \frac{\mathrm{cf}_X(U)}{2} \qquad (0 \leq i \leq 3).$$

Because $\mathrm{cf}_X(U) < 1$, one has

$$\mathrm{cf}_X(H_i) < \frac{1}{2} \qquad (0 \leq i \leq 3).$$

By Lemma 9.107, an ordinary horizontal section has discrepancy coefficient at least

$$\frac{1}{2}.$$

So every H_i is nonordinary. By Definition 9.105, this means every H_i is special. This proves (iii). □

Proposition 11.81 (The rigid four-section hub cannot occur)**.** *Assume the setup of Corollary 9.13, so that* $D_{\mathrm{hor}} = H_0+H_1+H_2+H_3$ *is the union of four horizontal* 1*-sections. Then there does not exist a vertical irreducible component* $U \subset D$ *meeting all four sections* H_0, H_1, H_2, H_3*.*

Proof. Assume for contradiction that such a component U exists. By Proposition 11.80, one has $U^2 = -2$, the component U has no vertical neighbor in D, and

$$\mathrm{cf}_X(H_i) = \frac{\mathrm{cf}_X(U)}{2} < \frac{1}{2} \qquad (0 \le i \le 3).$$

Write

$$H_i^2 = -a_i, \qquad a_i \ge 2.$$

We first claim that $\mathrm{cf}_X(U) > 0$. Indeed, if $\mathrm{cf}_X(U) = 0$, then also

$$\mathrm{cf}_X(H_i) = 0 \qquad (0 \le i \le 3).$$

Intersecting the discrepancy formula with H_i gives

$$a_i\, \mathrm{cf}_X(H_i) - \mathrm{cf}_X(U) - \sum_{V \sim H_i,\ V \ne U} \mathrm{cf}_X(V) = a_i - 2,$$

hence

$$-\sum_{V \sim H_i,\ V \ne U} \mathrm{cf}_X(V) = a_i - 2.$$

Therefore

$$a_i = 2 \qquad \text{and} \qquad \mathrm{cf}_X(V) = 0$$

for every neighbor V of H_i distinct from U. Repeating the same argument along the connected component of D containing U shows that every component in that connected component has discrepancy coefficient 0 and self-intersection -2. So the corresponding singularity is canonical, hence Du Val, and its exceptional graph must be of ADE type. But the component U has valence 4, impossible for an ADE graph. This contradiction proves $\mathrm{cf}_X(U) > 0$.

Now fix $i \in \{0, 1, 2, 3\}$ and let $V_{i,1}, \dots, V_{i,s_i}$ be the vertical neighbors of H_i distinct from U. Intersecting the discrepancy formula with H_i yields

$$a_i\, \mathrm{cf}_X(H_i) - \mathrm{cf}_X(U) - \sum_{j=1}^{s_i} \mathrm{cf}_X(V_{i,j}) = a_i - 2.$$

Substituting

$$\mathrm{cf}_X(H_i) = \frac{\mathrm{cf}_X(U)}{2}$$

gives

$$\sum_{j=1}^{s_i} \mathrm{cf}_X(V_{i,j}) = (a_i - 2)\Big(\frac{\mathrm{cf}_X(U)}{2} - 1\Big).$$

The left-hand side is nonnegative, while the factor

$$\frac{\mathrm{cf}_X(U)}{2} - 1$$

is strictly negative because $0 < \mathrm{cf}_X(U) < 1$. Hence necessarily

$$a_i = 2 \qquad \text{and} \qquad \sum_{j=1}^{s_i} \mathrm{cf}_X(V_{i,j}) = 0.$$

Thus $H_i^2 = -2$, and every $V_{i,j}$ has discrepancy coefficient 0.

Assume some $V_{i,j}$ exists. Intersecting the discrepancy formula with $V := V_{i,j}$ gives

$$(-V^2)\,\mathrm{cf}_X(V) - \mathrm{cf}_X(H_i) - \sum_{W\sim V,\ W\neq H_i} \mathrm{cf}_X(W) = (-V^2) - 2.$$

Since

$$\mathrm{cf}_X(V) = 0 \qquad \text{and} \qquad \mathrm{cf}_X(H_i) = \frac{\mathrm{cf}_X(U)}{2} > 0,$$

the left-hand side is strictly negative, while the right-hand side is nonnegative because $-V^2 \geq 2$. This contradiction shows that $s_i = 0$ for every i. So each H_i is a leaf, and the connected component of D containing U is exactly the five-vertex star with center U and leaves H_0, H_1, H_2, H_3, all of self-intersection -2.

Its intersection matrix is

$$\begin{pmatrix} -2 & 1 & 1 & 1 & 1 \\ 1 & -2 & 0 & 0 & 0 \\ 1 & 0 & -2 & 0 & 0 \\ 1 & 0 & 0 & -2 & 0 \\ 1 & 0 & 0 & 0 & -2 \end{pmatrix},$$

whose determinant is 0. This contradicts the negative definiteness of the intersection matrix of the exceptional divisor over a surface singularity. Therefore no such component U exists. □

Corollary 11.82 (Rigid all-special hub versus distributed-anchor failure)**.** *Assume the setup of Corollary 9.13, and among all height-4, maximal-width witnessing fibrations on the fixed smooth surface choose one minimizing $\lambda(p)$. Let F be a degenerate fiber with*

$$v(F) > 0, \qquad t(F) = 1.$$

Then exactly one of the following holds.

(i) *There exists a vertical irreducible component $U \subset D_F$ meeting all four horizontal sections. Then U is the unique component of D_F meeting D_{hor}, one has $U^2 = -2$, the component U has no vertical neighbor in D, and all four horizontal sections are special.*

(ii) *No vertical component of D_F meets all four horizontal sections. In this case, for every* *purely vertical connected component*

$$W \subset D_F, \qquad W \cap D_{\mathrm{hor}} = \varnothing,$$

the branch from the unique touched component of D_F to W is rooted, in the sense of Proposition 11.77, at either a packet-section point or a collision point of the horizontal packet on the clean-section ruled model.

Proof. If some vertical component $U \subset D_F$ meets all four horizontal sections, then Proposition 11.80 gives $U^2 = -2$ and shows that U has no vertical neighbor in D and that all four sections are special. Because $t(F) = 1$, the component U is then automatically the unique connected component of D_F meeting D_{hor}. This is alternative (i).

If no vertical component of D_F meets all four horizontal sections, then alternative (ii) is exactly Proposition 11.77 applied to each purely vertical connected component of D_F. □

Corollary 11.83 (Only the distributed-anchor alternative remains)**.** *Keep the setup of Corollary 11.82. Then alternative* (i) *of that corollary cannot occur. Equivalently, every one-touch bad fiber is already a distributed-anchor configuration on the clean-section ruled model.*

More precisely, for every purely vertical connected component

$$W \subset D_F, \qquad W \cap D_{\mathrm{hor}} = \varnothing,$$

the associated bad branch starts at the first blowup center over the marked fiber, and that first center is either

(i) *a packet-section point of the horizontal packet, or*

(ii) *a collision point of at least two horizontal branches.*

Proof. By Proposition 11.81, no vertical component of D_F can meet all four horizontal sections. So alternative (i) of Corollary 11.82 is impossible. Hence only alternative (ii) remains. The final description of the first center is exactly Corollary 11.78. □

Lemma 11.84 (Elementary transform at a packet-section point)**.** *Let $Y_0 \to \mathbf{P}^1$ be a smooth $\mathbf{P}^1$-bundle, let $f_0 \subset Y_0$ be a fiber, and let $\Sigma_0, \ldots, \Sigma_r \subset Y_0$ be sections. Assume that $q \in \Sigma_0 \cap f_0$ is a point through which no other section passes. Blow up q and denote by $\alpha\colon Y_1 \to Y_0$ the blowup, by $E \subset Y_1$ the exceptional curve, and by $f_1 \subset Y_1$ the strict transform of f_0. Let $\beta\colon Y_1 \to Y_0'$ be the contraction of f_1. Then:*

(i) *$Y_0' \to \mathbf{P}^1$ is again a smooth $\mathbf{P}^1$-bundle;*

(ii) *the images of $\Sigma_0, \ldots, \Sigma_r$ on Y_0' are sections;*

(iii) *the image $f_0' := \beta(E)$ is a smooth fiber;*

(iv) *the point*

$$\alpha^{-1}(q) \cap \Sigma_0^{\mathrm{str}} = E \cap \Sigma_0^{\mathrm{str}}$$

descends to the packet-section point $f_0' \cap \beta(\Sigma_0^{\mathrm{str}})$*;*

(v) *every point of* E *different from* $E \cap \Sigma_0^{\mathrm{str}}$ *descends to a free point of the fiber* f_0' *disjoint from all section images.*

Proof. Because q lies on the smooth fiber f_0, one has $E^2 = f_1^2 = -1$ on Y_1, so contracting f_1 gives another smooth $\mathbf{P}^1$-bundle $Y_0' \to \mathbf{P}^1$. Since no section other than Σ_0 passes through q, the strict transforms of all sections are disjoint from f_1, hence descend to sections on Y_0'. The curve E is sent to a smooth fiber $f_0' := \beta(E)$. Moreover Σ_0^{str} meets E transversely in one point, and this point becomes exactly the intersection of its image section with the new fiber f_0'. Any other point of E is disjoint from all section strict transforms and therefore descends to a smooth point of the fiber f_0' away from every section image. ☐

Lemma 11.85 (Local packet-section staircase)**.** *Let* $Y_0 \to \mathbf{P}^1$ *be a smooth* $\mathbf{P}^1$*-bundle, let* $f_0 \subset Y_0$ *be a fiber, and let* $\Sigma \subset Y_0$ *be a section meeting* f_0 *at a point* q_1 *through which no other section passes. For an integer* $a \geq 1$*, construct successively points* $q_1 \prec q_2 \prec \cdots \prec q_a$ *by declaring that for each* $1 \leq k < a$*, the point* q_{k+1} *is the intersection of the strict transform of* Σ *with the newest exceptional curve created by blowing up* q_k*. Let* $\tau_a \colon Y_a \to Y_0$ *be the resulting composition of blowups. Then the reduced fiber over* f_0 *on* Y_a *is a chain* $F - G_1 - \cdots - G_a$*, where*

$$F^2 = -1, \qquad G_i^2 = -2 \ \ (1 \leq i < a), \qquad G_a^2 = -1,$$

and the strict transform of Σ *meets* G_a *transversely at a point away from* $G_{a-1} \cap G_a$*.*

Proof. For $a = 1$, blowing up $q_1 = \Sigma \cap f_0$ produces two (-1)-curves in the fiber: the strict transform F of f_0 and the exceptional curve G_1, with Σ^{str} meeting G_1 transversely away from $F \cap G_1$. Assume the statement proved for $a - 1$. After the first $a - 1$ blowups, the point q_a lies on the intersection of the strict transform of Σ with the last exceptional G_{a-1}. Blowing up q_a creates a new exceptional curve G_a with $G_a^2 = -1$, changes G_{a-1}^2 from -1 to -2, and does not affect the self-intersections of the earlier curves in the chain. The strict transform of Σ now meets G_a transversely away from the rest of the fiber. This gives the displayed chain and self-intersections. ☐

Proposition 11.86 (Packet-section one-touch branches eventually collide or already give easy entry)**.** *Keep the setup of Corollary 11.83, and let*

$$W \subset D_F, \qquad W \cap D_{\mathrm{hor}} = \varnothing,$$

be a purely vertical connected component of a one-touch bad fiber. Assume that the branch from the unique touched connected component of D_F *to* W *starts, in the sense of Corollary 11.78, at a packet-section point on the horizontal packet. Then, after stripping finitely many maximal initial packet-section staircases by the elementary transforms of Lemma 11.84, one reaches exactly one of the following two outcomes.*

(i) *The next center is a collision point of at least two horizontal branches on the resulting ruled model.*

(ii) *No later packet center remains. Then the last exceptional curve of the final packet-section staircase survives on* $\widetilde{X}$ *as a singly-hit multiplicity-one nonexceptional tip whose unique vertical neighbor lies in* D. *Consequently Proposition 9.10 already gives Route C entry.*

In particular, on a residual no-entry counterexample, every packet-section-rooted one-touch bad branch becomes collision-rooted after stripping finitely many initial packet-section staircases.

Proof. Start with the given packet-section-rooted branch. On the current ruled model, let $q_1 \prec q_2 \prec \cdots \prec q_m$ be the corresponding chain of blowup centers over the marked fiber, and let $a \geq 1$ be maximal such that $q_1, \ldots, q_a$ are the successive packet-section points described in Lemma 11.85. By that lemma, the first a centers create a local chain $F - G_1 - \cdots - G_a$, where the strict transform of the chosen packet section meets G_a and no other component of this chain.

Apply the packet-section elementary transform of Lemma 11.84 successively at the points $q_1, \ldots, q_a$. This absorbs the whole initial staircase into a new ruled model with the same horizontal packet. If there is a next center, it descends to a point of the new smooth fiber. Because of the maximality of a, this point is not the packet-section point of the same horizontal branch. Hence there are only three possibilities for the next center on the new ruled model:

(a) it is a free point of the fiber, disjoint from the packet;

(b) it is a packet-section point of some *other* horizontal branch;

(c) it is a collision point of at least two horizontal branches.

Case (a) is impossible in a λ-minimal witness, by Lemma 11.76. So if a next center exists, only cases (b) and (c) remain. If case (c) occurs, we are in alternative (i) of the proposition. If case (b) occurs, we restart the same argument from that new packet-section point. Since the branch has only finitely many blowup centers, this iterative stripping process must terminate.

It remains to analyze the terminal noncollision outcome. Suppose the process terminates with a final maximal packet-section staircase and with no later packet center. Let $F - G_1 - \cdots - G_a$ be the corresponding final local staircase block. Then the last exceptional curve G_a is not blown up again and survives on $\widetilde{X}$ as a vertical (-1)-curve $L \not\subset D$. Because the branch under consideration ends at a purely vertical connected component of D_F, one must have $a \geq 2$, since for $a = 1$ this local staircase contains no vertical D-component at all. Thus the unique vertical neighbor of L is $G_{a-1} \subset D$. Every component of this staircase appears in the fiber with multiplicity one, because each blowup is performed at a smooth point of a multiplicity-one component. Moreover the chosen horizontal section meets L once and no other horizontal branch passes through the packet side of this final staircase. Therefore L is a singly-hit multiplicity-one nonexceptional tip whose unique vertical neighbor lies in D. Proposition 9.10 gives Route C entry, proving alternative (ii). □

Corollary 11.87 (The distributed-anchor one-touch problem is collision-rooted)**.** *Keep the setup of Corollary 11.83, and assume that Route C entry does not already occur. Then every one-touch bad branch is collision-rooted after stripping finitely many initial packet-section steps.*

Equivalently, the remaining maximal-width one-touch problem is concentrated in branches whose first nonremovable packet center is a collision point of at least two horizontal branches on the clean-section ruled model.

Proof. By Corollary 11.83, every one-touch bad branch starts either at a packet-section point or at a collision point of the packet. If it starts at a collision point, there is nothing to prove. If it starts at a packet-section point, Proposition 11.86 shows that either Route C entry already occurs or, after stripping finitely many initial packet-section steps by elementary transforms, the branch becomes collision-rooted. ☐

Proposition 11.88 (Collision-rooted bad branches reduce to isolated pair collisions)**.** *Keep the setup of Corollary 11.87, and assume that Route C entry does not already occur. Let*

$$W \subset D_F, \qquad W \cap D_{\mathrm{hor}} = \varnothing,$$

be a purely vertical connected component of a one-touch bad fiber, and follow the unique branch from the touched part of D_F to W. After stripping finitely many initial packet-section staircases as in Corollary 11.87, consider the resulting collision-rooted branch on the clean-section ruled model.

Then, after stripping in addition finitely many initial common collision prefixes *along that branch, one reaches a stage with the following properties:*

(i) *the current center is a collision point through which pass exactly two horizontal branch strict transforms;*

(ii) *no other horizontal branch strict transform passes through that point;*

(iii) *the remaining local block along the chosen bad branch is therefore the explicit order-m isolated pair-collision block of Proposition 9.119 for some $m \geq 1$.*

Equivalently, on a residual no-entry counterexample every one-touch bad branch is obtained from an isolated pair collision by adjoining only packet-section staircases and earlier peeled-off horizontal branches on the preceding local exceptional chain.

Proof. Fix the chosen bad branch after the initial packet-section stripping from Corollary 11.87. Let $q_1 \prec q_2 \prec \cdots \prec q_N$ be the successive blowup centers on the clean-section ruled model lying on that branch, up to the last center at which a horizontal strict transform still passes through the chosen infinitely near point. For each $1 \leq k \leq N$, let

$$\mathcal{H}_k := \{\text{horizontal branch strict transforms passing through } q_k\}.$$

Since the branch is collision-rooted, $\#\mathcal{H}_1 \geq 2$. Because every later center q_{k+1} lies on the unique descendant of q_k selected by the chosen bad branch, any horizontal branch strict transform which misses q_{k+1} after the blowup at q_k is separated forever from that branch and cannot reappear at a later chosen center. Hence

$$\mathcal{H}_{k+1} \subseteq \mathcal{H}_k \qquad (1 \leq k < N).$$

So the cardinalities $\#\mathcal{H}_k$ form a weakly decreasing sequence of positive integers.

There are at most four horizontal sections in total. Therefore, after finitely many common collision steps, one reaches an index k_0 for which $\#\mathcal{H}_{k_0} \in \{1, 2\}$, and this is the first time the cardinality drops to at most 2. We analyze the two possibilities.

If $\#\mathcal{H}_{k_0} = 1$, then q_{k_0} is a packet-section point of the unique horizontal branch in $\mathcal{H}_{k_0}$. Applying Proposition 11.86 from that stage onward, either Route C entry already occurs, contrary to our standing assumption, or after stripping a finite maximal packet-section staircase one reaches a later collision point on the same bad branch. Repeating this procedure is possible only finitely many times because the branch has only finitely many blowup centers. Hence, on a residual no-entry counterexample, the process cannot terminate in the packet-section case.

Therefore eventually one reaches a stage for which $\#\mathcal{H}_{k_0} = 2$. By the choice of k_0, no third horizontal branch passes through q_{k_0}. Thus q_{k_0} is an isolated collision point of exactly two horizontal branch strict transforms. Proposition 9.119 applies and gives the explicit order-m pair-collision block at that stage.

Finally, every horizontal branch discarded earlier in the process peeled off at one of the previous common collision steps, so it meets an earlier exceptional component of the same local exceptional tree and is disjoint from the terminal isolated pair-collision block. This is exactly the final description stated in the proposition. □

Corollary 11.89 (The one-touch gap is now an isolated-pair service problem)**.** *Keep the setup of Corollary 11.87, and assume that Route C entry does not already occur. Then every one-touch bad branch is obtained by the following finite recipe:*

(i) *start with an isolated order-m pair collision of exactly two horizontal branches, as in Proposition 9.119;*

(ii) *attach to it finitely many earlier peeled-off horizontal branches along the preceding local exceptional chain;*

(iii) *continue only by packet-section staircases and later collisions issued from the section points of the successive last nonexceptional curves.*

In particular, the remaining maximal-width one-touch problem is no longer an arbitrary multi-branch collision problem: it is a finite service problem on isolated pair-collision blocks.

Proof. This is just Proposition 11.88 together with Proposition 11.86, which controls every later packet-section phase. □

Proposition 11.90 (Terminal one-touch service blocks are explicit forks)**.** *Keep the setup of Corollary 11.89, and assume that Route C entry does not already occur. Fix a one-touch bad* *branch and let L be the last nonexceptional curve on that branch before the terminal purely vertical connected component*

$$W \subset D_F, \qquad W \cap D_{\mathrm{hor}} = \varnothing.$$

Then exactly one of the following two cases occurs.

(i) ***Primitive higher-order pair.*** *The terminal stage is already an isolated pair collision of order $p \geq 2$. In this case $L \cdot D = 3$, and the terminal local core is the primitive fork from Proposition 9.119.*

(ii) ***Serviced pair.*** *There exist integers*

$$u \geq 1, \qquad p \geq 1,$$

a smooth local fiber component M which is the previous last nonexceptional curve at the start of the final packet-service phase, and local coordinates (t, z) centered at the packet-section point of the active horizontal branch on M such that

$$M = \{t = 0\}, \qquad A_0 = \{z = 0\}, \qquad C_0 = \{z = t^{u+p}\},$$

where A_0 is the active horizontal branch and C_0 is the other horizontal branch involved in the final collision. The first u blowups are the successive packet-section blowups at the current point of the strict transform of A_0 and the newest exceptional curve, and the next p blowups are the forced order-p pair-collision blowups of the strict transforms of A_0 and C_0. The resulting reduced local fiber is the chain $M - G_1 - \cdots - G_u - F_1 - \cdots - F_p$, where

$$M^2 \leq -2, \qquad G_i^2 = -2 \ \ (1 \leq i \leq u), \qquad F_i^2 = -2 \ \ (1 \leq i < p), \qquad F_p^2 = -1.$$

The two final colliding horizontal branches meet $L := F_p$ transversely at distinct points, any earlier peeled-off horizontal branches meet only the preceding chain $M - G_1 - \cdots - G_u - F_1 - \cdots - F_{p-1}$, and one has

$$L \not\subset D, \qquad L \cdot D = 3.$$

In particular, the terminal local core is an explicit fork with center L, two horizontal arms, and one vertical chain arm.

Moreover, the primitive transverse pair case $(u, p) = (0, 1)$ cannot occur terminally on a one-touch bad branch.

Proof. By Corollary 11.89, the chosen branch is built from an isolated pair-collision block together with finitely many packet-section staircases and later collisions issued from the section points of the successive last nonexceptional curves. Take the *last* stage at which a nonexceptional curve lies on the chosen bad branch. We analyze the final local block at that stage.

If there is no final packet-service phase before that last nonexceptional curve is created, then the terminal stage is already the primitive isolated pair-collision block of Proposition 9.119 for some order $p \geq 1$. If $p = 1$, then the primitive last exceptional curve meets only the two colliding horizontal branches and has no vertical neighbor in D. Such a curve cannot be the last nonexceptional curve before a purely vertical connected component $W \subset D_F$, so the terminal one-touch branch cannot end in the primitive transverse case. Hence in the terminal primitive case one must have $p \geq 2$, and Proposition 9.119 gives exactly alternative (i).

Assume now that there is a final packet-service phase before the terminal nonexceptional curve is created. Let M be the previous last nonexceptional curve at the start of that phase, and let q be the chosen section point of the active horizontal branch on M. By definition of the final packet-service phase, there is an integer $u \geq 1$ such that the next u blowups are exactly the successive packet-section blowups at the current point of the active section and the newest exceptional curve, and after these blowups a final isolated collision of order $p \geq 1$ occurs between the strict transform of that active section and one other horizontal branch.

