

Erdős Problem #460: a trivial divergence and a nontrivial truncated lower bound

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Abstract

Erdős asked whether a reciprocal sum attached to a greedy coprimality construction diverges as $n \rightarrow \infty$. As usually quoted (with no restriction on the size of the greedy choices), the reciprocal series diverges for every fixed $n \geq 2$ by a simple prime-insertion argument: $a = n + p$ occurs for every prime $p > n$. We therefore focus on a natural nontrivial truncation, summing only those $a_k < n$ (or $a_k \leq n$). For this truncated problem we isolate a forced “rough” component, identify it explicitly, and prove that it has mean order $\gg \log \log n$; in particular the truncated reciprocal sum is unbounded along a subsequence of n in both standard formulations of the greedy rule.

1 The problem and two formulations

Fix $n \geq 2$ (actually we need $n \geq 3$ in Case A below). Erdős posed the following question (Erdős Problem #460 in the Erdős Problems database by T. Bloom [1]), with references [2, 3].

Case A (as in [3] and many restatements). Set $a_0 = n$ and $a_1 = 1$. For $k \geq 2$, let a_k be the least integer $> a_{k-1}$ such that

$$\gcd(n - a_k, n - a_i) = 1 \quad \text{for all } 1 \leq i < k. \quad (1)$$

Define $S_A(n) := \sum_{k \geq 1} 1/a_k$.

Case B (as in [2], equivalently forcing $(a_k, n) = 1$). Set $a_0 = 0$ and $a_1 = 1$. For $k \geq 2$, let a_k be the least integer $> a_{k-1}$ such that

$$\gcd(n - a_k, n - a_i) = 1 \quad \text{for all } 0 \leq i < k. \quad (2)$$

Since $n - a_0 = n$, this forces $\gcd(n - a_k, n) = 1$, i.e. $\gcd(a_k, n) = 1$ for all $k \geq 1$. Define $S_B(n) := \sum_{k \geq 1} 1/a_k$.

Remark 1. The two formulations can behave differently at small values. For example, for $n = 6$ one has initial segments 0, 1, 5 in Case B, while Case A begins 6, 1, 2.

2 A trivial divergence in the unrestricted problem

In both cases, if one sums over all k without any upper cut-off on a_k , then the reciprocal series diverges for every fixed $n \geq 2$.

Lemma 2 (Prime insertion). *Fix $n \geq 2$ and consider either Case A or Case B. For every prime $p > n$, the value $a = n + p$ occurs among the greedy choices.*

Proof. Fix a prime $p > n$ and set $a := n + p$, so $n - a = -p$. Consider the first index k with $a_{k-1} < a$. For every $i < k$ we have $a_i < a$, hence

$$n - a_i \in [n - (n + p - 1), n - 1] = [-p + 1, n - 1].$$

Since $p > n$, the only multiple of p in the interval $[-p + 1, n - 1]$ is 0. But $n - a_i \neq 0$ for $i \geq 1$ in both constructions, and in Case B we also have $n - a_0 = n \neq 0$. Thus $p \nmid (n - a_i)$ for all $i < k$, and therefore $\gcd(n - a, n - a_i) = \gcd(p, n - a_i) = 1$ for all required $i < k$. Hence a is admissible at step k , and by minimality the greedy algorithm cannot jump past it; therefore a is chosen at some step. $\square$

Corollary 3. *For every fixed $n \geq 2$, the full sums $S_A(n)$ and $S_B(n)$ diverge.*

Proof. By Lemma 2, both sequences contain $n + p$ for every prime $p > n$, so

$$S_A(n) \geq \sum_{p > n} \frac{1}{n + p}, \quad S_B(n) \geq \sum_{p > n} \frac{1}{n + p}.$$

Since $\sum_p 1/p = \infty$, the shifted prime sum $\sum_{p > n} 1/(n + p)$ diverges, and so do $S_A(n)$ and $S_B(n)$. $\square$

Remark 4. Because of Corollary 3, the problem only becomes substantive after imposing a natural truncation, e.g. summing only over those k with $a_k \leq n$ (or $a_k < n$). We work with the strict cut-off

$$S_{< n}(n) := \sum_{\substack{k \geq 1 \\ a_k \leq n-1}} \frac{1}{a_k},$$

in either Case A or Case B.

3 Rough and proper terms below n

For an integer $m \neq 0$, let $P^-(m)$ be the least prime divisor of $|m|$, with $P^-(\pm 1) = \infty$.

