

Greedy Coprimality Sequences and a Forced “Rough” Subsum

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January 13, 2026

Abstract

READ V2 FOR THE COMPLETE TREATMENT:

<https://www.ulam.ai/research/erdos-460-v2.pdf>

Fix $n \geq 2$ and define a sequence (a_k) greedily by requiring the integers $n - a_k$ to be pairwise coprime. Erdős in [1] asked whether the reciprocal series $\sum_k 1/a_k$ tends to infinity as $n \rightarrow \infty$, and similarly for two natural restricted subsums. In this note we isolate a canonical subsum—the “rough” part—which is forced by the greedy rule and admits an explicit description independent of the past history of the construction. The same description holds for a commonly suggested variant of the definition that also enforces coprimality with n .

1 The problem and two greedy constructions

Fix $n \geq 2$. There are two potential formulations of the Erdős problem #460, see [1].

1.1 Case A (Erdős’ formulation)

Set $a_0 = n$ and $a_1 = 1$. For $k \geq 2$, define a_k to be the least integer $> a_{k-1}$ such that

$$\gcd(n - a_k, n - a_i) = 1 \quad \text{for all } 1 \leq i < k. \quad (1)$$

Define the reciprocal sum

$$S_A(n) := \sum_{k \geq 1} \frac{1}{a_k}.$$

Erdős asked whether $S_A(n) \rightarrow \infty$ as $n \rightarrow \infty$, and also asked the same question for the two subsums obtained by splitting indices according to whether $n - a_k$ has a prime divisor $\leq a_k$.

Remark 1. In Case A the choice of a_0 is immaterial: the admissibility constraints (1) only involve indices $i \geq 1$, so once a_1 is fixed the sequence $(a_k)_{k \geq 1}$ is independent of a_0 .

1.2 Case B (including $i = 0$ with $a_0 = 0$)

Set $a_0 = 0$ and $a_1 = 1$. For $k \geq 2$, define a_k to be the least integer $> a_{k-1}$ such that

$$\gcd(n - a_k, n - a_i) = 1 \quad \text{for all } 0 \leq i < k. \quad (2)$$

Define

$$S_B(n) := \sum_{k \geq 1} \frac{1}{a_k}.$$

Remark 2. The extra constraint $i = 0$ in (2) is

$$\gcd(n - a_k, n - a_0) = \gcd(n - a_k, n) = 1.$$

Since $\gcd(n - a, n) = \gcd(a, n)$, Case B is precisely Case A with the additional standing requirement $\gcd(a_k, n) = 1$ for all k .

2 Proper and rough terms

For an integer $m \neq 0$, let $P^-(m)$ be the least prime divisor of $|m|$, and set $P^-(\pm 1) = \infty$.

Definition 3 (Proper/rough relative to n). For $1 \leq a \leq n - 1$ we call a *rough* (relative to n) if $P^-(n - a) > a$, equivalently if no prime $p \leq a$ divides $n - a$. Otherwise we call a *proper*, i.e. $P^-(n - a) \leq a$.

Define the rough set

$$R(n) := \{a \in \{1, 2, \dots, n - 1\} : P^-(n - a) > a\}.$$

For either Case A or Case B, let

$$I_{<n} := \{k : 1 \leq a_k \leq n - 1\} \quad \text{and} \quad I_{\text{rough}} := \{k \in I_{<n} : P^-(n - a_k) > a_k\}.$$

Define the corresponding subsums

$$S_{\text{rough}}(n) := \sum_{k \in I_{\text{rough}}} \frac{1}{a_k}, \quad S_{\text{prop}}(n) := \sum_{\substack{k \in I_{<n} \\ k \notin I_{\text{rough}}}} \frac{1}{a_k}.$$

Then on the natural range $a_k \leq n - 1$ we have

$$\sum_{k \in I_{<n}} \frac{1}{a_k} = S_{\text{rough}}(n) + S_{\text{prop}}(n).$$

3 The forced rough subsequence

The key point is that “rough” is a purely local condition which forces coprimality with all earlier candidates.