Choose local coordinates (t, z) centered at q so that

$$M = \{t = 0\}, \qquad A_0 = \{z = 0\}.$$

Let C_0 be the other horizontal branch participating in the final collision. Since the strict transform of C_0 passes through the first u chosen centers and then collides with the strict transform of A_0 of order p at the final packet point, one may write $C_0 = \{z = t^{u+p}\}$ after rescaling the local coordinate z. Now recursively write local coordinates on the successive packet blowups by

$$z = t^k w_k \qquad (1 \leq k \leq u).$$

Then after the first u packet blowups one has

$$A_u = \{w_u = 0\}, \qquad C_u = \{w_u = t^p\},$$

and the reduced local fiber is the chain $M - G_1 - \cdots - G_u$, where the first blowup lowers the self-intersection of M from -1 to at most -2, each G_i for $1 \leq i < u$ is blown up once more and hence has self-intersection -2, and $G_u^2 = -1$ just before the final isolated collision block. Applying Lemma 9.20 to the pair $A_u, \qquad C_u$ creates the further chain $F_1 - \cdots - F_p$, attached to G_u, with

$$F_i^2 = -2 \ \ (1 \leq i < p), \qquad F_p^2 = -1,$$

and lowers G_u^2 by one more, so in the final surface $G_u^2 = -2$. Thus the full reduced local fiber is exactly $M - G_1 - \cdots - G_u - F_1 - \cdots - F_p$.

The strict transforms of the two final colliding horizontal branches meet $L := F_p$ transversely at distinct points, and no earlier peeled-off horizontal branch passes through the final collision point; those earlier branches therefore meet only the preceding chain $M - G_1 - \cdots - G_u - F_1 - \cdots - F_{p-1}$. Since $F_p^2 = -1$, one has $L \not\subset D$, while

$$M,\ G_1, \ldots, G_u,\ F_1, \ldots, F_{p-1}$$

all have self-intersection at most -2 and hence lie in D. Therefore the only components of D meeting L are the two final colliding horizontal branches and the adjacent curve F_{p-1} if $p \geq 2$, or G_u if $p = 1$. In either case $L \cdot D = 3$, and the terminal local core is a fork. This proves alternative (ii).

The exclusion of the primitive transverse case $(u, p) = (0, 1)$ was already proved in the first paragraph, so the proposition follows. □

Corollary 11.91 (The one-touch isolated-pair gap is terminally fork-type)**.** *Keep the setup of Corollary 11.89, and assume that Route C entry does not already occur. Then every one-touch bad branch has a last nonexceptional curve $L \not\subset D$ with $L \cdot D = 3$. More precisely, the terminal local block of every such branch is one of the explicit fork blocks from Proposition 11.90; the primitive contact-2 case never occurs terminally on a one-touch bad branch.*

Proof. Immediate from Proposition 11.90. □

Remark 11.92 (What this means for the remaining one-touch gap). After Corollary 11.91, the residual height-4, maximal-width one-touch problem is no longer a service problem on arbitrary isolated pair-collision blocks. The last nonexceptional curve on every bad branch is forced into a fork-type local model with $L \cdot D = 3$, namely either the primitive higher-order pair fork or the serviced-pair fork of Proposition 11.90. So the remaining gap is now a genuinely explicit fork-survival problem rather than a mixed contact-2/fork problem.

Proposition 11.93 (Terminal one-touch forks already carry an old special side)**.** *Keep the setup of Corollary 11.91, and assume that Route C entry does not already occur. Let $L \not\subset D$ be the last nonexceptional curve on a one-touch bad branch, and let*

$$H_1, H_2 \subset D_{\mathrm{hor}}, \qquad E \subset D$$

be the three irreducible components of D meeting L, where H_1, H_2 are horizontal and E is vertical. Then:

(i) *at least one of the two horizontal components H_1, H_2 is a special horizontal section in the sense of Definition 9.105;*

(ii) *assume moreover that neither H_1 nor H_2 lies on a branching connected component of D, and that neither has reduced touched core equal to a touched tip of weight 2. Then*

$$\mathrm{cf}_X(E) < \frac{1}{3}.$$

Consequently, if the connected component of D containing E is a chain, then the 2-tail core of the touched pair relative to E is a touched tip of weight 2; if that connected component is not a chain, then one is already on the branching side.

Proof. By Corollary 11.91, the curve L meets exactly the three components H_1, H_2, E of D

transversely once each. Hence Proposition 9.133 gives

$$\mathrm{cf}_X(H_1) + \mathrm{cf}_X(H_2) + \mathrm{cf}_X(E) < 1.$$

Assume first that both

$$H_1 \qquad \text{and} \qquad H_2$$

are ordinary. Then Lemma 9.107 gives

$$\mathrm{cf}_X(H_1) \geq \frac{1}{2}, \qquad \mathrm{cf}_X(H_2) \geq \frac{1}{2},$$

so

$$\mathrm{cf}_X(H_1) + \mathrm{cf}_X(H_2) + \mathrm{cf}_X(E) \geq 1,$$

contrary to the strict contact-defect inequality above. Therefore at least one of H_1, H_2 is special. This proves (i).

Now assume the additional hypotheses of (ii). Let $H \in \{H_1, H_2\}$. Since H does not lie on a branching connected component of D, it lies on a chain component. If H is ordinary, then Lemma 9.107 gives

$$\mathrm{cf}_X(H) \geq \frac{1}{2} > \frac{1}{3}.$$

If H is special but its reduced touched core is not a touched tip of weight 2, then by Definition 9.105 its reduced touched core is the singleton chain [3]. By Lemma 9.106 the discrepancy coefficient on that reduced core is

$$1 - \mu([3], 1) = 1 - \frac{2}{3} = \frac{1}{3},$$

and Lemma 9.93 shows that adding terminal 2-tails away from the touched component can only decrease μ and hence can only increase cf_X. Thus in all cases covered by the hypotheses of (ii) one has

$$\mathrm{cf}_X(H_1) \geq \frac{1}{3}, \qquad \mathrm{cf}_X(H_2) \geq \frac{1}{3}.$$

Therefore the contact-defect inequality implies

$$\mathrm{cf}_X(E) < 1 - \frac{1}{3} - \frac{1}{3} = \frac{1}{3}.$$

If the connected component of D containing E is a chain, Lemma 11.45 shows that the 2-tail core of the touched pair relative to E is a touched tip of weight 2. If that connected component is not a chain, then by definition one is already on the branching side. This proves (ii). ☐

Corollary 11.94 (The terminal one-touch fork introduces no genuinely new side type)**.** *Keep the setup of Corollary 11.91, and assume that Route C entry does not already occur. Then every terminal one-touch fork already lands in one of the side types isolated elsewhere in the note:*

(i) *a horizontal special side, i.e. a branching horizontal side or a horizontal chain side whose reduced touched core is a touched tip of weight* 2 *or the singleton* [3];

(ii) *a vertical branching side;*

(iii) *a vertical chain side whose* 2*-tail core is a touched tip of weight* 2*.*

In particular, the terminal one-touch fork problem creates no genuinely new local corridor beyond the previously isolated special-tip / branching geometry.

Proof. This is exactly Proposition 11.93 together with the definition of a special horizontal section. ☐

Lemma 11.95 (Ordinary chain sides have discrepancy coefficient at least one half)**.** *Let* (A, E) *be a touched chain pair, and let* $(A^\sharp, E^\sharp)$ *be its* 2*-tail core from Definition 9.92. Assume that* $E^\sharp$ *is neither a touched tip of weight* 2 *nor the singleton chain* [3]*. Then*

$$\mu_X(E) \le \frac{1}{2} \qquad \text{and hence} \qquad \mathrm{cf}_X(E) \ge \frac{1}{2}.$$

Proof. By Lemma 9.106, $\mathrm{cf}_X(E) = 1 - \mu(A, E)$. If $E^\sharp$ is an interior vertex of $A^\sharp$, then Proposition 9.94 gives

$$\mu(A^\sharp, E^\sharp) \le \frac{1}{2}.$$

If $E^\sharp$ is a tip, then the hypothesis excludes exactly the two high-contact cases isolated by Proposition 9.96, so again

$$\mu(A^\sharp, E^\sharp) \le \frac{1}{2}.$$

Finally, Lemma 9.93 shows that removing terminal 2-tails away from the touched component can only increase the contact weight, hence

$$\mu(A, E) \le \mu(A^\sharp, E^\sharp) \le \frac{1}{2}.$$

Therefore

$$\mathrm{cf}_X(E) = 1 - \mu(A, E) \ge \frac{1}{2}.$$

☐

Proposition 11.96 (Every terminal one-touch fork has spherical adjacent weights)**.** *Keep the setup of Corollary 11.91, and assume that Route C entry does not already occur. Let* $L \not\subset D$ *be the last nonexceptional curve on a one-touch bad branch, and let*

$$H_1, H_2 \subset D_{\mathrm{hor}}, \qquad E \subset D$$

be the three irreducible components of D *meeting* L*. Write*

$$H_1^2 = -a, \qquad H_2^2 = -b, \qquad E^2 = -c, \qquad a, b, c \ge 2.$$

Then

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} > 1.$$

Consequently, after reordering so that $\alpha \le \beta \le \gamma$ *are the three numbers* a, b, c*, one has*

$$(\alpha, \beta, \gamma) = (2, 2, n)\ (n \ge 2), \qquad (2, 3, 3), \qquad (2, 3, 4), \qquad (2, 3, 5).$$

In particular, at least one of H_1, H_2, E *has self-intersection* -2*, and at least two of them have self-intersection in* $\{-2, -3\}$*.*

Proof. By Corollary 11.91, the curve L meets exactly the three components H_1, H_2, E of D transversely once each. Therefore Proposition 9.133 gives

$$\mathrm{cf}_X(H_1) + \mathrm{cf}_X(H_2) + \mathrm{cf}_X(E) < 1.$$

Applying Lemma 9.132 to each term yields

$$\Bigl(1 - \frac{2}{a}\Bigr) + \Bigl(1 - \frac{2}{b}\Bigr) + \Bigl(1 - \frac{2}{c}\Bigr) < 1,$$

which is equivalent to

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} > 1.$$

Now reorder so that $\alpha \le \beta \le \gamma$ are the three numbers a, b, c. If $\alpha \ge 3$, then

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} \le 1,$$

a contradiction. Hence $\alpha = 2$. If moreover $\beta \ge 4$, then

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} \le \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1,$$

again impossible. Thus $\beta \in \{2, 3\}$. If $\beta = 2$, then any $\gamma \ge 2$ works. If $\beta = 3$, then

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{\gamma} > 1$$

forces $\gamma < 6$, so $\gamma \in \{3, 4, 5\}$. This gives exactly the displayed list. The final statement is immediate from that list. □

Proposition 11.97 (Outside the old vertical corridor, the terminal one-touch fork is finite and double-special or exceptional)**.** *Keep the setup of Corollary 11.91, and assume that Route C entry does not already occur. Let* $L \not\subset D$ *be the last nonexceptional curve on a one-touch bad branch, and let*

$$H_1, H_2 \subset D_{\mathrm{hor}}, \qquad E \subset D$$

be the three irreducible components of D *meeting* L*. Then one of the following holds.*

- (i) *both horizontal components* H_1, H_2 *are special horizontal sections;*
- (ii) *the connected component of* D *containing* E *is already in the old vertical corridor, i.e. it is branching, or it is a chain and the* 2*-tail core of the touched pair relative to* E *is a touched tip of weight* 2*;*

(iii) *after renumbering the two horizontal sides, one has:*

  (a) H_1 *is special, but not of singleton-* [3] *type; equivalently,* H_1 *is either branching in its connected component of* D *or lies on a chain whose reduced touched core is a touched tip of weight* 2;

  (b) H_2 *is ordinary;*

  (c) *the connected component of* D *containing* E *is a chain whose reduced touched core relative to* E *is the singleton* [3];

  (d) *one has the numerical bounds*

$$H_1^2 = -2, \qquad E^2 = -3, \qquad H_2^2 \in \{-2, -3, -4, -5\};$$

  (e) *the discrepancy coefficient on the special side satisfies*

$$\mathrm{cf}_X(H_1) < \frac{1}{6}.$$

Proof. By Proposition 11.93, at least one of H_1, H_2 is special. If both are special, we are in alternative (i). Assume from now on that, after renumbering if necessary, H_1 is special and H_2 is ordinary.

If the connected component of D containing E is branching, or if it is a chain whose 2-tail core relative to E is a touched tip of weight 2, then alternative (ii) holds. Assume therefore that alternative (ii) does not hold.

We first claim that H_1 cannot be of singleton- [3] type. Indeed, under the present assumptions neither H_1 nor H_2 lies on a branching connected component of D, and neither has reduced touched core a touched tip of weight 2. So Proposition 11.93(ii) applies and gives

$$\mathrm{cf}_X(E) < \frac{1}{3}.$$

Then Lemma 11.45 forces the chain side containing E into the touched-tip- 2 corridor, contradicting the assumption that alternative (ii) does not hold. Thus H_1 is special of the non- [3] kind, proving (iii)(a).

Next we claim that the connected component of D containing E cannot be ordinary as a touched chain side. If it were, then Lemma 11.95 would imply

$$\mathrm{cf}_X(E) \geq \frac{1}{2}.$$

Since H_2 is ordinary, Lemma 9.107 gives

$$\mathrm{cf}_X(H_2) \geq \frac{1}{2}.$$

But Proposition 9.133, applied to the curve L, yields

$$\mathrm{cf}_X(H_1) + \mathrm{cf}_X(H_2) + \mathrm{cf}_X(E) < 1,$$

which is impossible if both

$$\mathrm{cf}_X(H_2) \geq \frac{1}{2} \qquad \text{and} \qquad \mathrm{cf}_X(E) \geq \frac{1}{2}.$$

Therefore the chain side containing E is not ordinary. Since alternative (ii) was excluded, the only remaining chain possibility is that its reduced touched core is the singleton [3]. This proves (iii)(c).

For a singleton- [3] core, Lemma 9.106 gives

$$\mathrm{cf}_X(E) = 1 - \mu([3], 1) = \frac{1}{3},$$

and Lemma 9.93 shows that adding terminal 2-tails away from the touched component can only increase the discrepancy coefficient. Hence

$$\mathrm{cf}_X(E) \geq \frac{1}{3}.$$

Together with

$$\mathrm{cf}_X(H_2) \geq \frac{1}{2}$$

and the contact-defect inequality, this gives

$$\mathrm{cf}_X(H_1) < 1 - \frac{1}{2} - \frac{1}{3} = \frac{1}{6},$$

which proves (iii)(e).

Now apply Lemma 9.132 to $H_1^2 = -a$. Since

$$\mathrm{cf}_X(H_1) < \frac{1}{6},$$

one has

$$1 - \frac{2}{a} < \frac{1}{6},$$

so

$$a < \frac{12}{5}.$$

Because $a \geq 2$ is an integer, it follows that $a = 2$, that is, $H_1^2 = -2$. Also, because the reduced touched core of the chain side containing E is the singleton [3], one has $E^2 = -3$. Finally, Proposition 11.96 applied to the triple $(-H_1^2, -H_2^2, -E^2) = (2, -H_2^2, 3)$ gives $-H_2^2 \in \{2, 3, 4, 5\}$. This proves (iii)(d) and completes the proof. □

Corollary 11.98 (The remaining terminal one-touch fork is reduced to multi-special horizontals or a finite exceptional corridor)**.** *Keep the setup of Corollary 11.91, and assume that Route C entry does not already occur. Then every terminal one-touch fork falls into one of the following three classes.*

(i) *the witness already contains at least two special horizontal sections;*

(ii) *the vertical side of the terminal fork is already in the old special vertical corridor (branching* *or touched tip of weight* 2 *after* 2*-tail reduction);*

(iii) *after renumbering the two horizontal arms, the local adjacent-weight pattern is the finite exceptional pattern*

$$(-H_1^2, -E^2, -H_2^2) = (2, 3, b), \qquad b \in \{2, 3, 4, 5\},$$

where H_1 is the unique special horizontal arm and is of non- [3] *type, while H_2 is ordinary.*

In particular, outside the old vertical corridor the unresolved maximal-width one-touch problem is no longer an infinite fork-survival problem: it is reduced to either a witness with at least two special horizontal sections, or the finite adjacent-weight corridor $(2, 3, b), \qquad b \in \{2, 3, 4, 5\}$.

Proof. This is exactly Proposition 11.97. □

Remark 11.99 (What now remains of the one-touch fork gap). Corollary 11.94 does not yet prove that the height-4, maximal-width one-touch corridor is empty. What it does prove is that the terminal fork-survival problem introduces no new local side shape. Every surviving terminal fork already falls into one of the old side types isolated earlier in the note: branching, a touched (-2)-tip, or the singleton [3]. So the residual issue is now purely global: show that these already-known side types cannot survive in the section-only, maximal-width one-touch geometry.

Remark 11.100 (What this improves in the two-touch program). The rigid-hub case is now eliminated rather than merely isolated. Thus the height-4, maximal-width one-touch problem is concentrated entirely in the distributed-anchor geometry: every bad branch starts on the packet itself, at either a packet-section point or a horizontal collision point. So the remaining two-touch problem is no longer about arbitrary vertical trees, and no longer contains the subcase of one vertical component carrying all four sections.

Corollary 11.101 (An ordinary section forces distributed anchors)**.** *Keep the setup of Corollary 11.82. If at least one of the four horizontal sections is ordinary, then only alternative* (ii) *of that corollary can occur. Equivalently, every one-touch bad fiber is already a distributed-anchor, packet/collision-rooted configuration on the clean-section ruled model.*

Proof. Alternative (i) of Corollary 11.82 makes all four horizontal sections special, by Proposition 11.80. So if one section is ordinary, only alternative (ii) can occur. □

Remark 11.102 (What remains of the maximal-width two-touch problem). The two new reductions are complementary. The λ-minimality argument removes free purely vertical first centers, so every one-touch bad branch must already start on the horizontal packet itself. The rigid-hub proposition then isolates the only subcase in which a single vertical component carries all four horizontal sections: that component is an isolated (-2)-hub and the whole horizontal packet is special.

So the height-4, maximal-width one-touch problem is no longer an unrestricted question about arbitrary vertical trees or arbitrary multibranch collisions. What remains is the distributed-anchor case, and after the collision-stripping reduction above every bad branch is controlled by an isolated pair-collision block together with finitely many packet services and earlier peeled-off horizontal branches on the same local exceptional chain. That is the sharpest honest reduction obtained here toward the two-touch part of Conjecture 10.1.

11.14. The finite exceptional terminal corridor is empty

The finite adjacent-weight corridor isolated in Corollary 11.98 can in fact be removed entirely. The reason is that the explicit terminal service model already controls the self-intersection of the vertical component adjacent to the last nonexceptional curve: it is always the component immediately preceding the last exceptional curve in the local chain arm, hence has self-intersection -2. Therefore the terminal one-touch fork is automatically already in the old special vertical corridor, and the exceptional pattern $(2, 3, b), \qquad b \in \{2, 3, 4, 5\}$, never occurs.

Proposition 11.103 (The vertical side of every terminal one-touch fork is already old-corridor)**.** *Keep the setup of Corollary 11.91, and assume that Route C entry does not already occur. Let $L \not\subset D$ be the last nonexceptional curve on a one-touch bad branch, and let $E \subset D$ be the unique vertical component of D meeting L. Then $E^2 = -2$. Moreover, the connected component of D containing E is already in the old special vertical corridor: either it is branching, or it is a chain and the 2-tail core of the touched pair relative to E is a touched tip of weight 2.*

Proof. By Proposition 11.90, exactly one of the following two cases occurs.

Case 1: primitive higher-order pair. Then the terminal stage is an isolated pair collision of order $p \geq 2$, and Proposition 9.119 gives a primitive local chain $F_1 - \cdots - F_p$ with

$$F_i^2 = -2 \quad (1 \leq i < p), \qquad L = F_p \not\subset D.$$

Hence the unique vertical component of D meeting L is $E = F_{p-1}$, so $E^2 = -2$. If no further component of D meets E besides the preceding chain arm, then the connected component of D containing E is a chain and E is already a touched tip of weight 2. If additional components of D do meet E, then that connected component is branching. So in either subcase the vertical side is in the old special vertical corridor.

Case 2: serviced pair. Then Proposition 11.90 gives integers

$$u \geq 1, \qquad p \geq 1,$$

and a reduced local fiber $M - G_1 - \cdots - G_u - F_1 - \cdots - F_p$ with

$$M,\ G_1, \ldots, G_u,\ F_1, \ldots, F_{p-1} \subset D, \qquad L := F_p \not\subset D,$$

$$G_i^2 = -2 \ \ (1 \leq i \leq u), \qquad F_i^2 = -2 \ \ (1 \leq i < p),$$

and such that the unique vertical component of D meeting L is

$$E = \begin{cases} F_{p-1}, & p \geq 2, \\ G_u, & p = 1. \end{cases}$$

Therefore again $E^2 = -2$.

By the same proposition, any earlier peeled-off horizontal branches meet only the preceding chain $M - G_1 - \cdots - G_u - F_1 - \cdots - F_{p-1}$. If one of them meets E, then the connected component of D containing E is branching. Otherwise that connected component is a chain, and E is the touched end-component of weight 2 adjacent to L. So again the vertical side lies in the old special vertical corridor.

This proves both claims. ☐

Corollary 11.104 (The finite exceptional terminal corridor is empty)**.** *Keep the setup of Corollary 11.91, and assume that Route C entry does not already occur. Then every terminal one-touch fork is already in the old special vertical corridor. Equivalently, alternative* (iii) *of Corollary 11.98 never occurs.*

In particular, the adjacent-weight pattern

$$(-H_1^2, -E^2, -H_2^2) = (2, 3, b), \qquad b \in \{2, 3, 4, 5\},$$

is impossible.

Proof. By Proposition 11.103, the vertical component E adjacent to the last nonexceptional curve always satisfies $E^2 = -2$ and its connected component is already in the old special vertical corridor. Hence the exceptional alternative of Corollary 11.98, which requires $E^2 = -3$, cannot occur. ☐

Remark 11.105 (What now remains of the maximal-width one-touch gap)*.* The terminal one-touch fork no longer contributes a finite residual exceptional list. After the packet-section stripping and isolated-pair reduction, the final surviving nonexceptional curve always lands on the old special vertical corridor: either the adjacent vertical side is branching, or it is a touched tip of weight 2. So the residual maximal-width one-touch problem is now purely the global survival problem for those already-known vertical side types.

11.15. The height-4 maximal-width corridor closes by a terminal-leaf count

The preceding one-touch analysis gives exactly the missing graph-theoretic input for the excess route. The key point is that a one-touch bad fiber always contains a *leaf component* of $F_{\mathrm{red}} - D$ adjacent to only one vertical connected component of D_F. This improves the degree count in Proposition 9.137 by one, which is enough to recover the desired bound $v(F) \leq \sigma(F) - 1$ for every degenerate fiber.

Proposition 11.106 (Every one-touch bad fiber contains a leaf component off D)**.** *Assume the setup of Corollary 9.13, and among all witnessing fibrations of height* 4 *and maximal width on the fixed smooth surface* $\widetilde{X}$ *choose one minimizing* $\lambda(p)$ *in the sense of Definition 11.75. Let* F *be a degenerate fiber with*

$$v(F) > 0, \qquad t(F) = 1.$$

Then there exists a connected component $N_* \subset F_{\text{red}} - D$ *consisting of a single irreducible curve* L *such that* $\deg_{\Gamma_F}(N_*) = 1$*. Equivalently,* $L \not\subset D$ *meets exactly one component of* D_F *inside the fiber.*

Proof. Fix a purely vertical connected component

$$W \subset D_F, \qquad W \cap D_{\text{hor}} = \varnothing,$$

and follow the unique bad branch from the touched part of D_F to W. By Corollary 11.83, the first center of that branch on the clean-section ruled model is either a packet-section point or a collision point of horizontal branches.