Definition 5 (Rough/proper relative to n). For $1 \leq a \leq n-1$ call a *rough* if $P^-(n-a) > a$ (equivalently, no prime $p \leq a$ divides $n-a$), and *proper* otherwise. Define

$$R(n) := \{a \in \{1, \dots, n-1\} : P^-(n-a) > a\}.$$

Lemma 6 (Automatic coprimality). *If $1 \leq a \leq n-1$ and $P^-(n-a) > a$, then $\gcd(n-a, n-a') = 1$ for every integer a' with $0 \leq a' < a$.*

Proof. If a prime p divides both $n-a$ and $n-a'$, then p divides their difference $a-a'$, hence $p \leq a-a' < a$, contradicting $P^-(n-a) > a$. $\square$

Proposition 7 (Forced rough subsequence). *In either Case A or Case B, every $a \in R(n)$ occurs among the greedy values a_k with $a_k \leq n-1$. Consequently,*

$$S_{<n}(n) \geq f(n) := \sum_{a=1}^{n-1} \frac{1}{a} \mathbf{1}_{\{P^-(n-a) > a\}}. \quad (3)$$

Proof. Let $a \in R(n)$ and suppose a is not attained by the greedy values below n . Let k be the first index with $a_{k-1} < a < a_k$. For each required $i < k$ one has $a_i \leq a_{k-1} < a$, so Lemma 6 gives $\gcd(n-a, n-a_i) = 1$, showing that a is admissible at step k . This contradicts the definition of a_k as the least admissible integer $> a_{k-1}$. Thus every $a \in R(n)$ must appear with some $a_k \leq n-1$, and summing reciprocals gives (3). $\square$

4 Average growth of the truncated sum

Let $\Phi(x, y)$ denote the number of positive integers $m \leq x$ with $P^-(m) > y$ (the y -rough integers up to x).

Theorem 8 (Mean order $\gg \log \log n$). *One has*

$$\frac{1}{N} \sum_{n \leq N} f(n) \gg \log \log N,$$

and hence $\limsup_{n \rightarrow \infty} f(n) = \infty$. In particular, in both Case A and Case B,

$$\limsup_{n \rightarrow \infty} S_{<n}(n) = \infty.$$

Proof. Start from the definition and change variables $m = n - a$:

$$\begin{aligned} \sum_{n \leq N} f(n) &= \sum_{n \leq N} \sum_{a=1}^{n-1} \frac{1}{a} \mathbf{1}_{\{P^-(n-a) > a\}} = \sum_{a \leq N-1} \frac{1}{a} \sum_{m \leq N-a} \mathbf{1}_{\{P^-(m) > a\}} \\ &= \sum_{a \leq N-1} \frac{1}{a} \Phi(N-a, a). \end{aligned}$$

Restrict to $a \in [N^{1/3}, N^{1/2}]$, for which $a \rightarrow \infty$ and $u := \frac{\log(N-a)}{\log a} \in [2, 3]$. By Buchstab's theorem (see, e.g., [4, Chapter III.6]),

$$\Phi(x, y) = \frac{x}{\log y} (\omega(u) + o(1)) \quad \text{uniformly when } x \geq y \rightarrow \infty \text{ and } u = \frac{\log x}{\log y} \in [2, 3],$$

where ω is Buchstab's function. On $[2, 3]$ one has $\omega(u) \geq \omega(2) = 1/2$, so for all large N and all $a \in [N^{1/3}, N^{1/2}]$,

$$\Phi(N-a, a) \gg \frac{N}{\log a}.$$

Therefore

$$\sum_{n \leq N} f(n) \geq \sum_{N^{1/3} \leq a \leq N^{1/2}} \frac{1}{a} \Phi(N-a, a) \gg N \sum_{N^{1/3} \leq a \leq N^{1/2}} \frac{1}{a \log a} \asymp N \log \log N.$$

Dividing by N gives the mean lower bound, which implies $\limsup_{n \rightarrow \infty} f(n) = \infty$. Finally, Proposition 7 gives $S_{<n}(n) \geq f(n)$, hence $\limsup S_{<n}(n) = \infty$. $\square$

Remark 9. Eggleton, Erdős, and Selfridge showed that in related work one has bounds of the form $a_k < k^{2+o(1)}$ for k large enough depending on n , and conjectured a sharper growth $a_k \ll k \log k$. These growth questions are orthogonal to the present note, which only extracts a universal truncated lower bound.