Lemma 4 (Automatic coprimality). *Let $1 \leq a \leq n - 1$. If $P^-(n - a) > a$, then for every integer a' with $0 \leq a' < a$ one has*

$$\gcd(n - a, n - a') = 1.$$

Proof. Suppose $\gcd(n - a, n - a') > 1$. Then there is a prime p dividing both $n - a$ and $n - a'$, and hence dividing their difference

$$(n - a') - (n - a) = a - a'.$$

Thus $p \leq a - a' < a$, so in particular $p \leq a$, contradicting the assumption that $n - a$ has no prime divisor $\leq a$. $\square$

Corollary 5. *If $a \in R(n)$ then $\gcd(n - a, n) = 1$, and hence $\gcd(a, n) = 1$.*

Proof. Apply Lemma 4 with $a' = 0$, for which $n - a' = n$. The identity $\gcd(n - a, n) = \gcd(a, n)$ gives the second claim. $\square$

Lemma 6 (Stagewise admissibility). *Let $a \in R(n)$. If $k \geq 1$ and $a > a_{k-1}$, then a satisfies the admissibility constraints for a_k in both Case A and Case B.*

Proof. For every relevant $i < k$ we have $a_i \leq a_{k-1} < a$, hence Lemma 4 applies with $a' = a_i$. (In Case B, the index $i = 0$ is allowed, and Lemma 4 also covers $a' = 0$.) $\square$

Lemma 7 (Greedy non-skipping). *Every $a \in R(n)$ occurs as $a = a_j$ for some $j \in I_{<n}$, in both Case A and Case B.*

Proof. Assume not, and let $a \in R(n)$ be the least element of $R(n)$ not attained by the sequence. Since (a_k) is strictly increasing and $a_1 = 1 \leq a$, there is a first index $k \geq 2$ such that

$$a_{k-1} < a < a_k.$$

By Lemma 6, the integer a is admissible for the choice of a_k . This contradicts the definition of a_k as the *least* admissible integer $> a_{k-1}$. $\square$

Proposition 8 (Identification of the rough subsequence). *In both Case A and Case B,*

$$\{a_k : k \in I_{\text{rough}}\} = R(n).$$

Consequently,

$$S_{\text{rough}}(n) = \sum_{a \in R(n)} \frac{1}{a}.$$

Proof. If $a \in R(n)$, Lemma 7 yields $j \in I_{<n}$ with $a_j = a$; then $j \in I_{\text{rough}}$ by definition, so $R(n) \subseteq \{a_k : k \in I_{\text{rough}}\}$. Conversely, if $k \in I_{\text{rough}}$, then $P^-(n - a_k) > a_k$, so $a_k \in R(n)$, giving the reverse inclusion. Since (a_k) is strictly increasing, each value appears at most once, and the reciprocal sums agree termwise. $\square$

4 Consequences for divergence questions

Corollary 9 (A canonical lower bound). *In either Case A or Case B one has, for each $n \geq 2$,*

$$\sum_{k \geq 1} \frac{1}{a_k} \geq \sum_{k \in I_{<n}} \frac{1}{a_k} \geq S_{\text{rough}}(n) = \sum_{\substack{1 \leq a \leq n-1 \\ P^-(n-a) > a}} \frac{1}{a}.$$

In particular, if the explicit “rough” sum on the right tends to ∞ as $n \rightarrow \infty$, then so does $\sum_k 1/a_k$.

Proof. The first inequality drops nonnegative terms, the second drops the proper part, and the final identity is Proposition 8. $\square$

Remark 10. Proposition 8 isolates a subsum that is independent of the evolving coprimality history: every $a < n$ with $P^-(n - a) > a$ is admissible as soon as it becomes available, hence cannot be skipped by a greedy minimality argument. Thus the original divergence problems reduce, at least for lower

bounds, to understanding the explicit quantity $\sum_{1 \leq a \leq n-1, P^-(n-a) > a} 1/a$, while the complementary “proper” subsum encodes the genuinely interactive part of the construction.

Remark 11. If one were to impose the Case B constraint (2) while taking $a_0 = n$, then $n - a_0 = 0$ and the condition $\gcd(n - a_k, 0) = 1$ forces $|n - a_k| = 1$, collapsing the process. This is why the variant is naturally paired with $a_0 = 0$.

References

- [1] Erdos Problem #460 at <https://www.erdosproblems.com/460>