We now run the same finite stripping procedure as in the proofs of Propositions 11.86, 11.88, and 11.90, but without assuming in advance that Route C entry fails. Whenever the current center is a packet-section point, strip the maximal initial packet staircase by the elementary transforms of Lemma 11.84. If that staircase is terminal, then Proposition 11.86(ii) produces a last exceptional curve $L \not\subset D$ which is a singly-hit multiplicity-one nonexceptional tip and whose unique vertical neighbor lies in D. In particular, inside the reduced fiber F_{red} the curve L has exactly one neighbor, belonging to D_F, so the corresponding connected component of $F_{\text{red}} - D$ is exactly $N_* := \{L\}$ and has degree 1 in Γ_F. Thus the proposition is proved in this case.

Assume therefore that after each stripped packet staircase a later collision center remains on the chosen branch. At every collision stage, apply the monotonicity argument from the proof of Proposition 11.88: along the chosen infinitely near branch, the set of horizontal branch strict transforms through the successive centers is weakly decreasing. Hence after finitely many common collision-prefix steps one reaches either a packet-section point again or an isolated collision of exactly two horizontal branches. In the first case we return to the previous paragraph and continue the same finite procedure. Because the branch has only finitely many blowup centers, this process must eventually terminate.

Consider the last stage at which a nonexceptional curve still lies on the chosen bad branch before the branch enters W. If this last stage is a terminal packet staircase, we are back in the first paragraph and obtain the desired leaf component.

Otherwise there is a final isolated pair-collision stage of exactly two horizontal branches. There are then two possibilities.

Primitive higher-order pair. If there is no final packet-service phase before the last nonexceptional curve is created, then the terminal local block is the primitive order-p pair-collision block of Proposition 9.119. The primitive transverse case $p = 1$ cannot be terminal here, because then the last exceptional curve meets only the two horizontal branches and no vertical component of D_F, so it cannot be the last nonexceptional curve before entering W. Hence one has $p \geq 2$, and the primitive local chain is

$$F_1 - \cdots - F_p, \qquad F_i^2 = -2 \ (1 \leq i < p), \qquad F_p^2 = -1, \qquad L := F_p.$$

Then the unique vertical neighbor of L inside the fiber is $F_{p-1} \in D_F$, so again $N_* := \{L\}$ is a degree-1 vertex of Γ_F.

Serviced pair. If there is a final packet-service phase before the terminal nonexceptional curve is created, let $u \geq 1$ be its length and let $p \geq 1$ be the order of the final isolated pair collision. The same local-coordinate calculation as in the proof of Proposition 11.90 then gives a reduced local fiber $M - G_1 - \cdots - G_u - F_1 - \cdots - F_p$, where

$$M,\ G_1, \ldots, G_u,\ F_1, \ldots, F_{p-1} \subset D, \qquad L := F_p \not\subset D,$$

and

$$G_i^2 = -2 \ (1 \leq i \leq u), \qquad F_i^2 = -2 \ (1 \leq i < p), \qquad F_p^2 = -1.$$

The unique vertical neighbor of L inside the fiber is

$$F_{p-1} \in D_F \qquad (p \geq 2)$$

or

$$G_u \in D_F \qquad (u \geq 1,\ p = 1).$$

So once more the connected component of $F_{\mathrm{red}} - D$ containing L is exactly $N_* := \{L\}$, and $\deg_{\Gamma_F}(N_*) = 1$.

This proves the proposition. ☐

Lemma 11.107 (Multiplicity-one components outside D are leaves)**.** *Let F be a degenerate fiber of a $\mathbf{P}^1$-fibration on the minimal resolution of a rank-one del Pezzo surface, and let $C \subset F_{\mathrm{red}} - D$ be an irreducible component of multiplicity one in F. Then C is a leaf of the reduced fiber F_{red}.*

Proof. By [7, Lemma 2.8(b)], every component of $F_{\mathrm{red}} - D$ is a vertical (-1)-curve. Hence $C^2 = -1$. Since C has multiplicity one in F, the fiber relation gives

$$0 = F \cdot C = C^2 + \sum_{E \sim C} \mathrm{mult}_F(E) = -1 + \sum_{E \sim C} \mathrm{mult}_F(E),$$

where the sum runs over the irreducible components E adjacent to C in the reduced fiber. All multiplicities are positive integers, so the sum must be exactly 1. Therefore C has exactly one adjacent component, hence is a leaf of F_{red}. ☐

Lemma 11.108 (A degree-one component of $F_{\mathrm{red}} - D$ carries exactly one leaf)**.** *Keep the notation of Proposition 11.109. Let N_i be a connected component of $F_{\mathrm{red}} - D$. Assume $D_F \neq 0$*

and that N_i contains a leaf of the whole reduced fiber. Then $\deg_{\Gamma_F}(N_i) = 1$. In particular, if N_i has degree one in Γ_F, then it contains exactly one leaf of the whole reduced fiber.

Proof. All irreducible components of N_i are (-1)-curves by [7, Lemma 2.8(b)], and every such (-1)-curve is non-branching in the reduced fiber by [7, Lemma 2.3(a)]. Hence N_i is a chain. If $\deg_{\Gamma_F}(N_i) = 2$, then both ends of this chain are adjacent to components of D_F, so no end of N_i is a leaf of the whole fiber. Therefore the presence of a leaf of F_{red} inside N_i forces $\deg_{\Gamma_F}(N_i) = 1$. If $\deg_{\Gamma_F}(N_i) = 1$, then N_i is a chain attached to D_F at exactly one end, so its opposite end is the unique leaf of the whole reduced fiber contained in N_i. □

Proposition 11.109 (Every degenerate fiber in the maximal-width corridor satisfies $v(F) \le \sigma(F)-1$)**.** *Assume the setup of Corollary 9.13, and among all witnessing fibrations of height 4 and maximal width on the fixed smooth surface $\widetilde{X}$ choose one minimizing $\lambda(p)$. Then for every degenerate fiber F one has $v(F) \le \sigma(F) - 1$.*

Proof. Keep the notation of Proposition 9.137: let $C_1, \dots, C_s$ be the connected components of D_F, let $N_1, \dots, N_r$ be the connected components of $F_{\text{red}} - D$, and let Γ_F be the contracted bipartite tree. As in Proposition 9.137, one has

$$r \le \sigma(F), \qquad |E(\Gamma_F)| = r + s - 1, \qquad \deg_{\Gamma_F}(N_i) \le 2 \quad (1 \le i \le r).$$

We distinguish cases according to $t(F)$.

Case 1: $t(F) = 0$. If $D_F = 0$, then $s = v(F) = 0$ and there is nothing to prove. So assume $D_F \ne 0$. Then every horizontal section meets F on a component of $F_{\text{red}} - D$. Because each horizontal component is a 1-section, every such intersection lies on a multiplicity-one component of the fiber. By Lemma 11.107, every multiplicity-one component of $F_{\text{red}} - D$ is a leaf of the whole reduced fiber.

Let C be such a component hit by a horizontal section. Then the connected component N_i of $F_{\text{red}} - D$ containing C has degree one in Γ_F by Lemma 11.108. By Corollary 9.13(1), no multiplicity-one component outside D can meet all four horizontal sections. Therefore the four horizontal intersections are supported on at least two distinct components of $F_{\text{red}} - D$, hence on at least two distinct multiplicity-one leaves of the whole fiber. By the final assertion of Lemma 11.108, these leaves lie in at least two distinct degree-one vertices N_{i_1}, N_{i_2} of Γ_F. Thus

$$\deg_{\Gamma_F}(N_{i_1}) = \deg_{\Gamma_F}(N_{i_2}) = 1.$$

Since every other N_i has degree at most 2, we get

$$|E(\Gamma_F)| = \sum_{i=1}^{r} \deg_{\Gamma_F}(N_i) \le 1 + 1 + 2(r-2) = 2r - 2.$$

Combining this with $|E(\Gamma_F)| = r + s - 1$ gives

$$r + s - 1 \le 2r - 2, \qquad \text{hence} \qquad s \le r - 1 \le \sigma(F) - 1.$$

Since $t(F) = 0$, one has $v(F) = s \leq \sigma(F) - 1$.

Case 2: $t(F) = 1$**.** By Proposition 11.106, there exists some connected component $N_* \subset F_{\mathrm{red}} - D$ with $\deg_{\Gamma_F}(N_*) = 1$. Hence

$$|E(\Gamma_F)| = \sum_{i=1}^{r} \deg_{\Gamma_F}(N_i) \leq 1 + 2(r-1) = 2r - 1.$$

Together with $|E(\Gamma_F)| = r + s - 1$ this yields

$$r + s - 1 \leq 2r - 1, \qquad \text{so} \qquad s \leq r \leq \sigma(F).$$

Because $t(F) = 1$, we obtain $v(F) = s - 1 \leq \sigma(F) - 1$.

Case 3: $t(F) \geq 2$**.** Here Proposition 9.137 gives directly $v(F) \leq \sigma(F) + 1 - t(F) \leq \sigma(F) - 1$.

This completes the proof. ☐

Theorem 11.110 (The height-4 maximal-width corridor closes unconditionally)**.** *Assume the setup of Corollary 9.13. Then* $\#\,\mathrm{Sing}(X) \leq 7$*. In particular, no benchmark counterexample exists inside the height-*4*, maximal-width corridor.*

Proof. Among all witnessing fibrations of height 4 and maximal width on the fixed smooth surface $\widetilde{X}$ choose one minimizing $\lambda(p)$. By Proposition 11.109, every degenerate fiber satisfies $v(F) \leq \sigma(F) - 1$. Summing over all degenerate fibers and using Corollary 9.13, we get

$$\sum_F v(F) \leq \sum_F (\sigma(F) - 1) = \Sigma = 3.$$

Let c_{hor} be the number of connected components of D which contain at least one horizontal component, and let c_{vert} be the number of connected components of D which are purely vertical. Then

$$\#\pi_0(D) = c_{\mathrm{hor}} + c_{\mathrm{vert}}.$$

Since $D_{\mathrm{hor}} = H_0 + H_1 + H_2 + H_3$, one has $c_{\mathrm{hor}} \leq 4$. Every purely vertical connected component of D lies in a unique degenerate fiber, so

$$c_{\mathrm{vert}} = \sum_F v(F) \leq 3.$$

Therefore

$$\#\,\mathrm{Sing}(X) = \#\pi_0(D) = c_{\mathrm{hor}} + c_{\mathrm{vert}} \leq 4 + 3 = 7.$$

☐

Corollary 11.111 (Reduction to the height-selection statement)**.** *To prove the characteristic-*3 *seven-point theorem, it is enough to prove the following statement.*

Every characteristic-3 *klt del Pezzo surface* X *of Picard rank* 1 *with* $\mathrm{ht}(X) \geq 4$ *and no descendant with elliptic boundary admits a witnessing* $\mathbf{P}^1$*-fibration of height* 4 *and maximal width.*

Equivalently, the fiberwise two-touch condition from the two-part endpoint formulation of Remark 10.1 is now automatic once one is in the height-4*, maximal-width corridor.*

Proof. Assume the boxed height-selection statement. Let X be a hypothetical characteristic-3 klt del Pezzo surface of Picard rank 1 with $\#\operatorname{Sing}(X) \geq 8$. By Theorem 9.1, the surface lies in the residual sector $\mathrm{ht}(X) \geq 4$ and has no descendant with elliptic boundary. The boxed statement then provides a witnessing fibration of height 4 and maximal width, so Theorem 11.110 applies and yields $\#\operatorname{Sing}(X) \leq 7$, a contradiction. □

Remark 11.112 (Status before the final section)*.* At this point in the paper, the two-touch part of the two-part endpoint formulation of Remark 10.1 is already proved. The only issue left for the final section is the *height-selection* statement: pass from the residual sector $\mathrm{ht}(X) \geq 4$ to the existence of a witnessing fibration of height 4 and maximal width. All later packet-section, isolated-pair, terminal-fork, and old-corridor arguments are now absorbed into the terminal-leaf count above.

12. The final height-selection theorem

By Corollary 11.111, the whole characteristic-3 seven-point theorem is now reduced to the existence of a witnessing fibration of height 4 and maximal width on every residual surface. The purpose of this final section is to isolate the remaining branches in one place and to record one additional numerical restriction on the section-only high-height side.

Proposition 12.1 (Final branch decomposition of the height-selection problem)**.** *Let* X *be a characteristic-*3 *klt del Pezzo surface of Picard rank* 1 *with* $\mathrm{ht}(X) \geq 4$ *and no descendant with elliptic boundary. To prove that* X *admits a witnessing* $\mathbf{P}^1$*-fibration of height* 4 *and maximal width, it is enough to treat the following three residual cases:*

(i) *the* several-special-horizontal branch*, which contains the section-only high-height packet / isolated-collision problem;*

(ii) *the* separable special-multisection branch*;*

(iii) *the* purely inseparable ordinary singleton [2]-root branch*, already reduced to a two-fiber service problem by Corollary 11.14.*

Proof. Corollary 11.1 reduces the height-selection theorem to two global branches: several special horizontal components, or a special horizontal multisection of degree at least 2. This already gives case (i).

On the special-multisection side, the purely inseparable branch is isolated by Corollary 11.8, which produces a surviving primitive ramification fork. Corollary 11.9 then shows that every ordinary surviving fork is cubic, and Corollary 11.14 reduces the remaining ordinary singleton [2]-root service patterns to at most two fibers. This is case (iii).

The rest of the special-multisection branch is therefore separable, giving case (ii). Finally, Proposition 11.16 and Corollary 11.73 show that the low-degree one-section mixed corridor does not survive as a separate fourth branch. ☐

Lemma 12.2 (After removing the special-multisection branches, case (i) is the packet branch)**.** *In the setup of Proposition 12.1, assume that case* (ii) *(the separable special-multisection branch) and case* (iii) *(the purely inseparable ordinary singleton* [2]*-root branch) do not occur. Then the remaining content of case* (i) *is precisely the several-special-horizontal packet branch closed later in this section.*

Proof. This is exactly the content of the proof of Proposition 12.1. That proof first isolates the two global branches from Corollary 11.1: several special horizontals, or a special horizontal multisection of degree at least 2. It then splits the special-multisection side into the separable branch (case (ii)) and the purely inseparable ordinary singleton [2]-root branch (case (iii)), while also observing that the low-degree one-section mixed corridor does not survive as a separate fourth branch. Therefore, once cases (ii) and (iii) are removed, the remaining content of case (i) is exactly the packet-side several-special-horizontal branch. ☐

12.1. The several-special-horizontal branch

The several-special-horizontal side includes the section-only high-height packet geometry. Earlier sections already show that, after choosing a clean section and preparing the packet along the negative section, any packet whose pairwise collisions are supported on at most three fibers is forced into the scalar branch. The next four results sharpen the scalar side of that packet geometry: a collision-root elementary transform removes one common collision layer, the whole common scalar core peels to a disjoint packet on $\mathbb{F}_0$, and each branch of that disjoint packet still carries at least two later service centers.

Lemma 12.3 (Elementary transform at a packet collision peels one collision layer)**.** *Let* $Y_0 \to \mathbf{P}^1$ *be a smooth* $\mathbf{P}^1$*-bundle, let* $f_0 \subset Y_0$ *be a fiber, and let* $\Sigma_0, \dots, \Sigma_r \subset Y_0$ *be sections. Assume that a point* $q \in f_0$ *lies on at least two of the sections, and that a smooth surface* Y *is obtained from* Y_0 *by a sequence of blowups whose centers lie over finitely many fibers and whose branch over* f_0 *starts with the center* q*. Then there exists another* $\mathbf{P}^1$*-bundle* $Y_0' \to \mathbf{P}^1$*, obtained from* Y_0 *by the elementary transform at* q*, and a factorization of* Y *as a sequence of blowups over the image fiber* $f_0' \subset Y_0'$ *with the following properties:*

- (i) *the images of* $\Sigma_0, \dots, \Sigma_r$ *on* Y_0' *are again sections;*
- (ii) *in local coordinates adapted to the ruling, every local branch equation* $z = \varphi(t)$ *through* q *is replaced on* Y_0' *by* $w = \varphi(t)/t$*; equivalently, every pairwise packet-collision order at* q *drops by exactly one;*

(iii) *every descendant of the blowup branch issued from q is still realized over f_0', but with the first collision layer removed;*

(iv) *no blowup branch over any other fiber is changed.*

Proof. Blow up the point q, say $\alpha\colon Y_1 \to Y_0$. Let $E \subset Y_1$ be the exceptional curve and let f_1 be the strict transform of f_0. Since $q \in f_0$ is a smooth point of the fiber, $f_1^2 = -1$, so contracting f_1 produces the elementary transform $\beta\colon Y_1 \to Y_0'$. Because every Σ_i is a section, its strict transform on Y_1 meets f_1 in at most one point and descends to a section on Y_0'. This proves (i).

Choose local coordinates (t, z) centered at q and adapted to the ruling, with $f_0 = \{t = 0\}$. The elementary transform is given on the standard chart by $(t, z) \mapsto (t, w)$ with $w = z/t$. So any local branch through q with equation $z = \varphi(t)$ becomes $w = \varphi(t)/t$. If two branches had local intersection multiplicity $m = \operatorname{ord}_t(\varphi_1 - \varphi_2)$, then after the transform their new intersection multiplicity is

$$\operatorname{ord}_t\Big(\frac{\varphi_1 - \varphi_2}{t}\Big) = m - 1.$$

This proves (ii).

Now fix the chosen blowup branch over f_0. All later centers on that branch are infinitely near to q and are defined by the transformed local equations of the packet branches together with the subsequent exceptional data. Because those transformed equations are obtained by dividing by t, all descendants of the q-branch descend to centers over f_0', with the first collision layer removed and no other change in the later local combinatorics. This gives (iii). Since the elementary transform is local over the single fiber f_0, all branches over every other fiber are untouched, proving (iv). ☐

Lemma 12.4 (One common scalar packet layer is removable)**.** *Assume the setup of Corollary 9.126, and suppose that the prepared packet is in the scalar branch of Proposition 9.121. Thus, after choosing homogeneous coordinates $[x : y]$ on the base and a fiber coordinate z on $\mathbb{F}_n - S_0 \cong \operatorname{Tot}\mathcal{O}_{\mathbf{P}^1}(n)$, there exist distinct scalars $\lambda_1, \dots, \lambda_r \in k$ and nonnegative integers a, b, c with $a + b + c = n$ such that*

$$S_i = \{z = \lambda_i g\}, \qquad g = x^a y^b (x - y)^c \qquad (1 \le i \le r).$$

Assume that $a > 0$, and let $q \in \{x = 0\}$ be the common packet point on that fiber. Then the elementary transform at q produces a new ruled surface $\mathbb{F}_{n-1}$ with negative section S_0' such that:

(1) *the images S_i' of the packet sections are again sections disjoint from S_0';*

(2) *on $\mathbb{F}_{n-1} - S_0' \cong \operatorname{Tot}\mathcal{O}_{\mathbf{P}^1}(n-1)$ with a suitable fiber coordinate w, one has*

$$S_i' = \{w = \lambda_i x^{a-1} y^b (x - y)^c\} \qquad (1 \le i \le r);$$

(3) *for every fixed factorization of $\widetilde{X}$ through the original prepared ruled model, the same smooth surface $\widetilde{X}$ factors through the new ruled model, with the first common collision layer over $x = 0$ removed and all other fibers unchanged.*

Proof. The point q lies off the negative section S_0. Blow up q and contract the strict transform of the fiber $\{x = 0\}$. Because the contracted fiber meets S_0 once and the center is not on S_0, the image S_0' of S_0 has self-intersection $S_0'^2 = S_0^2 + 1 = -(n-1)$, so the target ruled surface is $\mathbb{F}_{n-1}$.

On the affine chart $y \neq 0$ with local parameter $t = x/y$ and fiber coordinate z, the packet sections are $z = \lambda_i t^a u(t)$ with $u(t) = (1-t)^c$. The elementary transform is given locally by $w = z/t$, so the transformed packet branches are $w = \lambda_i t^{a-1} u(t)$. Globally this is exactly the displayed equation for S_i'. Hence each S_i' is a section of the class $S_0' + (n-1)f$, so it is disjoint from S_0'. This proves (1) and (2).

For (3), apply Lemma 12.3 to the common collision point q. That lemma says precisely that the same smooth surface $\widetilde{X}$ may be factored over the elementary transform, with the chosen blowup branch over $x = 0$ losing its first collision layer and every other fiber unchanged. ☐

Proposition 12.5 (Prepared scalar packets peel to a disjoint packet on $\mathbb{F}_0$)**.** *Assume the setup of Corollary 9.126, and suppose that the prepared packet is in the scalar branch of Proposition 9.121. Write it on the prepared ruled model as*

$$S_i = \{z = \lambda_i x^a y^b (x-y)^c\} \qquad (1 \leq i \leq r),$$

with distinct scalars $\lambda_i \in k$ and $a+b+c = n$. Then, after $a+b+c = n$ successive elementary transforms at the marked common packet points, one obtains a ruled surface $\mathbb{F}_0$ with sections $S_0^\flat, S_1^\flat, \ldots, S_r^\flat$ such that:

1. *the packet sections $S_1^\flat, \ldots, S_r^\flat$ are pairwise disjoint;*
2. *each $S_i^\flat$ is disjoint from $S_0^\flat$;*
3. *on $\mathbb{F}_0 - S_0^\flat \cong \mathbf{P}^1 \times \mathbb{A}^1$ with a suitable fiber coordinate w, one has $S_i^\flat = \{w = \lambda_i\}$ for $1 \leq i \leq r$;*
4. *the same smooth surface $\widetilde{X}$ factors through $\mathbb{F}_0$, and over each of the three marked fibers all common scalar-collision layers are removed; in particular, every later center lying on a packet section is a packet-section point of a unique section $S_i^\flat$.*

Proof. Apply Lemma 12.4 repeatedly:

(i) a times at the common packet point over $x = 0$;

(ii) b times over $y = 0$;

(iii) c times over $x - y = 0$.

Each step lowers the Hirzebruch index by one and decreases the corresponding exponent by one. After the total of $a+b+c = n$ steps, all common factors are gone and the ruled surface has become $\mathbb{F}_0$. The packet sections are then given by $w = \lambda_i$, so they are pairwise disjoint and disjoint from $S_0^\flat$. At each step Lemma 12.3 preserves the same smooth surface $\widetilde{X}$ and removes exactly the first common collision layer on the chosen fiber, without changing any other fiber.

Therefore, once all common scalar layers have been peeled, any remaining center on a packet section lies on a unique packet branch and is therefore a packet-section point. ☐

Corollary 12.6 (After peeling the scalar core, every packet section needs at least two later service centers)**.** *Keep the setup of Proposition 12.5. Let* $H_i \subset D$ *be the strict transform on* $\widetilde{X}$ *of the section* $S_i^\flat$ *for* $1 \leq i \leq r$*. Then the birational map* $\widetilde{X} \dashrightarrow \mathbb{F}_0$ *has at least two later blowup centers on each section* $S_i^\flat$*. Equivalently, after the common scalar packet has been peeled away, every horizontal packet branch still carries a packet-section service pattern of length at least two.*

Proof. On $\mathbb{F}_0$, each section $S_i^\flat$ has self-intersection 0. Every blowup center lying on $S_i^\flat$ lowers the self-intersection of its strict transform by one. Since $H_i \subset D$ is a component of the exceptional divisor of the minimal resolution, one has $H_i^2 \leq -2$. Therefore the factorization from Proposition 12.5 must contain at least two centers on each $S_i^\flat$. Because the packet sections are pairwise disjoint on $\mathbb{F}_0$, every such center is automatically a packet-section point of a unique branch. ☐

Proposition 12.7 (Away from the marked scalar fibers, packet services strip to collisions or to one explicit tip)**.** *Keep the setup of Proposition 12.5, and fix one factorization* $\upsilon \colon \widetilde{X} \to \mathbb{F}_0$ *produced there. Let* $T^\flat \subset \mathbf{P}^1$ *be the set of the three marked fibers coming from* $x = 0$*,* $y = 0$*, and* $x - y = 0$ *before peeling. Fix a fiber* $f \subset \mathbb{F}_0$ *with* $f \notin T^\flat$*, and assume that some blowup center of* υ^{-1} *lies on the running strict transform of one packet section over* f*. Then, after finitely many repetitions of the following two operations,*

(a) *stripping a maximal initial packet-section staircase by Lemma 11.84;*

(b) *stripping an initial free center by Lemma 11.76,*

one reaches exactly one of the following three outcomes.