5 The remaining pointwise question

Definition 10 (The explicit rough lower bound). For $n \geq 2$ define

$$f(n) := \sum_{a=1}^{n-1} \frac{1}{a} \mathbf{1}_{\{P^-(n-a) > a\}} = \sum_{\substack{1 \leq a \leq n-1 \\ P^-(n-a) > a}} \frac{1}{a}.$$

Remark 11 (What is proved and what remains open). Proposition 7 gives the pointwise inequality

$$S_{<n}(n) \geq f(n) \quad (n \geq 2)$$

in both Case A and Case B, while Theorem 8 shows that $f(n)$ (hence $S_{<n}(n)$) has mean order $\gg \log \log n$ and in particular is unbounded along a subsequence of n . Thus the strongest form of Erdős' *truncated* divergence question,

$$S_{<n}(n) \xrightarrow[n \rightarrow \infty]{} \infty,$$

would follow from the purely explicit statement

$$f(n) \xrightarrow[n \rightarrow \infty]{} \infty,$$

but the arguments here do not yield such a pointwise lower bound. In other words, the only obstruction left after the reduction is the possibility that for infinitely many n the interval $\{n - a : 1 \leq a \leq n - 1\}$ contains too few integers whose least prime factor exceeds the moving threshold a .

6 The dyadic strategy: what it proves, and what is missing

6.1 Dyadic decomposition and the pointwise target

Recall the auxiliary quantity

$$f(n) := \sum_{1 \leq a < n} \frac{1}{a} \mathbf{1}_{\{P^-(n-a) > a\}},$$

where $P^-(m)$ is the least prime factor of m (with the convention $P^-(1) = \infty$). In the truncated formulation of Erdős #460 (e.g. summing only over $a_i < n$), one has

$$S_{<n}(n) \geq f(n),$$

so that the pointwise divergence question reduces to proving

$$f(n) \longrightarrow \infty \quad (n \rightarrow \infty).$$

A standard dyadic decomposition is to group a by $2^j < a \leq 2^{j+1}$. Writing $m := n - a$, the condition $P^-(m) > a$ is implied by $P^-(m) > 2^{j+1}$ and $m \in (n - 2^{j+1}, n - 2^j]$. Define the dyadic rough-counts

$$R_j(n) := \#\{m \in (n - 2^{j+1}, n - 2^j] : P^-(m) > 2^{j+1}\}.$$

Then for each j ,

$$\sum_{2^j < a \leq 2^{j+1}} \frac{1}{a} \mathbf{1}_{\{P^-(n-a) > a\}} \geq \frac{1}{2^{j+1}} R_j(n),$$

and hence

$$f(n) \geq \sum_{j \geq 0} \frac{R_j(n)}{2^{j+1}}. \quad (4)$$

Heuristically, the set of y -rough integers has density $\asymp 1/\log y$ (Buchstab-de Bruijn), so for $y = 2^{j+1}$ one expects

$$\mathbb{E} R_j(n) \asymp \frac{2^j}{\log 2^{j+1}} \asymp \frac{2^j}{j}.$$

If, for a given n , one had the lower bound $R_j(n) \gg 2^j/j$ for “many” j , then (4) would yield

$$f(n) \gg \sum_{j \in \mathcal{G}(n)} \frac{1}{j},$$

and a harmonic-sized set of good scales $\mathcal{G}(n)$ would force $f(n) \rightarrow \infty$.

6.2 Why a naive uniform conjecture is impossible

A tempting strengthening is to posit a uniform-in- x lower bound for y -roughs in intervals of length $\asymp y$. However any statement of the form

$$\#\{m \in (x, x+y] : P^-(m) > y\} \geq 1 \quad \text{for all } x$$

is false for arbitrarily large y , because there exist long intervals in which *every* integer has a prime factor $\leq y$. This is measured by the classical “sieving length” function $Y(y)$ (equivalently Jacobsthal-type gaps for y -rough numbers); in particular, $Y(y) \gg y$ along a sequence of $y \rightarrow \infty$, so there are intervals of length $> y$ containing no y -rough integers at all. Consequently, any conjecture asserting *every* interval of length comparable to y contains $\gg y/\log y$ many y -rough integers is untenable.