(i) *The whole branch disappears; equivalently, after the stripping process no vertical component of* D *survives over that branch.*

(ii) *The next unstripped center is a collision point of at least two horizontal branches on the resulting ruled model.*

(iii) *The branch contributes some vertical component of* D *and no later packet center remains. Then the last exceptional curve of the final packet-section staircase survives on* $\widetilde{X}$ *as a singly-hit multiplicity-one nonexceptional tip whose unique vertical neighbor lies in* D*.*

Proof. Because $f \notin T^\flat$, the fiber f is one of the smooth unmarked fibers of the peeled model, and the packet sections $S_1^\flat, \ldots, S_r^\flat$ meet it at pairwise distinct points. Let $q_1 \prec q_2 \prec \cdots \prec q_m$ be the chain of blowup centers of the chosen branch over f, starting at the first center $q_1 \in S_i^\flat \cap f$ for some packet section.

As in Lemma 11.85, there is a maximal initial packet-section staircase $q_1, \ldots, q_a$ along the running strict transform of that packet section. Stripping that staircase by successive applications of Lemma 11.84 replaces the ruled model by another smooth $\mathbf{P}^1$-bundle carrying the same packet sections and removes those first a packet-section steps from the branch.

If a next center remains and is a free point of the current smooth fiber disjoint from the packet, then Lemma 11.76 removes that free step as well while preserving the same final smooth surface $\widetilde{X}$. Repeating these two stripping operations as long as possible must terminate because the branch has only finitely many centers.

If no center remains at the end, then the whole branch has been stripped away and contributes no vertical component of D. This is (i).

Assume now that some center remains. By construction, the first remaining center is neither a packet-section point of a single branch nor a free point of the fiber. Hence it must be a collision point of at least two horizontal branches on the resulting ruled model. This is (ii).

It remains to analyze the case in which the stripping process ends with a final maximal packet-section staircase and with no later packet center afterwards, but with some vertical component of D still contributed by the branch. Let $F - G_1 - \cdots - G_a$ be the final staircase block from Lemma 11.85. If $a = 1$, then both curves F and G_1 are (-1)-curves and lie outside D, so the branch contributes no vertical component of D, contrary to the present assumption. Hence $a \geq 2$. Then $G_{a-1} \subset D$ and $G_a^2 = -1$. Since no later packet center remains on this branch, the last exceptional curve G_a is not blown up again at its packet-side point and survives on $\widetilde{X}$ as a vertical (-1)-curve $L := G_a \not\subset D$. Every component of the staircase appears in the fiber with multiplicity one, and the chosen packet section meets L once while no other horizontal branch passes through the packet side of this final staircase. Therefore L is a singly-hit multiplicity-one nonexceptional tip whose unique vertical neighbor is $G_{a-1} \subset D$. This is (iii). □

Corollary 12.8 (Outside the marked scalar fibers, no independent smooth service staircase remains)**.** *Keep the setup of Proposition 12.5. Then every packet-side contribution to the exceptional divisor away from the three marked scalar fibers is either:*

(i) *collision-rooted on a smooth service fiber; or*

(ii) *carried by the explicit terminal singly-hit multiplicity-one tip of Proposition 12.7*(iii)*.*

Equivalently, after peeling the common scalar core, no independent smooth packet-section staircase branch remains away from the marked fibers.

Proof. Apply Proposition 12.7 to any packet-side branch over an unmarked fiber. Alternative (i) contributes nothing to the exceptional divisor, while (ii) and (iii) are exactly the two cases listed here. □

Lemma 12.9 (Packet staircases create complementary scalar packets)**.** *Let $Y_0 \to \mathbf{P}^1$ be a smooth $\mathbf{P}^1$-bundle, let $f_0 \subset Y_0$ be a fiber, and let $\Sigma_0, \Sigma_1, \ldots, \Sigma_r \subset Y_0$ be sections. Choose local coordinates (t, z) centered along $f_0 = \{t = 0\}$ so that $\Sigma_0 = \{z = 0\}$ and $\Sigma_i = \{z = \lambda_i\}$ for $1 \leq i \leq r$, where $\lambda_1, \ldots, \lambda_r$ are pairwise distinct nonzero constants. Let $q_1 \prec q_2 \prec \cdots \prec q_u$ be the packet-section staircase of length $u \geq 1$ on Σ_0 over f_0, and strip it by the successive elementary transforms of Lemma 11.84. Then on the resulting smooth $\mathbf{P}^1$-bundle $Y_0^{(u)} \to \mathbf{P}^1$ there exist local coordinates (t, w) near the transformed fiber such that the images of $\Sigma_1, \ldots, \Sigma_r$ are $\Sigma_i^{(u)} = \{w = \lambda_i^{-1} t^u\}$ for $1 \leq i \leq r$. In particular, all complementary sections pass through* *one common point* $o_u = (0,0)$*, any two of them meet there with local intersection multiplicity exactly* u*, and the image of* Σ_0 *meets the transformed fiber away from* o_u*.*

Proof. Set $\zeta_0 := z$. Near the chosen section point on the initial bundle one has $\Sigma_0 = \{\zeta_0 = 0\}$ and $\Sigma_i = \{\zeta_0 = \lambda_i\}$. After one elementary transform at $q_1 \in \Sigma_0 \cap f_0$, a standard local coordinate near the image of the chosen section is $\zeta_1 := \zeta_0/t$. Hence the complementary sections are given there by $\zeta_1 = \lambda_i/t$, so near the point where they meet on the new fiber it is convenient to use the reciprocal coordinate $w_1 := 1/\zeta_1 = t/\zeta_0$. Then $\Sigma_i^{(1)} = \{w_1 = \lambda_i^{-1}t\}$ for $1 \le i \le r$, which is the claimed formula for $u = 1$.

Assume inductively that after stripping a staircase of length $u-1$ one has local coordinates $\zeta_{u-1} = z/t^{u-1}$ near the transformed chosen section and reciprocal coordinate $w_{u-1} = 1/\zeta_{u-1} = t^{u-1}/z$ near the complementary collision point, with $\Sigma_i^{(u-1)} = \{w_{u-1} = \lambda_i^{-1}t^{u-1}\}$ for $1 \le i \le r$. The next elementary transform is again taken at the packet-section point of the chosen section, so the new local coordinate near that section is $\zeta_u := \zeta_{u-1}/t = z/t^u$. Passing to the reciprocal coordinate near the complementary collision point gives $w_u := 1/\zeta_u = t^u/z$. Therefore $\Sigma_i^{(u)} = \{w_u = \lambda_i^{-1}t^u\}$ for $1 \le i \le r$. This proves the displayed local form for every $u \ge 1$.

Since the constants λ_i^{-1} are pairwise distinct, all complementary sections pass through $o_u = (0,0)$ and, for $i \ne j$, one has

$$(\Sigma_i^{(u)} \cdot \Sigma_j^{(u)})_{o_u} = \operatorname{ord}_t\big((\lambda_i^{-1} - \lambda_j^{-1})t^u\big) = u.$$

Finally, the chosen section is given in the coordinate ζ_u by $\zeta_u = 0$, so it meets the transformed fiber at the point corresponding to $w_u = \infty$, hence away from o_u. ☐

Proposition 12.10 (Collision-rooted unmarked packet branches are complementary scalar packets)**.** *Keep the setup of Proposition 12.5, and fix one factorization* $\upsilon\colon \widetilde{X} \to \mathbb{F}_0$ *produced there. Let* $T^\flat \subset \mathbf{P}^1$ *be the set of the three marked scalar fibers, and let* $f \subset \mathbb{F}_0$ *be a fiber with* $f \notin T^\flat$*. Assume that a packet-side contribution over* f *falls into alternative* (ii) *of Proposition 12.7. Let* $u \ge 1$ *be the length of the last packet-section staircase stripped from that branch in the proof of Proposition 12.7. Then on the final ruled model produced there:*

(i) *the first remaining center over* f *is the common collision point of the complementary packet sections;*

(ii) *in suitable local coordinates its horizontal branches are exactly of the form* $w = \mu_i t^u$ *with pairwise distinct nonzero constants* μ_i*;*

(iii) *after the forced order-*u *common-collision blowups, the last exceptional curve* G_u *is the unique component meeting all complementary packet sections, so every later center of the chosen branch lies above* G_u*.*

Proof. By construction in Proposition 12.7, immediately before the first remaining center appears there is a last maximal packet-section staircase of length u on one active packet section over the smooth unmarked fiber f. Because the peeled packet on $\mathbb{F}_0$ consists of pairwise disjoint constant sections, we may choose local coordinates so that the active section is $\Sigma_0 = \{z = 0\}$

and the remaining packet sections are $\Sigma_i = \{z = \lambda_i\}$ with pairwise distinct nonzero constants λ_i. Lemma 12.9 then shows that after stripping that last staircase, the complementary packet sections are given by $w = \mu_i t^u$ and all meet at one common point $o = (0, 0)$, while the active section meets the transformed fiber away from o.

At that stage the first remaining center is neither a packet-section point nor a free point of the fiber, by the definition of the stripped model in Proposition 12.7. Since no other point of the transformed fiber lies on more than one complementary section, that first remaining center must be exactly the common collision point o. This proves (i) and (ii).

Now apply Proposition 9.125 to the local model $w = \mu_i t^u$. It identifies the forced order-u common-collision block and shows that its last exceptional curve G_u is the unique component met by all complementary packet sections; the earlier exceptional curves are disjoint from them. Consequently any later center of the chosen branch that still follows one of those packet sections must lie on or above G_u, proving (iii). ☐

Corollary 12.11 (Away from the marked scalar fibers, surviving collision roots are complementary scalar roots)**.** *Keep the setup of Proposition 12.5. Then every packet-side contribution to the exceptional divisor away from the three marked scalar fibers is one of the following:*

(i) *it disappears after the stripping process of Proposition 12.7;*

(ii) *it is carried by the explicit terminal singly-hit multiplicity-one tip of Proposition 12.7*(iii)*;*

(iii) *after the last stripped packet staircase, it is rooted at the last exceptional curve of a complementary scalar packet over a smooth service fiber.*

In case (iii)*, either that last exceptional curve survives as a vertical curve $L \not\subset D$, in which case the same contact-defect calculation as in Corollary 9.135 applies, or every later center of the branch lies above that complementary scalar root.*

Proof. Apply Proposition 12.7 to a packet-side branch over an unmarked fiber. Alternative (i) gives nothing, and alternative (iii) is exactly case (ii) here. If alternative (ii) occurs, then Proposition 12.10 identifies the remaining collision-rooted branch with a complementary scalar packet and places all later packet-side activity above its last exceptional curve. If that curve survives on $\widetilde{X}$ as a vertical (-1)-curve outside D, the proof of Corollary 9.135 applies verbatim to this local scalar-star model. ☐

Proposition 12.12 (Rooted complementary scalar packets reduce to surviving roots or isolated pairs)**.** *Keep the setup of Corollary 12.11, and assume that Route C entry does not already occur. Let f be a smooth service fiber, and suppose that over f one reaches a complementary scalar packet of some order $\nu \geq 1$ on packet sections $B_1, \dots, B_m$ with $m \geq 2$, whose last common exceptional curve is $R \not\subset D$. Assume that some later packet-side center still lies above R. Then, after finitely many repetitions of stripping a maximal packet-section staircase on one active packet section and passing to the next complementary scalar packet if a later center remains, one reaches exactly one of the following two outcomes:*

(i) *the current last exceptional curve survives on* $\widetilde{X}$ *as a vertical curve* $L \not\subset D$*, so the same scalar-star contact-defect calculation applies;*

(ii) *the remaining packet-side block is an isolated pair collision of some order* $p \geq 1$*, hence is governed by Proposition 9.119.*

Equivalently, rooted service above complementary scalar packets introduces no genuinely new many-branch corridor beyond surviving scalar packets and isolated pair-collision service blocks.

Proof. We argue by induction on the number m of packet sections meeting the current scalar root. For $m = 2$, the current complementary scalar packet is already an isolated pair collision: in suitable local coordinates the two sections are $w = \mu_1 t^\nu$ and $w = \mu_2 t^\nu$ with $\mu_1 \neq \mu_2$, so their difference has order exactly ν, and the local block is precisely the primitive order-ν isolated pair collision of Proposition 9.119. This gives alternative (ii).

Assume now that $m \geq 3$ and that the statement is known for all smaller values. Because R is a smooth rational curve and the packet sections meet it transversely at pairwise distinct points, a neighborhood of R is locally a smooth bundle with those packet sections given by constant sections over it. Choose one packet section on the chosen branch and strip a maximal packet-section staircase above its intersection with R by the same local elementary-transform procedure as in Lemma 11.84.

If no later center remained above the resulting fiber, then the last exceptional of that staircase would be a singly-hit multiplicity-one nonexceptional tip with unique vertical neighbor in D, so Proposition 9.10 would give Route C entry, contrary to the hypothesis. Therefore a next center does remain.

The same local calculation as in Lemma 12.9 then shows that, after stripping the chosen staircase, the remaining packet sections form a complementary scalar packet over the same service fiber, now on exactly $m - 1$ packet sections. Its next unstripped center is the common collision point of that new complementary scalar packet, and any further packet-side activity lies above the last exceptional of its forced common-collision block, exactly as in the proof of Proposition 12.10. Thus the same problem reappears with one fewer active packet section. Applying the induction hypothesis to that new complementary scalar packet yields either a surviving scalar packet root, giving alternative (i), or an isolated pair collision block, giving alternative (ii). □

Corollary 12.13 (On the packet side, smooth-fiber branches reduce to surviving scalar roots or isolated pairs)**.** *Keep the setup of Corollary 12.11, and assume that Route C entry does not already occur. Then every packet-side contribution beyond the marked scalar roots is governed by exactly one of the following two mechanisms:*

(i) *a surviving scalar packet root, to which the same scalar-star contact-defect bound applies;*

(ii) *an isolated pair-collision service block of Proposition 9.119.*

In particular, in the controlled scalar regime there is no genuinely many-branch smooth-fiber corridor remaining beyond surviving scalar packets and isolated pair collisions.

Proof. By Corollary 12.11, every packet-side contribution away from the marked fibers either disappears, gives the explicit singly-hit tip, or is rooted at a complementary scalar packet. The explicit tip alternative is excluded here by the no-entry hypothesis and Proposition 9.10. If the last exceptional of the complementary scalar packet survives, we are in alternative (i). Otherwise Proposition 12.12 yields alternative (ii). ☐

The next proposition adds a further global restriction on the resulting distributed service geometry.

Proposition 12.14 (No vertical component of D can meet five or more horizontal sections)**.** *Let* $p\colon \widetilde{X} \to \mathbf{P}^1$ *be a witnessing* $\mathbf{P}^1$*-fibration such that every horizontal component of* D *is a* 1*-section. Then there does not exist a vertical irreducible component* $U \subset D$ *meeting* $m \geq 5$ *distinct horizontal sections.*

Proof. Assume for contradiction that such a component U exists, and write $H_1, \dots, H_m$ for the horizontal sections meeting U. Set $U^2 = -b$ and $H_i^2 = -a_i$ for $1 \leq i \leq m$, so $b \geq 2$ and $a_i \geq 2$.

Write $\pi^* K_X = K_{\widetilde{X}} + \sum_{E \subset D} \mathrm{cf}_X(E)\, E$. Intersecting with H_i gives

$$a_i \,\mathrm{cf}_X(H_i) = a_i - 2 + \sum_{E \sim H_i} \mathrm{cf}_X(E) \geq a_i - 2 + \mathrm{cf}_X(U).$$

Hence $\mathrm{cf}_X(H_i) \geq \mathrm{cf}_X(U)/2$ for $1 \leq i \leq m$. Since the fibration is section-only, Lemma 9.108 yields

$$2 > \sum_{H \subset D_{\mathrm{hor}}} \mathrm{cf}_X(H) \geq \sum_{i=1}^{m} \mathrm{cf}_X(H_i) \geq \frac{m}{2}\,\mathrm{cf}_X(U).$$

Therefore $\mathrm{cf}_X(U) < 4/m < 1$.

Now let $V_1, \dots, V_s$ be the vertical neighbors of U inside D. Intersecting the discrepancy formula with U gives

$$b\ \mathrm{cf}_X(U) - \sum_{i=1}^{m} \mathrm{cf}_X(H_i) - \sum_{j=1}^{s} \mathrm{cf}_X(V_j) = b - 2.$$

Using the lower bound on the $\mathrm{cf}_X(H_i)$ and the nonnegativity of the $\mathrm{cf}_X(V_j)$, we get

$$\left(b - \frac{m}{2}\right) \mathrm{cf}_X(U) \geq b - 2.$$

If $b > \frac{m}{2}$, then $\mathrm{cf}_X(U) \geq \frac{b-2}{b-\frac{m}{2}}$. Since $\frac{m}{2} > 2$, the denominator is strictly smaller than $b-2$, so the right-hand side is greater than 1, contradicting $\mathrm{cf}_X(U) < 1$.

If $b = \frac{m}{2}$, then the left-hand side is 0, while the right-hand side is strictly positive because $m \geq 5$. This is impossible.

So the only remaining possibility is $b < \frac{m}{2}$. Then the coefficient $b - \frac{m}{2}$ is negative. Since the right-hand side is nonnegative, the displayed inequality can hold only when $b = 2$ and

$\mathrm{cf}_X(U) = 0$. Substituting these values into the discrepancy equation for U gives

$$0 = \sum_{i=1}^{m} \mathrm{cf}_X(H_i) + \sum_{j=1}^{s} \mathrm{cf}_X(V_j),$$

so $\mathrm{cf}_X(H_i) = 0$ for $1 \le i \le m$ and $\mathrm{cf}_X(V_j) = 0$ for $1 \le j \le s$.

Now intersect the discrepancy formula with some H_i. Since $\mathrm{cf}_X(H_i) = \mathrm{cf}_X(U) = 0$, one gets

$$- \sum_{W \sim H_i,\ W \neq U} \mathrm{cf}_X(W) = a_i - 2.$$

Hence $a_i = 2$, and every other neighbor of H_i also has discrepancy coefficient 0. Repeating this argument along the connected component of D containing U shows that every curve in that connected component has self-intersection -2 and discrepancy coefficient 0. So the corresponding singularity is canonical, hence Du Val, and its exceptional dual graph must be of ADE type. But the vertex corresponding to U has valence at least $m \geq 5$, which is impossible for an ADE graph. This contradiction proves the proposition. ☐

Corollary 12.15 (The section-only residue is a five-defect packet problem)**.** *Assume the section-only high-height branch of Proposition 12.1*(i)*. Choose a clean section $H_0 \subset D_{\mathrm{hor}}$ as in Corollary 9.116, and let $\mathcal{P} := \{H \subset D_{\mathrm{hor}}\ ;\ H \neq H_0\}$ be the packet of the remaining horizontal sections. Then at most five members of $\mathcal{P}$ are not strong in the sense of Definition 9.112. Equivalently, after removing a defect set of size at most* 5*, the residual packet consists entirely of strong sections.*

Proof. By Theorem 9.114, the total horizontal multiplicity carried by mild or ordinary horizontal components is at most 5. In the section-only case every horizontal component has multiplicity 1, so there are at most five non-strong horizontal sections in total. Removing the chosen clean section H_0 cannot increase that number inside $\mathcal{P}$. ☐

Lemma 12.16 (The first split of a horizontal collision tree is read off from the first non-common jet)**.** *Let $A_1, \ldots, A_r$ with $r \geq 2$ be pairwise distinct smooth horizontal branches through a point o of a smooth ruled surface $Y \to \mathbf{P}^1$. Choose local coordinates (t, z) at o adapted to the ruling, so the fiber through o is $\{t = 0\}$, and after translating by A_1 one has $A_1 = \{z = 0\}$ and $A_i = \{z = \phi_i(t)\}$ for $2 \le i \le r$, with $\phi_i \in t\,k[[t]]$. Let μ be the order of the common jet of the family, and write $\phi_i(t) = \alpha_i t^{\mu}$ + (higher order terms) with $\alpha_1 := 0$. Then the first μ blowups at the successive unique common points of the strict transforms are forced, and on the last exceptional curve the packet-section points are indexed exactly by the distinct values among $\alpha_1, \ldots, \alpha_r$. Equivalently, after those μ common blowups the branches split into disjoint clusters according to their first non-common coefficient.*

Proof. Because every difference $\phi_i - \phi_j$ vanishes to order at least μ, all branches have the same current point for the first μ blowups, so those blowups are forced. In the standard chart after these common blowups one has $z = t^{\mu} w$. Hence the strict transform of A_i is given by $w = \psi_i(t) := t^{-\mu}\phi_i(t)$ with $\psi_i(0) = \alpha_i$. The last exceptional curve is $\{t = 0\}$, and the strict transform of A_i meets it at the point $w = \alpha_i$. So the distinct packet-section points on that exceptional curve are indexed exactly by the distinct coefficients among the α_i, and the branches through each such point are precisely the branches in the corresponding cluster. ☐

Corollary 12.17 (Every multibranch horizontal collision tree reduces to binary and unary descendants)**.** *Let $A_1, \dots, A_r$ be pairwise distinct smooth horizontal branches through a point of a smooth ruled surface. Iterating Lemma 12.16 on every cluster of size at least 3 decomposes the full local factorization into finitely many descendants of only two kinds:*

(i) *unary descendants, supported on a single horizontal branch;*

(ii) *binary descendants, supported on exactly two horizontal branches.*

Equivalently, there is no irreducible many-branch horizontal local block beyond binary pair-collision trees and single-branch service or unibranch descendants.

Proof. Proceed by induction on r. The statement is tautological for $r \leq 2$. Assume $r \geq 3$ and apply Lemma 12.16 to the first non-common jet of the family. After the forced common blowups, later horizontal centers over the original point are supported independently over the smaller clusters determined by the first non-common coefficient. By induction, each such cluster produces only unary and binary descendants. Since the original local factorization is the union of these independent cluster descendants together with possible purely vertical blowups away from all horizontal branches, the same conclusion holds for the original family. ☐

Corollary 12.18 (The strong packet side is a five-defect binary service problem)**.** *Assume the section-only high-height branch of Proposition 12.1*(i)*. Choose a clean section and the packet $\mathcal{P}$ as in Corollary 12.15, peel the scalar core as in Proposition 12.5, and remove the defect set of at most 5 non-strong packet members. Then every local collision tree on the residual strong packet decomposes into binary pair-collision trees and unary packet-section service branches. In particular, the several-special-horizontal branch contributes no new many-branch local horizontal corridor beyond the primitive binary pair-collision-tree problem on strong packet sections.*

Proof. By Corollary 12.15, all but at most 5 packet members are strong. After Proposition 12.5, the common scalar core has been removed and the remaining packet geometry is supported on disjoint packet sections of $\mathbb{F}_0$. Any later local collision of those sections is therefore a multibranch horizontal collision tree on a smooth ruled surface, so Corollary 12.17 applies directly. The unary descendants are exactly packet-section service branches, while the binary descendants are pair-collision trees. Since the residual packet has already been reduced to strong sections outside the defect set, no new many-branch local side type remains. ☐

Proposition 12.19 (A later packet-point blowup on a scalar hub gives a tip-entry curve)**.** *Keep the setup of Proposition 9.125. Thus on a smooth ruled surface one has a primitive scalar packet of order $\nu \geq 1$ on a smooth fiber, with last common exceptional curve G_ν meeting packet sections $B_1, \dots, B_m$ with $m \geq 2$ transversely at pairwise distinct packet-section points. Assume that in the chosen factorization of the inverse birational map to $\widetilde{X}$ a later blowup center lies at one of those packet-section points on G_ν. Then $\widetilde{X}$ contains a vertical curve $L \not\subset D$ of multiplicity* *one such that* $L \cdot D_{\mathrm{hor}} = 1$*, and the unique connected component of the reduced fiber minus* L *meeting* L *belongs to* D*. Consequently Proposition 9.10 applies to* L*.*

Proof. Choose the first later blowup center lying at one of the packet-section points of G_ν, say at the point where the strict transform of B_1 meets G_ν. By Proposition 9.125(3), no other packet section passes through that point. So near that point the local picture is a smooth horizontal branch crossing a smooth multiplicity-one fiber component. Repeated blowups above that point create a chain $E_1 - \cdots - E_r$ attached to the strict transform of G_ν, where the strict transform of B_1 meets the last exceptional curve $L := E_r$ transversely once and no other horizontal component meets L.

Because the first later blowup occurs on G_ν itself, the strict transform of G_ν becomes a (-2)-curve and therefore belongs to D; likewise every earlier exceptional curve on this later branch belongs to D. Since the whole branch is created by successive blowups at smooth points of a multiplicity-one fiber component, the terminal exceptional curve L also has multiplicity one in the final fiber. Therefore $\mathrm{mult}(L) = 1$ and $L \cdot D_{\mathrm{hor}} = 1$, while the unique vertical connected component of the reduced fiber meeting L belongs to D. Proposition 9.10 now applies. □

Proposition 12.20 (Continuing complementary scalar packets are at most five-ary)**.** *Keep the setup of Proposition 12.12, and assume that Route C entry does not already occur. Let* $R \not\subset D$ *be the last common exceptional curve of a complementary scalar packet on packet sections* $B_1, \dots, B_m$ *with* $m \geq 2$ *over a smooth service fiber. Assume that some later packet-side center still lies above* R*. Then* $m \leq 5$*.*

Proof. By assumption, the packet is section-only, so every packet section is a 1-section. Choose the first later packet-side center above R, say at the packet-section point of B_1. Blowing up that point lowers the self-intersection of R from -1 to -2, so the strict transform $U \subset D$ of R on the final surface belongs to the exceptional divisor. Because the chosen blowup uses only the packet-section point of B_1, the strict transforms of $B_2, \dots, B_m$ still meet U transversely at pairwise distinct points. Thus U meets at least $m - 1$ distinct horizontal sections.

Proposition 12.14 says that no vertical component of D can meet five or more horizontal sections. Hence $m - 1 \leq 4$, that is, $m \leq 5$. □

Marked scalar service and surviving scalar roots.

Proposition 12.21 (Marked-fiber scalar service strips to a complementary scalar packet)**.** *Keep the setup of Proposition 12.5, and fix one of the three marked scalar fibers* $f \in T^\flat$*. Assume that in the chosen factorization* $v \colon \widetilde{X} \to \mathbb{F}_0$ *a packet-side branch over* f *starts with a maximal packet-section staircase of length* $u \geq 1$ *on one packet section, say on* $S_1^\flat$*. Then, after stripping that staircase by Lemma 11.84, the remaining packet sections* $S_2^\flat, \dots, S_r^\flat$ *form a complementary scalar packet of order* u *over the transformed fiber. More precisely, on the resulting smooth ruled model there are local coordinates* (t, w) *at the new common point such that* $S_i' = \{w = \mu_i t^u\}$ *for* $2 \leq i \leq r$*, where the* μ_i *are pairwise distinct nonzero constants. In particular, any later* *packet-side activity above the marked fiber is rooted at a complementary scalar packet on one fewer active packet section.*

Proof. Near the chosen point of $S_1^\flat \cap f$ choose local coordinates (t, z) with $f = \{t = 0\}$, $S_1^\flat = \{z = 0\}$, and $S_i^\flat = \{z = \lambda_i\}$ for $2 \le i \le r$, where the λ_i are pairwise distinct nonzero constants. Stripping the maximal packet-section staircase of length u is exactly the situation of Lemma 12.9, which gives local coordinates (t, w) on the transformed ruled model such that $S_i' = \{\lambda_i^{-1} t^u\}$ for $2 \le i \le r$. Thus the remaining packet sections form a complementary scalar packet of order u over the transformed fiber, and every later packet-side center above the marked fiber lies over the common point of that complementary scalar packet. ☐

Corollary 12.22 (Marked-fiber scalar service reduces to surviving scalar roots or binary pair blocks)**.** *Keep the setup of Proposition 12.5. Every remaining packet-side contribution over one of the three marked scalar fibers reduces, after finitely many maximal packet-section staircase strips, to exactly one of the following outputs:*

(i) *a surviving scalar root;*

(ii) *an isolated pair-collision service block whose terminal binary collision is governed by Proposition 9.119.*

Equivalently, marked-fiber scalar service contributes no endpoint beyond the same surviving-root / binary-pair outputs already present on smooth service fibers.

Proof. Take a packet-side branch over a marked fiber. If repeated staircase stripping removes the whole branch, then it contributes nothing further to D. Otherwise Proposition 12.21 shows that after stripping the first maximal packet-section staircase one reaches a complementary scalar packet on one fewer active packet section. Now Proposition 12.12 applies and gives exactly the two displayed alternatives. ☐

Proposition 12.23 (Surviving many-ary scalar roots already feed the old horizontal corridor)**.** *Let a surviving scalar root on the packet side meet horizontal packet sections $B_1, \dots, B_m \subset D_{\mathrm{hor}}$ with $m \ge 3$, and let $L \subset \widetilde{X} \setminus D$ be the surviving strict transform of its last common exceptional curve. Then at least one of the packet sections B_i already lies in the old horizontal Route C corridor: either the connected component of D containing B_i is not a chain, or it is a chain whose reduced touched core is a touched tip of weight 2. In particular, surviving scalar roots of arity at least 3 do not contribute an independent packet-side endpoint.*

Proof. Since L is a surviving scalar root, it meets each of the packet sections $B_1, \dots, B_m$ once. Therefore Corollary 9.135 applies and gives $\sum_{i=1}^m \mathrm{cf}_X(B_i) < 1$. Because $m \ge 3$, at least one index i satisfies $\mathrm{cf}_X(B_i) < \frac{1}{3}$. If the connected component of D containing B_i is not a chain, then B_i already lies on a branching component of D. If that connected component is a chain, then Lemma 11.45 shows that its reduced touched core is a touched tip of weight 2. So in either case one already lands in the old horizontal Route C corridor. ☐

Proposition 12.24 (Binary scalar roots are exactly primitive isolated pair collisions)**.** *Let a surviving scalar root on the packet side meet exactly two horizontal packet sections $B_1, B_2 \subset D_{\mathrm{hor}}$, and let $u \geq 1$ be its scalar order. Then the first u blowups over the root are exactly the order-u isolated collision resolution of Proposition 9.119 for the pair B_1, B_2. Consequently the surviving binary scalar root is precisely the primitive contact-2 or primitive fork output attached to that isolated pair collision. In particular, a binary scalar root contributes no separate scalar branch beyond the already isolated binary pair-collision model.*

Proof. Choose local coordinates (t, w) at the scalar root so that $B_1 = \{\mu_1 t^u\}$ and $B_2 = \{\mu_2 t^u\}$ with $\mu_1 \neq \mu_2$. After replacing w by $w - \mu_1 t^u$ and rescaling, one may rewrite the two branches as $B_1 = \{z = 0\}$ and $B_2 = \{z = t^u\}$ in suitable local coordinates (t, z). Thus the two packet sections have local intersection multiplicity exactly u at the root. Proposition 9.119 therefore applies and shows that the first u blowups are exactly the forced order-u two-branch collision resolution, with primitive last exceptional curve equal to a contact-2 curve when $u = 1$ and to a primitive fork center when $u \geq 2$. But this is exactly the primitive scalar-root chain for the binary packet. □

Theorem 12.25 (The scalar-root branch adds no independent packet endpoint)**.** *Assume the setup of Corollary 12.18. Then every packet-side branch governed by a scalar mechanism is absorbed as follows.*

1. *Every remaining marked-fiber scalar service contribution reduces to a surviving scalar root or an isolated pair-collision service block, by Corollary 12.22.*
2. *Every surviving scalar root of arity at least 3 already feeds the old horizontal Route C corridor, by Proposition 12.23.*
3. *Every surviving scalar root of arity 2 is exactly the primitive isolated pair-collision output of Proposition 9.119, by Proposition 12.24.*

Hence the packet side has no separate scalar-root residue: marked-fiber scalar service and surviving scalar roots contribute nothing beyond the already isolated old horizontal corridor and the binary pair-collision service blocks.

Proof. Take any packet-side branch governed by a scalar mechanism. If it lies over one of the three marked scalar fibers, then Corollary 12.22 reduces it to either a surviving scalar root or a binary pair-collision block. If it lies over a smooth service fiber, then Corollary 12.13 already gives the same dichotomy. So it remains only to analyze surviving scalar roots.

If the surviving scalar root has arity at least 3, then Proposition 12.23 shows that one of its horizontal packet arms already lies in the old horizontal Route C corridor. If instead the surviving scalar root has arity 2, then Proposition 12.24 identifies it with the primitive isolated pair-collision model. □

Defect-supported outputs.

Proposition 12.26 (Realized binary packet descendants have only primitive or branching terminal outputs)**.** *Assume the setup of Corollary 12.18, and assume moreover that Route C entry does not already occur. Let $\mathcal{T}$ be a realized binary packet descendant supported on exactly two packet sections $H_i, H_j \subset D_{\mathrm{hor}}$. Choose a terminal leaf of the blowup tree of $\mathcal{T}$, and on that leaf let q_c be the last collision center of the strict transforms of the two active horizontal branches. Let $u \geq 0$ be the length of the maximal packet-section staircase immediately preceding q_c on the chosen leaf, and let $p \geq 1$ be the order of the isolated pair collision at q_c. Then the exceptional curve L_0 created by the last blowup of that order-p collision block satisfies exactly one of the following two alternatives.*

(i) *No later blowup center on the chosen leaf lies on L_0. Then the final transform of L_0 on $\widetilde{X}$ is a nonexceptional curve $L \subset \widetilde{X} \setminus D$ meeting H_i and H_j transversely once each, and*

$$L \cdot (D - H_i - H_j) = \begin{cases} 0, & (u,p) = (0,1), \\ 1, & \textit{otherwise}. \end{cases}$$

In particular, L is either a surviving primitive contact-2 curve on H_i, H_j or a surviving fork center whose horizontal arms are H_i, H_j.

(ii) *Some later blowup center on the chosen leaf lies on L_0. Then the final strict transform $E \subset D$ of L_0 is a branching component of D: it meets both packet sections H_i, H_j and at least one later vertical component.*

Proof. Because $\mathcal{T}$ is a binary descendant in the sense of Corollary 12.18, every horizontal center on the chosen leaf is supported on the strict transforms of exactly two packet sections, namely H_i and H_j. So the last collision center q_c is well defined.

Let M be the current smooth multiplicity-one fiber component at the start of the final service phase immediately preceding q_c, and let A_0, C_0 be the current strict transforms of H_i, H_j. By construction there are exactly u successive packet-section blowups on A_0 before the final collision, and then an isolated order-p collision of the two active branches. Hence one may choose local coordinates (t,z) centered at the starting packet-section point of that final phase so that

$$M = \{t = 0\}, \qquad A_0 = \{z = 0\}, \qquad C_0 = \{z = t^{u+p}\}.$$

After the first u packet blowups, the reduced local fiber is $M - G_1 - \cdots - G_u$, and in suitable local coordinates the strict transforms of the two active horizontal branches are $A_u = \{w_u = 0\}$ and $C_u = \{w_u = t^p\}$. Applying Lemma 9.20 to the pair A_u, C_u adds the order-p pair-collision chain $F_1 - \cdots - F_p$, with $F_k^2 = -2$ for $1 \leq k < p$ and $F_p^2 = -1$, and the strict transforms of A_u and C_u meeting F_p transversely at distinct points. Set $L_0 := F_p$.

We next show that no later packet-section center can occur after the last collision q_c. Suppose otherwise. Because q_c is the last collision center on the chosen leaf, every later horizontal center would be a packet-section point of one of the two horizontal branches H_i, H_j. Take the first such later packet-section center and follow the maximal packet-section staircase issued from it. Since no later collision exists, this staircase is terminal on the chosen leaf. By the same local calculation as in Lemma 11.85, the last exceptional curve of that terminal staircase survives as a singly-hit multiplicity-one nonexceptional tip with unique vertical neighbor in D. Proposition 9.10 then gives Route C entry, contrary to the standing hypothesis. Hence there is no later packet-section center after q_c.

Therefore every later blowup center on the chosen leaf is purely vertical and is disjoint from the two horizontal contact points of L_0. If no later center lies on L_0, then the final transform of L_0 survives on $\widetilde{X}$ as a curve $L \subset \widetilde{X} \setminus D$ meeting H_i and H_j once each. The local description above shows that $L \cdot (D - H_i - H_j) = 0$ in the primitive transverse case $(u, p) = (0, 1)$, while in every other case exactly one vertical component of the local block meets L, namely F_{p-1} when $p \geq 2$ and G_u when $u \geq 1$ and $p = 1$. This is alternative (i).

Assume now that some later blowup center on the chosen leaf does lie on L_0, and let r be the first such later center. By the previous paragraph, r is purely vertical and is away from the two horizontal contact points of L_0. After blowing up r, the strict transform of L_0 has self-intersection at most -2. Therefore its final transform cannot remain a vertical component outside D, because [7, Lemma 2.8(b)] says that every vertical component of a degenerate fiber not contained in D is a (-1)-curve. Thus the final transform $E \subset D$ is an exceptional component on $\widetilde{X}$. Because the blowup at r is away from the two horizontal contact points, the final transform E still meets the strict transforms of H_i and H_j. It also meets the exceptional curve created over r, and therefore has valence at least 3 in D. So E is branching. This is alternative (ii). □

Proposition 12.27 (Every realized binary packet output is already old or impossible)**.** *Keep the setup of Corollary 12.18. Let $L \subset \widetilde{X} \setminus D$ be a surviving nonexceptional (-1)-curve arising from a binary packet output on packet sections $H_i, H_j \subset D_{\text{hor}}$. Then the configuration is already absorbed by previously isolated Route C geometry or is impossible. In particular, the finite defect set of non-strong packet sections contributes no independent realized binary output.*

Proof. By Corollary 12.13, a realized binary packet output comes either from a binary scalar root or from an isolated pair-collision service block. Apply Proposition 12.26 to the terminal leaf supporting L. Since L is a surviving nonexceptional curve, alternative (ii) of that proposition does not occur. Hence L is of exactly one of the following two types:

(1) a primitive contact-2 curve meeting H_i and H_j transversely once each and no other component of D;

(2) a primitive fork center meeting H_i and H_j transversely once each and one further component $A \subset D$.

If at least one of H_i, H_j is strong in the sense of Definition 9.112, then by definition that horizontal arm already lies in the old branching / touched-[2] corridor. So assume from now on that neither H_i nor H_j is strong. Then each of H_i, H_j is mild or ordinary.

Assume first that L is a fork output. If both H_i and H_j are ordinary, then Lemma 9.113 gives $\mathrm{cf}_X(H_i) \geq \frac{1}{2}$ and $\mathrm{cf}_X(H_j) \geq \frac{1}{2}$. Applying Proposition 9.133 to L gives $\mathrm{cf}_X(H_i) + \mathrm{cf}_X(H_j) + \mathrm{cf}_X(A) < 1$, a contradiction. So in the fork case at least one arm is mild. A mild arm is special in the sense of Definition 11.2 and is not of touched-[2] type. Relative to that arm one has $m = 1$ and $L \cdot (D - H) = 2$, so Proposition 12.36 applies and shows that the fork is already old geometry.

Assume now that L is a primitive contact-2 output. If both H_i and H_j are ordinary, then Lemma 9.113 again gives $\mathrm{cf}_X(H_i) \geq \frac{1}{2}$ and $\mathrm{cf}_X(H_j) \geq \frac{1}{2}$. Applying Proposition 9.133 to L now gives $\mathrm{cf}_X(H_i) + \mathrm{cf}_X(H_j) < 1$, again a contradiction. So in the contact-2 case at least one arm is mild. If the connected component of D containing that mild arm is not a chain, then one is already on the branching side. If it is a chain, then the mild arm is special and not of touched-[2] type, so after 2-tail reduction its reduced touched core is [3]. Thus Corollary 9.97 places the resulting bridge in the old special-tip corridor. □

Corollary 12.28 (Every realized binary packet descendant is already old or impossible)**.** *Assume the setup of Corollary 12.18, and assume moreover that Route C entry does not already occur. Let $\mathcal{T}$ be a realized binary packet descendant supported on exactly two packet sections $H_i, H_j \subset D_{\mathrm{hor}}$. Then $\mathcal{T}$ is already absorbed by previously isolated Route C geometry or is impossible. In particular, the finite defect set of non-strong packet sections contributes no independent realized binary packet output.*

Proof. Choose a terminal leaf of the blowup tree of $\mathcal{T}$ and apply Proposition 12.26. If alternative (i) holds, then the corresponding surviving nonexceptional curve $L \subset \widetilde{X} \setminus D$ is already absorbed or excluded by Proposition 12.27. If alternative (ii) holds, then the final transform $E \subset D$ is a vertical branching component meeting both H_i and H_j, so it is already old vertical Route C geometry. Thus no realized binary packet descendant contributes an independent residual packet-side output. □

Theorem 12.29 (The defect-supported packet sector contributes no independent residual endpoint)**.** *Assume the setup of Corollary 12.18. Then the several-special-horizontal packet branch admits the following further reduction.*

(1) *If a later blowup center hits a packet-section point on a primitive scalar hub, then Proposition 12.19 gives direct Route C entry.*

(2) *Every remaining scalar mechanism is already absorbed by Theorem 12.25.*

(3) *Every realized binary packet descendant is already old geometry or impossible, by Corollary 12.28.*

Consequently the packet side no longer has a separate defect-supported binary endpoint. Equivalently, the defect set of size at most 5 is harmless: it contributes no independent packet-side residue.

Proof. If Route C entry already occurs, there is nothing to prove, so assume from now on that it does not. By Theorem 12.25, after peeling the common scalar core the only packet-side outputs not already absorbed by the scalar analysis are binary pair-collision blocks. If a later blowup center hits a packet-section point on a primitive scalar hub, then Proposition 12.19 gives item (1). The remaining scalar mechanisms are handled by Theorem 12.25, giving item (2). Every realized binary packet descendant is already old geometry or impossible by Corollary 12.28, which gives item (3). Thus the defect set no longer contributes a separate packet-side residue. ☐

Theorem 12.30 (The several-special-horizontal packet side has no new local corridor)**.** *Assume the setup of Corollary 12.18. After removing the defect set and peeling the common scalar core, every packet-side contribution is already one of the following:*

1. *direct Route C entry;*
2. *old geometry already absorbed by the previously isolated horizontal or vertical Route C corridors;*
3. *an impossible configuration;*
4. *the genuinely global realization/survival problem for a binary pair output on the residual strong packet.*

Equivalently, the several-special-horizontal packet branch contributes no genuinely new packet-specific local corridor.

Proof. The scalar mechanisms are absorbed by Theorem 12.25. The defect-supported residue is eliminated by Theorem 12.29. So the only unresolved packet-side content is the global extraction/survival step for a binary pair output on the residual strong packet. ☐

Closing the residual strong packet.

Lemma 12.31 (After removing terminal unary staircases, the last essential binary block on a packet branch is either surviving or already branching)**.** *Assume we are in the residual strong-packet regime of Theorem 12.30. Choose a packet-side branch whose remaining local tree is binary in the sense of Corollary 12.17. Assume moreover that terminal packet-section staircases on this branch have already been discarded as direct Route C entry outputs via Proposition 9.10. Then there exist two packet sections $H_i, H_j \subset D_{\text{hor}}$ and integers $u \geq 0$, $p \geq 1$ such that, on the chosen branch, the last isolated pair collision of horizontal strict transforms has local model $M = \{t = 0\}$, $A_0 = \{z = 0\}$, and $C_0 = \{z = t^{u+p}\}$, where the first u blowups are packet-section blowups on A_0 and the next p blowups are the forced order-p collision blowups of A_0 and C_0. At the end of this last horizontal block the reduced local fiber is $M - G_1 - \cdots - G_u - F_1 - \cdots - F_p$, and the strict transforms of A_0 and C_0 meet F_p transversely at distinct points. Their strict transforms on $\widetilde{X}$ are exactly H_i and H_j. Moreover exactly one of the following holds.*

1. *No later center lies on F_p. Then $L := F_p$ survives on $\widetilde{X}$ as a nonexceptional curve with $L \cdot H_i = L \cdot H_j = 1$ and*

$$L \cdot (D - H_i - H_j) = \begin{cases} 0, & (u, p) = (0, 1), \\ 1, & \textit{otherwise.} \end{cases}$$

In particular, L is either a surviving primitive contact-2 curve on the two packet sections or a surviving fork center with those two packet sections as its horizontal arms.

(2) *Some later center lies on F_p. Then every later center on the transform of F_p is purely vertical, the final transform $U \subset D$ of F_p is a vertical branching component, and U still meets H_i and H_j.*

Proof. Start with the chosen packet-side branch. Whenever the current tail starts at a packet-section point, apply Proposition 11.86 to that tail. Because direct Route C entry has been excluded in the present residual regime, that proposition shows that after stripping finitely many maximal initial packet-section staircases the next essential horizontal event on the tail is a collision point. Repeating this along the chosen branch, and using the finiteness of the blowup tree, one reaches a last horizontal collision on the branch. Since the remaining local tree is binary in the sense of Corollary 12.17, that last collision involves exactly two packet sections, say H_i and H_j.

Let $u \geq 0$ be the length of the maximal packet-section staircase immediately preceding the last isolated pair collision on this branch, and let $p \geq 1$ be the order of that collision. Choose local coordinates (t, z) at the start of this final packet-service phase so that $M = \{t = 0\}$, $A_0 = \{z = 0\}$, and $C_0 = \{z = t^{u+p}\}$. After the first u packet blowups one reaches local coordinates with $A_u = \{w_u = 0\}$ and $C_u = \{w_u = t^p\}$, and the reduced local fiber is $M - G_1 - \cdots - G_u$. Applying Lemma 9.20 to the pair A_u, C_u then adds the chain $F_1 - \cdots - F_p$, with $F_i^2 = -2$ for $1 \leq i < p$ and $F_p^2 = -1$. Thus at the end of the last horizontal block the reduced local fiber is exactly $M - G_1 - \cdots - G_u - F_1 - \cdots - F_p$, and the strict transforms of A_0 and C_0 meet F_p transversely at distinct points. Their final strict transforms on $\widetilde{X}$ are precisely H_i and H_j.