Remark 12 (Why “uniform in x ” must fail). Fix $y \geq 2$ and let $P(y) := \prod_{p \leq y} p$ be the primorial. An integer m satisfies $P^-(m) > y$ if and only if $\gcd(m, P(y)) = 1$. Let $j(P(y))$ denote Jacobsthal’s function, i.e. the maximal length of a run of consecutive integers each having a nontrivial common factor with $P(y)$; equivalently, $j(P(y)) - 1$ is the maximal length of an

interval containing no integer coprime to $P(y)$. This quantity is also commonly denoted $Y(y)$ in the Erdős–Rankin literature on covering congruences and prime gaps. It is known unconditionally that $Y(y)$ is superlinear (indeed $Y(y) \gg y^{\frac{\log y \log \log \log y}{\log \log y}}$; see [9, 10]), so for infinitely many y there exist intervals of length $> y$ containing *no* y -rough integers at all. Therefore any conjecture asserting a positive lower bound for $\#\{m \in (x, x+y] : P^-(m) > y\}$ for all x is false in general.

Thus the dyadic strategy for Erdős #460 cannot be closed by a “uniform in x ” short-interval density statement; it must instead control how often a *single endpoint* n can be exceptional across the dyadic scales.

6.3 What existing sieve technology actually gives (and why it falls short)

What is available unconditionally is a strong *almost-all* theory for rough numbers in short intervals, obtained by bounding variances (or higher moments) of suitable sieve weights. For example, Friedlander–Iwaniec [5] prove (in the regime $y \leq X^c$ and $H/\log y \rightarrow \infty$) that for almost all $x \in [X, 2X]$ one has

$$\sum_{x < m \leq x+H} \mathbf{1}_{\{P^-(m) > y\}} \asymp H \prod_{p \leq y} \left(1 - \frac{1}{p}\right) \asymp \frac{H}{\log y},$$

and more recently Gorodetsky [7] has obtained unconditional asymptotics for the variance of $\mathbf{1}_{\{P^-(m) > y\}}$ in short intervals in a range that is qualitatively best possible.

Specializing such results to $H \asymp y$ (or $H \asymp 2^j$ and $y \asymp 2^{j+1}$) yields that for each *fixed* scale j and large X , the set of $n \in [X, 2X]$ for which $R_j(n) \ll 2^j/j$ has small density. Summing over j then proves lower bounds for the *average* of $f(n)$ and hence

$$\limsup_{n \rightarrow \infty} f(n) = \infty,$$

but it does *not* force $f(n) \rightarrow \infty$ pointwise, because a single n could in principle lie in the exceptional set for infinitely many j . This is the precise gap: average-in- n control at each fixed scale does not preclude an infinite exceptional set of endpoints n that are simultaneously bad on many scales.

6.4 A plausible conjectural closure: multiscale (endpoint-wise) non-exceptionality

The dyadic lower bound (4) shows that the following multiscale statement would imply the desired pointwise divergence.

Conjecture 13 (Multiscale roughness at a fixed endpoint). *There exists $c > 0$ such that for all sufficiently large integers n ,*

$$\sum_{\substack{1 \leq j \leq \lfloor \log_2 n \rfloor \\ R_j(n) \geq c 2^j / j}} \frac{1}{j} \geq c \log \log n.$$

Indeed, Conjecture 13 and (4) together give $f(n) \gg \log \log n$, hence $f(n) \rightarrow \infty$, and therefore $S_{<n}(n) \rightarrow \infty$.

6.5 How close are we? Poissonian heuristics and moment technology

A natural route to Conjecture 13 is to upgrade the known “almost-all” short-interval results into *strong lower-tail bounds* for $R_j(n)$, and then combine these bounds across j .

Heuristically, rough numbers behave “Poissonian” in short intervals after local congruence effects are accounted for, so one expects concentration of measure for $R_j(n)$ around its mean $\asymp 2^j/j$. This philosophy is made quantitative in recent work of Gafni–Tao [8] on rough numbers in intervals attached to prime gaps: the main technical hurdle is precisely a concentration estimate for rough-number counts in short intervals, where second moments alone lose strength and higher moments (encoded by averages of singular series) become decisive. Their analysis explicitly builds on the higher-moment technology developed by Montgomery–Soundararajan [11] for primes in short intervals, and it yields power-saving bounds for the number of exceptional prime-gap intervals lacking the predicted rough integer.

From the perspective of Erdős #460, these developments indicate that the remaining obstacle is not understanding $R_j(n)$ for a fixed j (where variance methods already give “almost-all” control), but rather understanding *simultaneous* lower-tail control for $R_j(n)$ across the dyadic scales for a single endpoint n —i.e. ruling out a scenario where one endpoint is exceptional at a positive proportion of scales. Conjecture 13 isolates exactly this missing multiscale correlation input.

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