Suppose first that no later center lies on F_p. Then $L := F_p$ survives on $\widetilde{X}$ as a nonexceptional curve, and the two horizontal branches meeting L are exactly H_i and H_j. If $(u, p) = (0, 1)$, then L is the primitive last exceptional of a transverse isolated pair collision, so $L \cdot (D - H_i - H_j) = 0$. In every other case, the local chain preceding L contains exactly one adjacent vertical component of D, namely F_{p-1} when $p \geq 2$ or G_u when $u \geq 1$ and $p = 1$. No other component of D meets L, so $L \cdot (D - H_i - H_j) = 1$. This proves alternative (1).

Assume now that some later center lies on F_p. We claim that no later center can lie at either of the two packet-section points where the current transforms of H_i and H_j meet the current transform of F_p. Indeed, if the first such later center existed, then the tail starting at that point would be a packet-section-rooted branch with no later collision, because the chosen collision is the last one on the branch. Applying Proposition 11.86 to that tail would then force direct Route C entry, contrary to the residual no-entry hypothesis. So every later center on the transform of F_p is purely vertical. Let q be the first such center. Blowing up q lowers the self-intersection of the current transform of F_p from -1 to -2. Therefore the final transform U of F_p cannot remain a vertical component outside D, because [7, Lemma 2.8(b)] says that every vertical component of a degenerate fiber not contained in D is a (-1)-curve. Thus $U \subset D$. The two horizontal intersection points were never blown up, so U still meets H_i and H_j. Moreover the first later blowup on the transform of F_p creates a new vertical exceptional curve meeting the new transform of F_p, and after any further blowups centered on the current transform, the new transform still has at least one vertical neighbor. So U is a vertical branching component of D. ☐

Proposition 12.32 (The last essential binary block on strong packet sections is already old)**.** *Keep the setup of Lemma 12.31, and assume that H_i and H_j are strong packet sections. Then the last essential binary block already lands in previously isolated Route C geometry. More precisely:*

(i) *in alternative (1) of the lemma, the curve L is either a surviving primitive contact-2 bridge or a surviving fork center on the strong sections H_i and H_j, so it is already absorbed by the old branching / touched-[2] corridor;*

(ii) *in alternative (2) of the lemma, the final transform $U \subset D$ is a vertical branching component meeting H_i and H_j, so it is already old vertical geometry.*

Proof. In alternative (1), the two subcases are exactly the intersection patterns furnished by Lemma 12.31. If $(u, p) = (0, 1)$, then L is a surviving primitive contact-2 bridge on the strong packet sections H_i and H_j. If $(u, p) \neq (0, 1)$, then L has one additional vertical neighbor in D, so the terminal local core is a fork with horizontal arms H_i and H_j. Because H_i and H_j are strong in the sense of Definition 9.112, each already lies on one of the old horizontal side classes. So alternative (1) is already old horizontal geometry.

Alternative (2) is exactly old vertical branching geometry inside D. ☐

Theorem 12.33 (The several-special-horizontal packet side contributes no separate residual gap)**.** *Assume the setup of Corollary 12.18. Then the several-special-horizontal packet branch contributes no separate unresolved packet problem. Equivalently, the packet side is closed as an independent branch of the height-selection analysis.*

Proof. By Theorem 12.30, the only unresolved packet-side content is the global realization/survival problem for a binary pair output on the residual strong packet. If no residual strong packet section remains after removing the defect set, then Theorem 12.29 has already closed the packet side. So assume the residual strong packet is nonempty. By Corollary 12.6, every remaining strong packet section still carries at least two later blowup centers, and by Corollary 12.17 every remaining local tree is binary after discarding terminal unary staircases that already give direct entry. Choose a packet-side leaf branch in this residual strong packet. Lemma 12.31 applies to its last essential binary block. If the lemma yields alternative (1), then Proposition 12.32 absorbs the surviving output into the old branching / touched-[2] corridor. If it yields alternative (2), the same proposition shows that the final transform is already old vertical branching geometry. Hence no separate packet-specific residual branch remains. ☐

Remark 12.34 (What remains after closing the packet side). Proposition 9.123 and Corollary 9.126 already show that, after preparation, every three-fiber section-only packet of height at least 6 is forced into the scalar branch. Corollary 12.15 shows that, after choosing a clean section, all but at most 5 packet members are strong. Proposition 12.5 removes the whole common scalar core and replaces it by a disjoint packet on $\mathbb{F}_0$, while Corollary 12.6 shows that every packet branch still needs at least two later service centers. Corollary 12.11 and Corollary 12.13 reduce every surviving smooth-fiber branch to either a surviving scalar root or an isolated pair-collision block, and Corollary 12.18 removes the remaining many-branch ambiguity on the residual strong packet. Proposition 12.14 bounds every continuing complementary scalar packet by the same five-defect budget. Theorem 12.25 removes the marked-scalar and surviving-scalar-root residue, Theorem 12.29 removes the defect-supported binary residue, and Theorem 12.33 closes the remaining strong binary packet regime. So the packet side is now closed as an independent branch of the height-selection analysis.

12.2. The special-multisection branch

The special-multisection side now breaks into three concrete pieces: the smooth-image separable branch, the one-section singular-image separable branch, and the purely inseparable ordinary singleton [2]-root branch. The next results first show that any ramification or singular-image output with the expected extra contact is already old geometry, then collapse the smooth-image side to an explicit low-degree corridor, close the one-section singular-image side, and finally remove the purely inseparable [2]-root branch altogether.

Lemma 12.35 (A special horizontal three-arm contributes at least one third)**.** *Assume the setup of Definition 11.2. Let $H \subset D_{\mathrm{hor}}$ be a horizontal component such that the connected component of D containing H is a chain and, after 2-tail reduction relative to H, the reduced touched core is the singleton chain* [3]*. Then $\mathrm{cf}_X(H) \geq \frac{1}{3}$.*

Proof. Let $(C^\sharp, H^\sharp)$ be the reduced touched pair obtained from the chain component C containing H. Lemma 9.106 gives $\mathrm{cf}_X(H) = 1-\mu(C,H)$, while Lemma 9.93 gives $\mu(C,H) \leq \mu(C^\sharp, H^\sharp)$. Since $(C^\sharp, H^\sharp) = ([3], \text{its unique component})$, one has $\mu(C^\sharp, H^\sharp) = \frac{2}{3}$, hence $\mathrm{cf}_X(H) \geq 1 - \frac{2}{3} = \frac{1}{3}$. ☐

Proposition 12.36 (A non-touched-two special horizontal arm with enough extra contact is already old)**.** *Let $L \subset \widetilde{X}$ be a (-1)-curve with $L \not\subset D$. Assume that for some horizontal component $H \subset D_{\mathrm{hor}}$ one has $m := L \cdot H \in \{1,2\}$ and $L \cdot (D-H) \geq 3-m$. Assume moreover that H is special in the sense of Definition 11.2, and that H is not of touched-*[2] *type. Then one already falls into one of the previously isolated old Route C side types:*

(a) *a branching side; or*

(b) *a chain side whose reduced touched core is a touched tip of weight* 2*.*

Proof. If the connected component of D containing H is not a chain, then one is already on the branching side. So assume it is a chain. Because H is special but not of touched-[2] type, Definition 11.2 says that after 2-tail reduction relative to H the reduced touched core is [3]. Hence Lemma 12.35 gives $\mathrm{cf}_X(H) \geq \frac{1}{3}$.

Apply Proposition 9.133 to L. One gets $m\ \mathrm{cf}_X(H) + \sum_{E\subset D-H}(L\cdot E)\ \mathrm{cf}_X(E) < 1$, so $\sum_{E\subset D-H}(L\cdot E)\ \mathrm{cf}_X(E) < 1 - \frac{m}{3}$. If every contacted component $E \subset D-H$ satisfied $\mathrm{cf}_X(E) \geq \frac{1}{3}$, then the left-hand side would be at least $\frac{L\cdot(D-H)}{3} \geq \frac{3-m}{3} = 1 - \frac{m}{3}$, a contradiction. Therefore some contacted component $E \subset D - H$ has $\mathrm{cf}_X(E) < \frac{1}{3}$. If the connected component of D containing E is not a chain, then one is on the branching side. If it is a chain, Lemma 11.45 shows that the reduced touched core relative to E is a touched tip of weight 2. □

Corollary 12.37 (The ramification-fork frontier for separable special multisections is already old)**.** *Keep the setup of Corollary 11.7, and assume that the horizontal multisection $H \subset D_{\mathrm{hor}}$ is special. Then every ramification point $q \in S \setminus B_\nu$ of local degree at least 2 produces a surviving primitive ramification fork that is already absorbed by the old Route C geometry. Equivalently, the smooth-image ramification side of the separable special-multisection branch contributes no new local horizontal corridor.*

Proof. By Corollary 11.7, the primitive ramification fork over q survives on $\widetilde{X}$. Its last exceptional curve L meets H once and has two units of additional vertical contact, so $L \cdot H = 1$ and $L \cdot (D - H) = 2$. If H is of touched-[2] type, or if the connected component of D containing H is not a chain, then the horizontal side is already old by definition. Otherwise Proposition 12.36 applies with $m = 1$ and gives the same conclusion. □

Corollary 12.38 (The separable special-multisection branch adds no new isolated local fork corridor)**.** *Let $H \subset D_{\mathrm{hor}}$ be a special horizontal multisection, and let $L \subset \widetilde{X} \setminus D$ be a surviving nonexceptional (-1)-curve arising from an isolated local block over the ruled-model image of H. Assume that $L \cdot H \in \{1, 2\}$. Then:*

(i) *if $L \cdot (D - H) \geq 3 - L \cdot H$, then the configuration is already absorbed by the old branching / touched-[2] geometry;*

(ii) *the only isolated local output not immediately reabsorbed is the pure same-component contact-2 case $L \cdot H = 2$ and $L \cdot (D - H) = 0$. In that remaining case one has $\mathrm{cf}_X(H) < \frac{1}{2}$, so H cannot be ordinary. Hence, unless the connected chain component containing H has reduced touched core [3], the horizontal side is already old.*

Equivalently, the separable special-multisection branch contributes no new isolated local fork corridor. The only still-distinctive isolated local output is the pre-existing same-component contact-2 case on a mild special multisection arm.

Proof. If $L \cdot H = 1$, then (i) follows from Proposition 12.36 when H is not already of touched-[2] or branching type, and is tautological otherwise. The same argument applies when $L \cdot H = 2$ and $L \cdot (D - H) \geq 1$. So only the pure same-component contact-2 case can remain.

In that case Proposition 9.133 gives $2\,\mathrm{cf}_X(H) < 1$, hence $\mathrm{cf}_X(H) < \frac{1}{2}$. By Lemma 11.3, an ordinary horizontal component satisfies $\mathrm{cf}_X(H) \geq \frac{1}{2}$, so H cannot be ordinary. Thus H is special. If its connected component is not a chain, or if it is a chain whose reduced touched core is a touched tip of weight 2, then the horizontal side is already old. The only nonold possibility is therefore the chain case whose reduced touched core is [3]. □

Proposition 12.39 (The mild same-component residue on a special multisection is already old)**.** *Keep the setup of Corollary 12.38. Assume that $L \cdot H = 2$, $L \cdot (D - H) = 0$, and that the connected component of D containing H is a chain whose 2-tail reduction relative to H is the singleton chain* [3]*. Then this same-component contact-2 configuration is already locally sandwiched of $\delta = 1$, hence root-contractible, and therefore it gives a descendant with elliptic boundary by Theorem 9.79. In particular, it cannot occur on a residual counterexample.*

Proof. Because the reduced touched core relative to H is [3], the full connected chain component containing H has the form $A = [(2)^u, 3, (2)^v]$ for some integers $u, v \geq 0$, with H corresponding to the unique (-3)-component. By Corollary 12.38, one already knows that $\mathrm{cf}_X(H) < \frac{1}{2}$. On the other hand, Lemma 9.106 gives $\mu_X(H) = \frac{d([2^u])+d([2^v])}{d(A)}$. Since $d([2^u]) = u+1$, $d([2^v]) = v+1$, and $d(A) = uv + 2u + 2v + 3$, one obtains $\mathrm{cf}_X(H) = 1 - \mu_X(H) = \frac{(u+1)(v+1)}{uv+2u+2v+3}$. The inequality $\mathrm{cf}_X(H) < \frac{1}{2}$ therefore forces $uv < 1$, so $uv = 0$. Up to reversing the chain, one may write $A = [3, (2)^m]$ for some $m \geq 0$.

Let q be the singular point of the ruled-model image of H whose isolated local block produces the surviving curve L. Because this block comes from resolving one curve singularity on a smooth ruled surface, there is a birational morphism from a neighborhood of the divisor $E := A + L$ on $\widetilde{X}$ to a smooth surface germ, contracting every component of $E - H$ and mapping H birationally onto the corresponding local branch of the ruled-model image. By Proposition 9.28(1), one has $p_a(E) = 1$. Hence Lemma 9.41 shows that E is locally sandwiched of $\delta = 1$, equivalently that the rooted graph on A with double contact on H is root-contractible.

Now Proposition 9.34(3) says that on a chain $[3, (2)^m]$ a double-contact pattern is root-contractible only in the one-vertex case $[3]^{|1,1|}$, that is, only when $m = 0$. So in fact $A = [3]$. This rooted graph is root-contractible, again by Proposition 9.34(3), and Theorem 9.79 therefore yields a descendant with elliptic boundary. □

Corollary 12.40 (The separable special-multisection branch contributes no new isolated local residue)**.** *Keep the setup of Corollary 12.38. Then every isolated local output on the separable special-multisection side is already absorbed by the old Route C geometry or by the earlier same-component tie program. Equivalently, the separable special-multisection branch contributes no new isolated local corridor at all.*

Proof. Corollary 12.38 already shows that every isolated local output is old unless one is in the mild pure same-component contact-2 case. That remaining case is excluded by Proposition 12.39. □

Proposition 12.41 (Extraction criteria on the separable special-multisection side)**.** *Let $H \subset D_{\mathrm{hor}}$ be a special horizontal multisection, and let $\nu \colon \widetilde{X} \to Y$ be a birational morphism onto a smooth relatively minimal ruled surface, chosen so that ν is an isomorphism at the generic point of H. Put $S := \nu(H)$. Assume that one of the following holds.*

(i) *There exists a ramification point $q \in S \setminus B_\nu$ of local degree at least 2, where B_ν is the finite bad set from Corollary 11.7.*

(ii) *There exists a surviving nonexceptional* (-1)*-curve* $L \subset \widetilde{X} \setminus D$ *arising from an isolated local block over a singular point of* S *and satisfying* $L \cdot H \in \{1, 2\}$.

Then the separable special-multisection branch already lands in previously treated geometry: in case (i) *in the old branching / touched-*[2] *corridor, and in case* (ii) *in the old branching / touched-*[2] *corridor or in the already-closed same-component tie program. In particular, the unresolved content of the separable special-multisection branch is only the global extraction problem of producing one of the outputs* (i) *or* (ii).

Proof. In case (i), Corollary 12.37 applies directly and shows that the surviving primitive ramification fork is already old Route C geometry. In case (ii), Corollary 12.40 applies and shows that every such isolated local output is already absorbed by the old branching / touched-[2] geometry or by the same-component tie program. □

Proposition 12.42 (Smooth-image special multisections leave only a ramification extraction problem)**.** *Keep the setup of Proposition 12.41, and assume in addition that* $S = \nu(H)$ *is smooth on the ruled model. Then the only unresolved content of the separable special-multisection branch is the existence of a ramification point* $q \in S \setminus B_\nu$ *of local degree at least* 2*. Any such point already lands in old Route C geometry by Corollary 12.37.*

Proof. If S is smooth, then there is no singular point of the ruled-model image, so alternative (ii) of Proposition 12.41 cannot occur. Therefore the only possible unresolved output on the separable special-multisection side is the ramification alternative (i). □

Proposition 12.43 (The all-bad smooth-image separable special-multisection side is an at–most-two-point threaded [3]-root corridor)**.** *Keep the setup of Proposition 12.41, and assume in addition that* $S = \nu(H)$ *is smooth and that the map* $S \to \mathbf{P}^1$ *is separable of degree* $d := F \cdot H \geq 2$*. Assume further that the smooth-image branch is not already absorbed by the old branching / touched-*[2] *corridor, and that all isolated local outputs on the separable special-multisection side have already been discarded by Corollary 12.40.*

If every ramification point of $S \to \mathbf{P}^1$ *lies in* B_ν*, then the remaining smooth-image branch is forced into a clean threaded* [3]*-root model of degree* $d \leq 5$ *supported on at most two ramification points. More precisely, if* $q_1, \dots, q_r \in S$ *are the ramification points and* $a_i \geq 2$ *is the local degree at* q_i*, then* $r \leq 2$*. Hence the residual all-bad smooth-image branch consists of one root chain containing* H *whose* 2*-tail reduction is* [3]*, together with at most two detached threaded ramification chains; in particular it is supported on at most two fibers of the ruled model and has multisection degree at most* 5.

Proof. Because S is smooth and $\nu|_H \colon H \to S$ is birational, it is an isomorphism. Thus $H \cong S \cong \mathbf{P}^1$. Since we have excluded the already old branching / touched-[2] corridor, the connected component of D containing H is a chain whose 2-tail reduction relative to H is the singleton chain [3]. In particular, $H^2 = -3$. Lemma 12.35 therefore gives $\mathrm{cf}_X(H) \geq \frac{1}{3}$. Applying Lemma 9.108 to a general fiber and discarding all horizontal contributions except that of H gives $2 > d\, \mathrm{cf}_X(H) \geq \frac{d}{3}$, so $d < 6$ and hence $d \leq 5$.

Now regard S as a smooth rational multisection of degree d on the relatively minimal ruled surface Y. The adjunction computation from the proof of Proposition 11.12 gives $S^2 = 2d$. Each blowup centered on the current strict transform of S lowers its self-intersection by 1. Therefore, passing from $S^2 = 2d$ to $H^2 = -3$ forces exactly $2d+3$ blowup centers on successive strict transforms of S:

$$\#\{\text{centers on successive transforms of } S\} = 2d+3. \tag{1}$$

Let $q_1, \ldots, q_r$ be the ramification points of the separable map $S \to \mathbf{P}^1$, and let $a_i \geq 2$ be the local degree at q_i. For each i, the first a_i centers of ν^{-1} over q_i are the forced order-a_i tangency-resolution centers from Proposition 9.128. After that forced stage there is a unique current nonexceptional (-1)-curve at the H-end of the local ramification branch.

Because every ramification point is assumed to lie in B_ν, there is at least one later center on that current (-1)-curve over every q_i. We claim that, on the still-unresolved branch, every such later center must be the thread center, namely the unique point where that current (-1)-curve meets the current transform of H. Indeed, if some later center on the current H-end (-1)-curve were chosen away from H, then the corresponding descendant over q_i would split off as an isolated unary local block over the smooth point $q_i \in S$. But Corollary 12.40 already absorbs every isolated local output on the separable special-multisection side. Hence such a nonthreaded continuation is impossible on the residual all-bad branch. Repeating the same argument after each thread blowup shows that every bad ramification branch is clean threaded at the H-end.

Consequently, for each ramification point q_i, one has at least $a_i + 1$ centers on successive strict transforms of S: the a_i forced tangency-resolution centers, plus at least one thread center because $q_i \in B_\nu$. Summing over all ramification points and using (1), we obtain

$$\sum_{i=1}^{r} (a_i + 1) \leq 2d + 3. \tag{2}$$

On the other hand, Riemann–Hurwitz for the separable map $S \cong \mathbf{P}^1 \to \mathbf{P}^1$ gives $-2 = -2d + \deg R$, so $\deg R = 2d - 2$. If δ_i denotes the different exponent at q_i, then $\delta_i \geq a_i - 1$, whence

$$\sum_{i=1}^{r} (a_i - 1) \leq 2d - 2. \tag{3}$$

Subtracting (3) from (2) yields $2r \leq (2d+3) - (2d-2) = 5$, hence $r \leq 2$.

Finally, the last paragraph of the proof of Proposition 11.12 is a purely local statement about a thread blowup: it lengthens the detached ramification chain over that point by one terminal (-2)-curve and does not create a new connected component of D. Thus each ramification point contributes one detached threaded ramification chain, while all nonramification centers on successive transforms of S lie on the unique root chain containing H. Therefore the remaining all-bad smooth-image branch is exactly an at-most-two-point threaded [3]-root corridor, supported on at most two fibers of the ruled model. □

Corollary 12.44 (Smooth-image separable special multisections reduce to good ramification or to an explicit at-most-two-point corridor)**.** *Keep the setup of Proposition 12.41, and assume that $S = \nu(H)$ is smooth. After discarding the already old branching / touched-*$[2]$ *corridor and the already old isolated local outputs from Corollary 12.40, exactly one of the following holds.*

(i) *There exists a ramification point $q \in S \setminus B_\nu$ of local degree at least 2. In this case the branch is already old by Corollary 12.37.*

(ii) *Every ramification point of $S \to \mathbf{P}^1$ lies in B_ν, and the remaining smooth-image branch is the explicit low-degree at-most-two-point threaded* $[3]$*-root corridor of Proposition 12.43.*

In particular, the smooth-image separable special-multisection side is no longer a general extraction problem: outside old geometry, its only residual content is this explicit low-degree at-most-two-point threaded $[3]$*-root corridor.*

Proof. If the first alternative holds, apply Corollary 12.37. Otherwise every ramification point lies in B_ν, and Proposition 12.43 applies. □

Lemma 12.45 (A unique special multisection leaves at most three units of ordinary horizontal degree)**.** *Assume that a witnessing fibration does not lie in the several-special-horizontal corridor, so that $H \subset D_{\mathrm{hor}}$ is the unique special horizontal multisection. Let $d := F \cdot H \geq 2$, and write $M := \sum_{T \subset D_{\mathrm{hor}} - H} (F \cdot T) = \mathrm{ht}(X) - d$. Then $d\,\mathrm{cf}_X(H) + \frac{1}{2}M < 2$. In particular, $M \leq 3$ and hence $d \geq \mathrm{ht}(X) - 3$.*

Proof. By Lemma 9.108, one has $\sum_{T \subset D_{\mathrm{hor}}} \mathrm{cf}_X(T)\,(F \cdot T) < 2$. Split this sum into the unique special component H and the remaining ordinary horizontal components. For every ordinary component $T \subset D_{\mathrm{hor}} - H$, Lemma 11.3 gives $\mathrm{cf}_X(T) \geq \frac{1}{2}$. Therefore $2 > d\,\mathrm{cf}_X(H) + \sum_{T \subset D_{\mathrm{hor}} - H} \mathrm{cf}_X(T)\,(F \cdot T) \geq d\,\mathrm{cf}_X(H) + \frac{1}{2}M$. Since $d\,\mathrm{cf}_X(H) \geq 0$, it follows that $\frac{1}{2}M < 2$, so $M < 4$. As M is an integer, one gets $M \leq 3$. The formula $d = \mathrm{ht}(X) - M$ then gives $d \geq \mathrm{ht}(X) - 3$. □

Proposition 12.46 (High-height unique special multisections are horizontal (-2)-curves)**.** *Keep the setup of Lemma 12.45, and assume that $\mathrm{ht}(X) \geq 6$. Then $H^2 = -2$.*

Proof. Assume for contradiction that $H^2 = -\beta$ with $\beta \geq 3$. By Lemma 9.132, $\mathrm{cf}_X(H) \geq 1 - \frac{2}{\beta} \geq \frac{1}{3}$. Now Lemma 12.45 gives $2 > d\,\mathrm{cf}_X(H) + \frac{1}{2}M \geq \frac{d}{3} + \frac{M}{2}$. Using $d = \mathrm{ht}(X) - M$, we obtain $2 > \frac{\mathrm{ht}(X) - M}{3} + \frac{M}{2} = \frac{2\,\mathrm{ht}(X) + M}{6}$. But $\mathrm{ht}(X) \geq 6$ and $M \geq 0$, so the right-hand side is at least 2, a contradiction. Therefore $\beta = 2$, that is, $H^2 = -2$. □

Corollary 12.47 (High-height special-multisection witnesses are already on an old side)**.** *Keep the assumptions of Proposition 12.46. Then exactly one of the following holds.*

(i) *the connected component of D containing H is not a chain, so one is already on the branching side;*

(ii) *the connected component of D containing H is a chain, and its 2-tail core relative to H is a touched tip of weight 2.*

In particular, outside the several-special-horizontal corridor, the special-multisection branch is no longer an independent high-height corridor.

Proof. If the connected component of D containing H is not a chain, then by definition one is on the branching side. Assume it is a chain. Because H is special in the sense of Definition 11.2, after 2-tail reduction the reduced touched core is either a touched tip of weight 2 or the singleton chain [3]. But Proposition 12.46 gives $H^2 = -2$. Removing terminal 2-tails away from the touched component does not change the self-intersection of the touched component, so the reduced touched core cannot be [3]. Hence the chain case is necessarily a touched tip of weight 2. ☐

Corollary 12.48 (Only a finite low-height list can support a non-(-2) unique special multisection)**.** *Keep the setup of Lemma 12.45, and assume that* $H^2 \neq -2$*. Then necessarily* $\mathrm{ht}(X) \leq 5$*. More precisely:*

(i) *if* $\mathrm{ht}(X) = 5$*, then* $M \leq 1$ *and* $d \in \{4, 5\}$*, so the only possible horizontal degree partitions are* 5 *or* $4 + 1$*;*

(ii) *if* $\mathrm{ht}(X) = 4$*, then* $d \in \{2, 3, 4\}$*, so the only possible horizontal degree partitions are* 4*,* $3 + 1$*,* $2 + 2$*, or* $2 + 1 + 1$*.*

Equivalently, outside the several-special-horizontal corridor the only possible non-(-2) *special-multisection partitions form the finite list* 5*,* $4 + 1$*,* 4*,* $3 + 1$*,* $2 + 2$*,* $2 + 1 + 1$*.*

Proof. If $H^2 \neq -2$, then writing $H^2 = -\beta$ one has $\beta \geq 3$. So Lemma 9.132 gives $\mathrm{cf}_X(H) \geq \frac{1}{3}$. Applying Lemma 12.45 again, we get $2 > d\,\mathrm{cf}_X(H) + \frac{1}{2}M \geq \frac{d}{3} + \frac{M}{2} = \frac{2\,\mathrm{ht}(X)+M}{6}$. Therefore $2\,\mathrm{ht}(X) + M < 12$. In particular, $\mathrm{ht}(X) \leq 5$. If $\mathrm{ht}(X) = 5$, then the last inequality becomes $10 + M < 12$, so $M \leq 1$. Hence $d = 5 - M \in \{4, 5\}$, and the only possible horizontal degree partitions are 5 or $4 + 1$. If $\mathrm{ht}(X) = 4$, then $d = 4 - M$ with $0 \leq M \leq 2$, so $d \in \{2, 3, 4\}$. When $d = 4$, one gets the partition 4; when $d = 3$, one gets $3 + 1$; and when $d = 2$, the ordinary remainder has total degree 2, so the partition is either $2 + 2$ or $2 + 1 + 1$. This proves the claim. ☐

Proposition 12.49 (The one-section separable special-multisection residue is finite and explicit)**.** *Assume the setup of Corollary 11.19, and assume that the unique multisection* H *is special and separable. Let* $C := \nu(H) \subset \mathbb{F}_n$*. Then the unresolved extraction problem on this one-section separable branch has the following form.*

(i) *If there exists a ramification point* $q \in C \setminus B_\nu$ *of local degree at least* 2*, then Proposition 12.41 already closes the branch.*

(ii) *Otherwise one may restrict attention to the singular points of* C*. Every such singular point is of one of the following two explicit types:*

   (a) *a multibranch collision point, and any isolated local output there is governed by the primitive two-branch collision model of Proposition 9.119;*

(b) *a unibranch point, and any surviving terminal output there is one of the three low-order forks listed in Corollary 11.40.*

Moreover every surviving isolated local output over such a singular point is already old Route C geometry or belongs to the already-closed same-component tie program by Corollary 12.40.

Consequently, the one-section separable special-multisection branch is no longer a general multisection extraction problem: after discarding the smooth-image ramification alternative, its unresolved content is only the survival of one of finitely many explicit low-degree singular-image local blocks.

Proof. Corollary 11.19 shows that C is a singular rational curve of degree 2 or 3 on $\mathbb{F}_n$ with $n \geq 2$. If a ramification point $q \in C \setminus B_\nu$ of local degree at least 2 exists, then Proposition 12.41 closes the branch immediately. So assume not.

Now every singular point of C is, by Lemma 11.18, either a multibranch collision point or a unibranch point of local degree at least 2. At an isolated multibranch collision point, Proposition 9.119 gives the primitive local collision model. At a unibranch point, Corollary 11.35 reduces the terminal endgame to corner blowups and order-2/3 tangency blocks, and Corollary 11.40 identifies any surviving terminal output with one of exactly three explicit low-order fork types.

Finally, Corollary 12.40 shows that every surviving isolated local output on the separable special-multisection side is already absorbed by old Route C geometry or by the same-component tie program. ☐

Corollary 12.50 (Many-branch singular-image points on the separable special-multisection side add no new endpoint)**.** *Keep the setup of Proposition 12.41, and let $q \in S$ be a point through which the total horizontal image on the ruled model has at least three local branches. Then the local factorization over q decomposes into binary descendants and unary descendants in the sense of Corollary 12.17. Consequently:*

(i) *the many-branch ambiguity itself disappears;*

(ii) *the remaining unary descendants belong to the ramification or unibranch side already isolated in Proposition 12.41;*

(iii) *the remaining binary descendants are pair-collision trees, and in the one-section height-4 partitions they are among the explicit low-degree local blocks recorded in Proposition 12.49.*

In particular, the separable special-multisection branch has no separate many-branch local endpoint.

Proof. Apply Corollary 12.17 to the family of all local horizontal branches through q. This removes the many-branch ambiguity itself: every descendant block is unary or binary. A unary descendant is supported on a single horizontal branch, so it lies on the already isolated ramification or unibranch side from Proposition 12.41. A binary descendant is a pair-collision tree.

In the one-section height-4 partitions, the possible surviving local outputs were already reduced in Proposition 12.49 to an explicit finite low-degree list. So no separate many-branch endpoint remains. □

Closing the degree-two one-section branch. The direct-cubic closure draft shows that the degree-two one-section branch can be pushed further than the finite local fork package alone suggests. The key point is that the singular-point elementary-transform sequence already lifts from $\widetilde{X}$, and after that lift the residual conic branch also feeds into the same singular plane-cubic output.

Proposition 12.51 (The singular-point transforms of the ruled-model bisection lift from $\widetilde{X}$)**.** *Assume the setup of Corollary 11.19, and suppose that $d = 2$. Let $\nu_0 := \nu\colon \widetilde{X} \to \mathbb{F}_n$ and $C_0 := \nu_0(H)$. Suppose that $q_0 \in C_0$ is a singular point, and let $\sigma_0\colon \mathbb{F}_n \dashrightarrow \mathbb{F}_{n-1}$ be the elementary transform centered at q_0. Then there exists a birational morphism $\mu_1\colon \widetilde{X} \to \mathbb{F}_{n-1}$ such that $\mu_1 = \sigma_0 \circ \nu_0$ as rational maps.*

More generally, any successive sequence of elementary transforms centered at singular points of the current image of H lifts inductively to birational morphisms $\mu_r\colon \widetilde{X} \to \mathbb{F}_{n-r}$.

Proof. Factor the birational map $\nu_0^{-1}\colon \mathbb{F}_n \dashrightarrow \widetilde{X}$ into a sequence of point blowups away from the negative section $S_0 := \nu_0(H_0)$. Because $H \subset \widetilde{X}$ is smooth while its image C_0 is singular at q_0, at least one blowup center of this factorization lies over q_0. The first such center must be the point q_0 itself. Hence ν_0 factors through the blowup $\varepsilon_0\colon \mathrm{Bl}_{q_0}(\mathbb{F}_n) \to \mathbb{F}_n$. Write $\widehat{\nu}_0\colon \widetilde{X} \to \mathrm{Bl}_{q_0}(\mathbb{F}_n)$ for the induced birational morphism. Let $\beta_0\colon \mathrm{Bl}_{q_0}(\mathbb{F}_n) \to \mathbb{F}_{n-1}$ be the contraction of the strict transform of the fiber through q_0. Then $\mu_1 := \beta_0 \circ \widehat{\nu}_0$ is the required birational morphism. The inductive statement follows by repeating the same argument at each later singular point of the current image of H. □

Corollary 12.52 (The conic-line endpoint of the degree-two branch already gives an elliptic descendant)**.** *Assume the setup of Corollary 11.19, suppose that $d = 2$ and $s = 0$, and perform a lifted singular-point transform sequence of Proposition 12.51 until the image of H is the smooth conic $Q \equiv 2S_0 + 2f$ on $\mathbb{F}_1$. Assume that the final image of the second horizontal section H_1 has class $L \equiv S_0 + f$. Then X has a descendant with elliptic boundary.*

Proof. Let $\phi\colon \widetilde{X} \to \mathbb{F}_1$ be the lifted morphism. By construction, $\phi(H) = Q$ and $\phi(H_1) = L$, and the image of H_0 is the negative section of $\mathbb{F}_1$. Let $\tau\colon \mathbb{F}_1 \to \mathbf{P}^2$ be the contraction of that negative section. Since $Q \equiv 2S_0 + 2f$ and $L \equiv S_0 + f$, the images $\tau(Q)$ and $\tau(L)$ are respectively a smooth plane conic and a line. Moreover, $Q \cdot L = (2S_0 + 2f) \cdot (S_0 + f) = 2$, so the reduced divisor $T := \tau(Q) + \tau(L)$ is connected and has arithmetic genus $p_a(T) = 0 + 0 + 2 - 1 = 1$. The only other boundary components on $\widetilde{X}$ are H_0 and vertical components, and under $\tau \circ \phi$ these map to points. Hence T is a connected component of the reduced image of the boundary. By [7, Definition 2.10, Definition 2.11, Lemma 2.12], this gives a descendant with elliptic boundary. □

Corollary 12.53 (Residual exclusion of the conic-line endpoint)**.** *In the residual sector* $\mathrm{ht}(X) \geq 4$ *and* X *has no descendant with elliptic boundary, the configuration of Corollary 12.52 cannot occur.*

Proof. This is immediate from Corollary 12.52. □

Proposition 12.54 (The residual conic branch birationally reduces to a lifted singular plane-cubic model)**.** *Assume the setup of Corollary 11.19, suppose that* $d = 2$ *and* $s = 0$*, and perform a lifted singular-point transform sequence of Proposition 12.51 until the image of* H *is the smooth conic* $Q \equiv 2S_0 + 2f$ *on* $\mathbb{F}_1$*. Let* $\phi\colon \widetilde{X} \to \mathbb{F}_1$ *be the resulting morphism, and write* $B := \phi(H_1) \equiv S_0 + bf$ *with* $b \geq 2$*. Then there exists a birational morphism* $\eta\colon \widetilde{X} \to \mathbf{P}^2$ *such that* $\eta(H) \subset \mathbf{P}^2$ *is an irreducible singular plane cubic.*

Proof. Since $Q \cdot B = (2S_0 + 2f) \cdot (S_0 + bf) = 2b \geq 4$, while distinct irreducible components of the reduced divisor D meet at most once on $\widetilde{X}$, there exists a point $q \in Q \cap B$ over which ϕ^{-1} is not a local isomorphism. Because $Q \cdot S_0 = 0$, one has $q \notin S_0$. Blow up q and contract the strict transform of the fiber through q. Exactly as in the standard elementary-transform class computation (Lemma 11.26), this produces a birational morphism $\psi\colon \widetilde{X} \to \mathbb{F}_0$ such that, if $Q' := \psi(H)$, $B' := \psi(H_1)$, and $S := \psi(H_0)$, then $Q' \equiv 2S + f$ and $B' \equiv S + (b-1)f$.

Now choose a point $q_0 \in S \setminus (Q' \cup B')$ and perform one further elementary transform centered at q_0. Again by Lemma 11.26, one obtains a birational morphism $\chi\colon \widetilde{X} \to \mathbb{F}_1$ whose image $Q_{\mathrm{cub}} := \chi(H)$ satisfies $Q_{\mathrm{cub}} \equiv 2S_0 + 3f$. Contracting the negative section of this final $\mathbb{F}_1$ gives a birational morphism $\eta\colon \widetilde{X} \to \mathbf{P}^2$ whose image of H is a plane cubic. Because $p_a(Q_{\mathrm{cub}}) = 1$ and the normalization of H is rational, that plane cubic is singular. □

Proposition 12.55 (A birational singular plane-cubic model already yields an elliptic-boundary descendant)**.** *Let* $H \subset \widetilde{X}$ *be a smooth rational curve. Assume that there exists a birational map* $\eta\colon \widetilde{X} \dashrightarrow \mathbf{P}^2$ *such that the induced rational map* $\eta|_H\colon H \dashrightarrow C$ *is birational onto an irreducible singular plane cubic* $C \subset \mathbf{P}^2$*. Then* X *has a descendant with elliptic boundary.*

Proof. Choose a sequence of point blowups $\sigma\colon Y \to \widetilde{X}$ resolving the indeterminacy of η and any remaining singularities of the total transform of C, minimal with the property that the induced map $f := \eta \circ \sigma\colon Y \to \mathbf{P}^2$ is a morphism and the reduced total transform $E := (f^{-1}(C))_{\mathrm{red}}$ is a simple normal crossings divisor. Let $R \subset E$ be the strict transform of H. Then E is connected, every irreducible component of E is a smooth rational curve, the restriction $f|_R\colon R \to C$ is birational, and every component of $E - R$ is f-exceptional.

Because C is an irreducible singular plane cubic, one has $p_a(C) = 1$ and C is rational. The fibers of the induced map $E \to C$ are connected trees of rational curves, so $f_*\mathcal{O}_E = \mathcal{O}_C$ and $R^1 f_*\mathcal{O}_E = 0$. Hence $p_a(E) = p_a(C) = 1$. Therefore E is a connected reduced simple normal crossings divisor with rational components and arithmetic genus one. Lemma 9.41 applied with root R shows that E contains an elliptic tie. By [7, Definition 2.10, Definition 2.11, Lemma 2.12], this gives a descendant with elliptic boundary. □

Theorem 12.56 (The degree-two one-section branch is closed)**.** *Assume the setup of Corollary 11.19, and suppose that $d = 2$. Then the partition $2 + 1 + 1$ does not occur in the residual characteristic-3 sector.*

Proof. Choose the singular-point elementary-transform sequence from Proposition 11.27. By Proposition 12.51, that sequence lifts to a birational morphism from $\widetilde{X}$ to the final ruled surface.

If $s = 1$, then Proposition 11.27 shows that the final image of H on $\mathbb{F}_1$ contracts to an irreducible singular plane cubic. So Proposition 12.55 gives a descendant with elliptic boundary, contradicting residuality.

If $s = 0$, write the final image of H_1 on $\mathbb{F}_1$ as $S_0 + bf$ with $b \geq 1$. If $b = 1$, then Corollary 12.53 excludes the branch. If $b \geq 2$, then Proposition 12.54 produces a lifted birational morphism to $\mathbf{P}^2$ whose image of H is an irreducible singular plane cubic, so Proposition 12.55 again gives a descendant with elliptic boundary.

All cases contradict residuality. $\square$

Sharpening the degree-three one-section branch. The new cubic-triple reduction removes one more ambiguity from the residual degree-three branch: after at most one double singular-point transform, everything feeds into a pure-triple cubic branch.

Proposition 12.57 (Singular-point transforms in the one-section cubic branch lift from $\widetilde{X}$)**.** *Assume the setup of Corollary 11.19, and suppose that $d = 3$. Let $\nu_0 := \nu\colon \widetilde{X} \to \mathbb{F}_n$ and $C_0 := \nu_0(H)$. Then any successive sequence of elementary transforms centered at singular points of the current image of H lifts inductively to birational morphisms $\mu_r\colon \widetilde{X} \to \mathbb{F}_{n-r}$. At each stage:*

(i) *every chosen singular point lies off the current negative section and has multiplicity $m \in \{2, 3\}$;*

(ii) *if the current image has class $3S_0 + (3\ell + s)f$ on $\mathbb{F}_\ell$, then after the elementary transform at such a point of multiplicity m the new image has class $3S_0' + (3(\ell - 1) + s + 3 - m)f' = 3S_0' + (3\ell + s - m)f'$ on $\mathbb{F}_{\ell-1}$;*

(iii) *the arithmetic genus drops by $2m - 3$, and the normalization remains $\mathbf{P}^1$.*

Proof. The proof is the same local class computation as in the degree-two branch, together with the same lifting argument. Indeed, if q is a singular point of the current image $C \equiv 3S_0 + (3\ell + s)f$, then $q \notin S_0$ because $C \cdot S_0 = s \leq 1$, and a singular point on S_0 would contribute local intersection multiplicity at least 2. Since $C \cdot f = 3$, the multiplicity at q is at most 3, hence belongs to $\{2, 3\}$.

Blowing up q and contracting the strict transform of the fiber through q produces $\mathbb{F}_{\ell-1}$. If m is the multiplicity of C at q, the same elementary-transform class calculation as in the proof of Proposition 12.51 gives the new class $3S_0' + (3\ell + s - m)f'$, and adjunction shows that the arithmetic genus drops by $2m - 3$. Because the transform is birational, the normalization remains $\mathbf{P}^1$.

For the lifting statement, factor the inverse map from the current ruled model to $\widetilde{X}$ into point blowups. Since $H \subset \widetilde{X}$ is smooth while its current ruled-model image is singular at the chosen center, the first blowup center over that singular point is the point itself, so the elementary transform factors through $\widetilde{X}$. Iterating gives the required birational morphisms μ_r. □

Corollary 12.58 (Exact double/triple count in the one-section cubic branch)**.** *Assume the setup of Corollary 11.19, and suppose that* $d = 3$*. Let* $s := H \cdot H_0 \in \{0, 1\}$*. Consider any full lifted singular-point transform sequence of Proposition 12.57, continued until the image of* H *becomes smooth. Let* N_2 *and* N_3 *denote the numbers of centers where the current image has multiplicity* 2 *and* 3*, respectively. Then* $N_2 = 1 - s$ *and* $N_3 = n + s - 1$*. Equivalently:*

(i) *if* $s = 1$*, then every step is triple and there are exactly* n *triple steps;*

(ii) *if* $s = 0$*, then there is exactly one double step and the remaining* $n - 1$ *steps are triple.*

Moreover, the final smooth model is always $\mathbb{F}_0$ *with smooth image* $Q_{\mathrm{sm}} \equiv 3S + f$*.*

Proof. By Proposition 12.57, every singular-point transform lowers the Hirzebruch index by one and lowers the arithmetic genus by 1 or 3 according as the chosen center has multiplicity 2 or 3. Let the final smooth model lie on $\mathbb{F}_m$ and have class $Q_{\mathrm{sm}} \equiv 3S + (3m + r)f$. Since the normalization is rational and the final curve is smooth, adjunction on $\mathbb{F}_m$ gives $0 = p_a(Q_{\mathrm{sm}}) = 3m + 2r - 2$. Hence necessarily $m = 0$ and $r = 1$, so the final smooth model is $\mathbb{F}_0$ with $Q_{\mathrm{sm}} \equiv 3S + f$. Therefore the total number of singular-point transforms is exactly $N_2 + N_3 = n$. On the other hand, the total drop in arithmetic genus equals the initial genus $p_a(C) = 3n + 2s - 2$, so $N_2 + 3N_3 = 3n + 2s - 2$. Solving the two linear equations gives the displayed formulas. □

Corollary 12.59 (The 3+1 one-section branch reduces to the pure-triple cubic branch)**.** *Assume the setup of Corollary 11.19, and suppose that* $d = 3$*. Then:*

(i) *if* $s = 1$*, every singular point of every intermediate lifted ruled-model image of* H *has multiplicity* 3*;*

(ii) *if* $s = 0$*, there is a unique double step, and immediately after that step the current image of* H *has class* $3S_0 + (3m + 1)f$ *on some* $\mathbb{F}_m$*, while every remaining singular-point transform is triple.*

In particular, the whole one-section cubic branch reduces to the pure-triple branch with parameter 1*: singular rational curves of class* $3S_0 + (3m + 1)f$ *all of whose later singular-point transforms are triple. Moreover, in that pure-triple branch every singular point is locally of one of the following two types:*

(a) *a collision of exactly three smooth horizontal branches over one fiber;*

(b) *a unibranch ramification point of local degree exactly* 3*.*

Proof. The first two claims are exactly Corollary 12.58. If $s = 0$, let the unique double step occur after a triple steps. After those a triple steps, Proposition 12.57 keeps the parameter equal to 0, so the current image has class $3S_0 + 3(n - a)f$ on $\mathbb{F}_{n-a}$. Applying the unique double transform then gives a curve of class $3S_0+(3(n-a)-2)f = 3S_0+(3(n-a-1)+1)f$ on $\mathbb{F}_{n-a-1}$. Thus, after the unique double step, the branch is of the same form as the initial $s=1$ branch.

For the local classification, let q be a singular point in the pure-triple branch. By Proposition 12.57, one has $\mathrm{mult}_q = 3$. Lemma 11.18 shows that q is either multibranch or unibranch ramification. In the multibranch case, each local branch is smooth and contributes multiplicity 1, so multiplicity 3 forces exactly three branches. In the unibranch case, if the local degree over the base were 2, then a parametrization $t = u^2$, $z = \varphi(u)$ would give multiplicity at most 2, contrary to $\mathrm{mult}_q = 3$. Hence the local degree is exactly 3. □

Proposition 12.60 (The smooth $3S+f$ endpoint birationally reduces to a singular plane cubic)**.** *Assume the setup of Corollary 11.19. Assume that there exists a birational morphism $\phi\colon \widetilde{X} \to \mathbb{F}_0$ such that $Q := \phi(H) \subset \mathbb{F}_0$ is smooth and satisfies $Q \equiv 3S+f$. Then there exists a birational map $\eta\colon \widetilde{X} \dashrightarrow \mathbf{P}^2$ whose restriction to H is birational onto an irreducible singular plane cubic.*

Proof. Choose a smooth point $q \in Q$. Let $T \in |S|$ and $F \in |f|$ be the two ruling curves on $\mathbb{F}_0$ passing through q. Then $Q\cdot T = 1$, $Q \cdot F = 3$, and $I_q(Q,T) = 1$.

Let $\sigma\colon Z := \mathrm{Bl}_q(\mathbb{F}_0) \to \mathbb{F}_0$ be the blowup of q, with exceptional curve E. Write $\widetilde{T}$, $\widetilde{F}$, and $\widetilde{Q}$ for the strict transforms of T, F, and Q. Since T and F meet only at q, the curves $\widetilde{T}$ and $\widetilde{F}$ are disjoint. Moreover, $\widetilde{T}^2 = \widetilde{F}^2 = -1$. Hence one may contract them successively to obtain a birational morphism $\tau\colon Z \to W$ onto a smooth rational surface W with Picard rank 1. Therefore $W \cong \mathbf{P}^2$. Set $\rho := \tau \circ \sigma^{-1}\colon \mathbb{F}_0 \dashrightarrow \mathbf{P}^2$.

Now

$$\widetilde{Q}\cdot\widetilde{T} = Q\cdot T - I_q(Q,T) = 1-1 = 0,$$

while $\widetilde{Q}\cdot\widetilde{F} \geq 0$. So $\widetilde{Q}$ is not contracted by τ, and its image $\Gamma := \tau(\widetilde{Q}) = \rho(Q)$ is irreducible. Since H is rational and $\phi|_H$ is birational onto the smooth curve Q, one has $Q \cong \mathbf{P}^1$. Therefore, because ρ is birational, the rational map $\rho|_Q\colon Q \dashrightarrow \Gamma$ is birational, so Γ is a rational plane curve.

To compute the degree of Γ, note that on Z the divisor $L := \sigma^*S + \sigma^*f - E$ has self-intersection 1 and intersects both $\widetilde{T}$ and $\widetilde{F}$ trivially, so L is the pullback of a line on $W \cong \mathbf{P}^2$. Because q is a smooth point of Q, one has $\widetilde{Q} \equiv \sigma^*Q - E = 3\sigma^*S + \sigma^*f - E$. Therefore

$$\deg\Gamma = L\cdot\widetilde{Q} = (\sigma^*S+\sigma^*f-E)\cdot(3\sigma^*S+\sigma^*f-E) = 3.$$

Thus $\Gamma \subset \mathbf{P}^2$ is an irreducible plane cubic. Because Γ is rational while every irreducible plane cubic has arithmetic genus 1, the curve Γ is necessarily singular.

Finally set $\eta := \rho\circ\phi\colon \widetilde{X} \dashrightarrow \mathbf{P}^2$. Since $\phi|_H\colon H \to Q$ is birational and $\rho|_Q$ is birational, the rational map $\eta|_H\colon H \dashrightarrow \Gamma$ is birational as well. □

Theorem 12.61 (The pure-triple cubic branch is closed)**.** *Assume the setup of Corollary 12.59. Suppose that after the possible unique double step of Corollary 12.59 one is in its pure-triple* *branch, so the current image of H has class $3S_0 + (3m+1)f$ on some $\mathbb{F}_m$ and every later singular-point elementary transform is centered at a triple point. Then X has a descendant with elliptic boundary. In particular, the pure-triple cubic branch cannot occur in the residual characteristic-*3 *sector.*

Proof. Continue the lifted singular-point transform sequence until the image of H becomes smooth. By Proposition 12.57, every step lifts from $\widetilde{X}$. Because every remaining step is triple, the parameter 1 is preserved throughout the sequence, and after exactly m steps one arrives at a birational morphism $\phi\colon \widetilde{X} \to \mathbb{F}_0$ whose image $Q := \phi(H)$ is smooth of class $Q \equiv 3S + f$; this is also the final smooth model identified in Corollary 12.58.

Proposition 12.60 now gives a birational map $\eta\colon \widetilde{X} \dashrightarrow \mathbf{P}^2$ whose restriction to H is birational onto an irreducible singular plane cubic. Applying Proposition 12.55 yields a descendant with elliptic boundary. This excludes the pure-triple branch in the residual sector. ☐

Corollary 12.62 (The one-section singular-image side of the separable special-multisection branch is closed)**.** *Assume the setup of Proposition 12.49, and assume that H is special and separable. Then the one-section singular-image side contributes no residual branch. More precisely:*

(i) *the degree-two partition $2+1+1$ is excluded by Theorem 12.56;*

(ii) *the degree-three branch $3+1$ is excluded by Corollary 12.59 together with Theorem 12.61.*

Consequently, on the separable special-multisection side only the smooth-image ramification / two-point corridor of Corollary 12.44 remains.

Proof. The degree-two branch is closed by Theorem 12.56. For the degree-three branch, Corollary 12.59 reduces the whole one-section cubic side to the pure-triple branch, and Theorem 12.61 excludes that branch. Hence no one-section singular-image residue remains. The last sentence is exactly Corollary 12.44. ☐

Closing the remaining smooth-image corridor. We now show that outside the several-special-horizontal corridor the explicit low-degree at-most-two-point threaded [3]-root model cannot occur either.

Lemma 12.63 (In the residual low-degree smooth-image corridor the [3]-root is one-sided)**.** *Keep the setup of Corollary 12.44, and assume in addition that we are outside the several-special-horizontal corridor, that we are in the residual sector* $\mathrm{ht}(X) \geq 4$ *with no descendant with elliptic boundary, and that alternative* (ii) *of Corollary 12.44 holds.*

Let $A = [(2)^u, 3, (2)^v]$ be the connected component of D containing H, with H the unique (-3)-component. Then $uv = 0$.

Proof. By Proposition 12.43, one has $H^2 = -3$ and $d := F \cdot H \leq 5$. Since we are outside the several-special-horizontal corridor, H is the unique special horizontal component. Moreover, Corollary 12.47 removes the high-height part, while Corollary 12.48 leaves only the finite low-height list 5, $4+1$, 4, $3+1$, $2+2$, $2+1+1$. Among those, the one-section partitions $3+1$ and $2+1+1$ belong to the one-section singular-image branch and are excluded by Corollary 12.62. Hence the only residual partitions still left here are 5, $4+1$, 4, and $2+2$, so in particular $d \in \{2, 4, 5\}$.

Assume first that $d \in \{4, 5\}$. If $u, v \geq 1$, then exactly as in the proof of Proposition 12.39 one has

$$\mathrm{cf}_X(H) = \frac{(u+1)(v+1)}{uv+2u+2v+3} \geq \frac{1}{2},$$

because $2(u+1)(v+1) - (uv+2u+2v+3) = uv-1 \geq 0$. Lemma 12.45 therefore gives $2 > d\,\mathrm{cf}_X(H) + \frac{1}{2}M \geq \frac{d}{2} \geq 2$, a contradiction. Hence $uv = 0$ in the cases $d = 4, 5$.

It remains to treat $d = 2$. Then $S \to \mathbf{P}^1$ is a separable double cover of $\mathbf{P}^1$, so by Riemann–Hurwitz it has exactly two ramification points, both of local degree 2. By Proposition 12.43, the total number of blowup centers on successive transforms of S equals $2d + 3 = 7$. Over each of the two ramification points, the all-bad hypothesis forces the two tangency-resolution centers together with at least one thread center, so already 6 of those 7 centers are prescribed. By the last paragraph of Proposition 12.43, every ramification point contributes a detached threaded ramification chain, while every nonramification center on a successive transform of S lies on the root chain containing H. A nonempty 2-tail on one side of H requires at least one such nonramification center on that side. Thus a two-sided chain $[(2)^u, 3, (2)^v]$ with $u, v \geq 1$ would require at least two nonramification centers on successive transforms of S, but only one center is left after the mandatory 6 ramification-side centers. This is impossible, so again $uv = 0$. ☐

Lemma 12.64 (A one-sided threaded [3]-root ramification branch gives low height or the closed $2+1+1$ package)**.** *Let Z be a smooth surface with reduced snc divisor D_Z. Assume that D_Z contains a chain $A = [3, (2)^s] = H - U_1 - \cdots - U_s$ with $s \geq 0$, and another chain $B = [a, (2)^m] = E_1 - E_2 - \cdots - E_{m+1}$ with $a \geq 2$ and $m \geq 2$, where $H^2 = -3$, $U_i^2 = -2$, $E_1^2 = -a$, and $E_j^2 = -2$ for $j \geq 2$. Assume moreover that there is a (-1)-curve $L \not\subset D_Z$ meeting H and E_2 once each and otherwise disjoint from D_Z, and that every component of $D_Z - (A + B)$ is disjoint from $A \cup B \cup L$.*

Let $\sigma\colon Z \to Y$ be the contraction of L, E_2, E_3, and put $F := \sigma_* H$. Then:

- (i) *F is a smooth rational curve with $F^2 = 0$, hence $|F|$ defines a $\mathbf{P}^1$-fibration on Y;*
- (ii) *$\sigma_* E_1$ is a bisection of that ruling;*
- (iii) *if $s \geq 1$, then $\sigma_* U_1$ is a section;*
- (iv) *if $m \geq 3$, then $\sigma_* E_4$ is a section;*
- (v) *every other component of $\sigma_* D_Z$ is vertical for $|F|$.*

Consequently a general fiber G of $|F|$ satisfies $G \cdot \sigma_* D_Z \leq 2 + \mathbf{1}_{\{s \geq 1\}} + \mathbf{1}_{\{m \geq 3\}}$. In particular:

- (a) *if $m = 2$, then $G \cdot \sigma_* D_Z \leq 3$;*
- (b) *if $m \geq 3$ and $s = 0$, then $G \cdot \sigma_* D_Z \leq 3$;*

(c) *if* $m \geq 3$ *and* $s \geq 1$*, then the horizontal degree partition of the ruling* $|F|$ *is exactly* $2+1+1$.

Proof. Contract L first. Then the image of H has self-intersection -2, and the image of E_2 is a (-1)-curve. Contracting E_2 raises the self-intersection of the image of H to -1 and makes the image of E_3 a (-1)-curve. Contracting E_3 raises the self-intersection of the image of H once more, so the final curve $F = \sigma_* H$ satisfies $F^2 = 0$. Since $H \cong \mathbf{P}^1$ is smooth, so is F, and Lemma 9.83 gives the ruling $|F|$. This proves (i).

After contracting L and then E_2, the image of H meets the image of E_1 once and the image of E_3 once. Contracting E_3 adds one further intersection between the images of H and E_1, so $F \cdot \sigma_* E_1 = 2$. Thus $\sigma_* E_1$ is a bisection, proving (ii).

If $s \geq 1$, then U_1 is disjoint from the contracted curves except for its original transverse intersection with H, so $F \cdot \sigma_* U_1 = 1$. Hence $\sigma_* U_1$ is a section, proving (iii). Likewise, if $m \geq 3$, then after the contraction of E_3 the image of E_4 meets F once, so $\sigma_* E_4$ is a section, proving (iv).

Every remaining component of A or B is disjoint from F, hence vertical for $|F|$. By assumption every component of $D_Z - (A + B)$ is disjoint from the whole local cluster and therefore also disjoint from F, so those components are vertical as well. This proves (v) and the displayed bound for a general fiber. The three final cases are then immediate. □

Lemma 12.65 (The chosen clean-threaded ramification cluster is isolated)**.** *Keep the setup of Corollary 12.44, and assume that alternative* (ii) *holds. Let* A *be the full root chain containing* H*. Choose a bad ramification point* q *of* $S \to \mathbf{P}^1$*, and let* $B_q = [a, (2)^m]$ *be the full detached threaded ramification chain over* q *furnished by Proposition 12.43 together with the threading statement from the proof of Proposition 11.12. Let* $L_q \not\subset D$ *be the unique* (-1)*-curve at the* H*-end of that local cluster meeting* H *and the second component of* B_q *once each. Then every component of* $D - (A + B_q)$ *is disjoint from* $A \cup B_q \cup L_q$*.*

Proof. By Proposition 12.43, every component of D lies either in the unique root chain A containing H or in one of at most two detached threaded ramification chains. Because B_q is taken to be the full detached threaded chain over q, any component of $D - (A + B_q)$ lies in a second detached threaded ramification chain corresponding to a different bad ramification point. Such a chain is disjoint from A and from B_q because detached connected components of D are pairwise disjoint. Moreover L_q is created entirely inside the local branch over q and meets only H and the second component of B_q. Hence every component of $D - (A + B_q)$ is also disjoint from L_q. □

Proposition 12.66 (The explicit low-degree smooth-image corridor is impossible outside the several-special-horizontal corridor)**.** *Keep the setup of Corollary 12.44, and assume in addition that we are outside the several-special-horizontal corridor, that we are in the residual sector* $\mathrm{ht}(X) \geq 4$ *with no descendant with elliptic boundary, and that alternative* (ii) *of Corollary 12.44 holds. Then this is impossible.*

Proof. Assume for contradiction that alternative (ii) holds. By Lemma 12.63, the connected component of D containing H is one-sided; after possibly reversing it we may write $A = [3, (2)^s]$

with $s \geq 0$ and with H the initial (-3)-curve.

Choose a bad ramification point q of $S \to \mathbf{P}^1$. By Proposition 9.128, the forced tangency-resolution stage over q produces a local bridge whose detached side starts with a chain $[a, (2)^{a-1}]$, where a is the local degree at q. By Proposition 12.43 together with the threading statement from the proof of Proposition 11.12, the all-bad branch over q is clean threaded at the H-end. So on $\widetilde{X}$ itself there is a detached chain $B = [a, (2)^m]$ with $a \geq 2$ and $m \geq 2$, where a is the local degree at q, together with a (-1)-curve $L \not\subset D$ meeting H and the second component of B once each. By Lemma 12.65, every component of $D - (A + B)$ is disjoint from $A \cup B \cup L$. Hence Lemma 12.64 applies with $Z = \widetilde{X}$.

We split according to $d = F \cdot H$. If $d = 2$, then $S \to \mathbf{P}^1$ is a separable double cover, so every ramification point has local degree $a = 2$. If $m = 2$, Lemma 12.64(a) gives a witnessing ruling of height at most 3, contradicting $\mathrm{ht}(X) \geq 4$. If $m \geq 3$ and $s = 0$, Lemma 12.64(b) again gives height at most 3, the same contradiction. If $m \geq 3$ and $s \geq 1$, then Lemma 12.64(c) gives a witnessing ruling of height 4 whose horizontal degree partition is exactly $2+1+1$, contradicting Theorem 12.56. Thus $d = 2$ is impossible.

Assume next that $d \in \{4, 5\}$. If every ramification point had local degree 2, then every ramification point would be tame with different exponent 1. Since Proposition 12.43 gives at most two ramification points, this would yield $\deg R \leq 2$, contrary to Riemann–Hurwitz $\deg R = 2d - 2 \geq 6$. So there exists a ramification point q with local degree $a \geq 3$. Then its detached chain satisfies $m \geq 3$.

If $s = 0$, Lemma 12.64(b) gives a witnessing ruling of height at most 3, contradicting $\mathrm{ht}(X) \geq 4$. If $s \geq 1$, then Lemma 12.64(c) gives a witnessing ruling of height 4 whose horizontal degree partition is exactly $2 + 1 + 1$. This is again excluded by Theorem 12.56.

All cases contradict residuality. ☐

Corollary 12.67 (Outside the several-special-horizontal corridor, the smooth-image separable special-multisection side is already old)**.** *Keep the setup of Corollary 12.44, and assume that we are outside the several-special-horizontal corridor. Then alternative* (ii) *of Corollary 12.44 cannot occur. Consequently there exists a ramification point $q \in S \setminus B_\nu$ of local degree at least* 2*, and the smooth-image separable special-multisection branch is already old by Corollary 12.37.*

Proof. The high-height part is already old by Corollary 12.47, so only the residual low-degree corridor of Corollary 12.44(ii) remained. That corridor is impossible by Proposition 12.66. Therefore alternative (i) of Corollary 12.44 must hold, and the last assertion is exactly Corollary 12.37. ☐

12.3. The purely inseparable ordinary singleton [2]-root branch

The two-fiber support bound from Corollary 11.14 can in fact be sharpened to a complete closure of the branch. The key point is that on the cubic purely inseparable backbone, one actual service center already produces a lifted singular plane-cubic model.

Lemma 12.68 (The cubic purely inseparable backbone is actually on $\mathbb{F}_0$)**.** *Keep the setup of Corollary 11.14, and assume that we are on the purely inseparable ordinary singleton* [2]*-root branch. Then the horizontal map* $S \to \mathbf{P}^1$ *has degree* 3*. Writing the relatively minimal ruled surface as* $Y = \mathbb{F}_n$ *and* $S \equiv 3S_0 + bf$*, one has* $n = 0$ *and* $b = 1$*. In particular,* $Y \cong \mathbb{F}_0$ *and* $S \equiv 3\Sigma + f$ *for every section* $\Sigma \subset \mathbb{F}_0$*.*

Proof. By Corollary 11.8, some point of S produces a surviving primitive ramification fork. Since the branch is ordinary, Corollary 11.9 forces the local ramification degree at that point to be 3. For a purely inseparable map $S \cong \mathbf{P}^1 \to \mathbf{P}^1$, every point has the same local degree, so the global degree is also 3.

Now write $Y = \mathbb{F}_n$ and $S \equiv 3S_0 + bf$. The first paragraph of Proposition 11.12 with $a = 3$ gives $S^2 = 6$. Hence $-9n + 6b = 6$, i.e. $-3n + 2b = 2$. Because S does not contain the negative section, one has $0 \leq S \cdot S_0 = b - 3n$. If $n \geq 1$, then $b \geq 3n$, so $2 = -3n + 2b \geq -3n + 6n = 3n \geq 3$, a contradiction. Therefore $n = 0$, and then $2b = 2$, so $b = 1$. Thus $Y \cong \mathbb{F}_0$ and $S \equiv 3\Sigma + f$ for every section Σ of the chosen ruling on $\mathbb{F}_0$. □

Proposition 12.69 (A service fiber on the cubic backbone yields a lifted singular plane cubic)**.** *Keep the setup of Lemma 12.68. Let* $\mathcal{T}$ *be the set of fibers from Corollary 11.14, and choose any* $f_0 \in \mathcal{T}$*. Let* $q \in S \cap f_0$ *be the first blowup center of* ν^{-1} *lying over* f_0*. Then there exists a birational morphism* $\eta\colon \widetilde{X} \to \mathbf{P}^2$ *such that* $\Gamma := \eta(H)$ *is an irreducible singular plane cubic and* $\eta|_H\colon H \to \Gamma$ *is birational.*

Proof. By Lemma 12.68, we may write $Y = \mathbb{F}_0$ and $S \equiv 3\Sigma + f$ for a section Σ passing through q. After commuting disjoint initial blowups if necessary, factor the inverse birational map ν^{-1} so that its first step is the blowup of q:

$$Y \xleftarrow{\sigma} Z := \mathrm{Bl}_q Y \xleftarrow{\rho} \widetilde{X}.$$

Let F be the fiber of Y through q, and let $F' \subset Z$ be its strict transform. Since $q \in \Sigma$ and $\Sigma^2 = 0$, contracting F' gives the elementary transform $\tau\colon Z \to \mathbb{F}_1$. Set $\mu := \tau \circ \rho\colon \widetilde{X} \to \mathbb{F}_1$ and $C := \mu(H)$. Because $\rho|_H$ and the elementary transform are birational on the image of H, the map $\mu|_H\colon H \to C$ is birational.

We now compute the class of C. Apply Lemma 11.63 on $\mathbb{F}_0$ with $a = 3$, $b = 1$, and $m = \mathrm{mult}_q(S) = 1$. This gives $C \equiv 3S_0 + 3f$ on $\mathbb{F}_1$, where $S_0^2 = -1$ is the negative section. Hence $C \cdot S_0 = (3S_0 + 3f) \cdot S_0 = 0$, so C is disjoint from S_0. Contracting the negative section, $\kappa\colon \mathbb{F}_1 \to \mathbf{P}^2$, therefore yields an irreducible plane cubic $\Gamma := \kappa(C) \subset \mathbf{P}^2$, and the composite $\eta := \kappa \circ \mu\colon \widetilde{X} \to \mathbf{P}^2$ is a birational morphism whose restriction to H is birational onto Γ.

It remains to show that Γ is singular. Since $H \cong \mathbf{P}^1$ and $\mu|_H$ is birational, the normalization of C is rational. But adjunction on $\mathbb{F}_1$ gives $p_a(C) = p_a(3S_0 + 3f) = 1$. Thus C is singular, and because κ is an isomorphism near C, the plane cubic Γ is singular as well. □

Theorem 12.70 (The purely inseparable ordinary singleton [2]-root branch is not residual)**.** *Keep the setup of Corollary 11.14, and assume that we are on the purely inseparable ordinary* *singleton* [2]*-root branch. Then* X *has a descendant with elliptic boundary. In particular, this branch cannot occur on a residual characteristic-*3 *counterexample.*

Proof. By the first paragraph of Proposition 11.12 with $a = 3$, exactly 8 blowup centers lie on successive strict transforms of S. So $\mathcal{T} \neq \varnothing$. Choose $f_0 \in \mathcal{T}$. Proposition 12.69 then produces a birational morphism $\eta\colon \widetilde{X} \to \mathbf{P}^2$ such that $\eta(H)$ is an irreducible singular plane cubic and $\eta|_H$ is birational. Applying Proposition 12.55 gives a descendant with elliptic boundary. ☐

Corollary 12.71 (The height-selection problem loses the purely inseparable ordinary singleton [2]-root branch)**.** *In Proposition 12.1, case* (iii) *does not occur. Equivalently, after the closure of the maximal-width corridor, the height-selection analysis at this stage is reduced to the several-special-horizontal branch and the separable special-multisection branch.*

Proof. Immediate from Theorem 12.70. ☐

Theorem 12.72 (Height selection in characteristic 3)**.** *Every characteristic-*3 *klt del Pezzo surface* X *of Picard rank* 1 *with* $\mathrm{ht}(X) \geq 4$ *and no descendant with elliptic boundary admits a witnessing* $\mathbf{P}^1$*-fibration of height* 4 *and maximal width.*

Proof. Assume for contradiction that such a surface X does not admit a witnessing $\mathbf{P}^1$-fibration of height 4 and maximal width. By Proposition 12.1, it is enough to treat three residual cases.

Case (ii)*: separable special multisection.* The one-section singular-image side is closed by Corollary 12.62. It remains only to consider the smooth-image side. Since case (i) is the several-special-horizontal branch and we are here in case (ii), we are outside that corridor. Corollary 12.67 then shows that the smooth-image separable special-multisection branch is already old as well. Thus case (ii) is closed.

Case (iii)*: purely inseparable ordinary singleton* [2]*-root.* This branch is excluded by Theorem 12.70.

Case (i)*: several special horizontals.* By Lemma 12.2, after removing cases (ii) and (iii) this is exactly the several-special-horizontal packet branch. Theorem 12.33 closes that branch.

All residual cases of Proposition 12.1 are therefore closed, contrary to the assumption that X fails to admit a witnessing $\mathbf{P}^1$-fibration of height 4 and maximal width. ☐

Corollary 12.73 (The characteristic-3 seven-point theorem)**.** *Let* X *be a characteristic-*3 *klt del Pezzo surface of Picard rank* 1*. Then* $\#\,\mathrm{Sing}(X) \leq 7$*.*

Proof. By Corollary 11.111, it is enough to prove the boxed height-selection statement there. That statement is exactly Theorem 12.72. ☐

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