A note on Erdős Problem 1092

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Abstract

Erdős Problem 1092 asks whether the local error term in an "r-partite plus few edges" condition may grow linearly with the order of subgraphs while still forcing $(r+1)$-colourability. The comments on the problem page point out that Rödl's construction of nearly bipartite graphs with large chromatic number gives a negative answer for $r = 2$. We record the short reduction showing that the same construction also gives a negative answer for every fixed $r \geq 2$: join Rödl's graph with a clique of size $r - 2$. Thus the proposed lower bound $f_r(n) \gg_r n$ fails for all fixed $r \geq 2$.

1 Statement and notation

All graphs in this note are finite. For a graph H and an integer $r \geq 1$, write

$$d_r(H) := \min\{|F| : F \subseteq E(H),\ \chi(H - F) \leq r\}.$$

Equivalently, $d_r(H) \leq t$ if and only if H can be written as

$$H = B \cup E_0,$$

where B is r-colourable and $|E_0| \leq t$. Thus, for $r = 2$, $d_2(H)$ is the minimum number of edge deletions needed to make H bipartite; in particular

$$d_2(H) = |E(H)| - \mathrm{MaxCut}(H).$$

We use "r-colourable" in the usual sense "chromatic number at most r". This is the interpretation under which the problem is non-vacuous for small subgraphs.

Erdős Problem 1092 asks whether the maximal local error threshold $f_r(n)$ can satisfy

$$f_2(n) \gg n,$$

and, more generally, whether for each fixed r one has

$$f_r(n) \gg_r n.$$

The answer is negative for every fixed $r \geq 2$.

2 The input from Rödl

We shall use the following consequence of Rödl's theorem on nearly bipartite graphs with large chromatic number.

Theorem 1 (Rödl [2]). *For every $\varepsilon > 0$ and every integer q there exists a graph X such that*

$$\chi(X) \geq q$$

and every subgraph $J \subseteq X$ satisfies

$$d_2(J) \leq \varepsilon |V(J)|.$$

Rödl states this through explicit graphs $K^*(n,k)$ with chromatic number $k+2$, for n sufficiently large depending on ε and k. The comments on the Erdős Problem 1092 page observe that this already disproves the $r = 2$ linear-growth claim. The only additional observation needed for the full fixed-r statement is that the construction can be lifted from bipartite to r-partite by taking a join with a clique.

3 The join reduction

Recall that the join $G_1 \vee G_2$ of two vertex-disjoint graphs is obtained by adding every edge between $V(G_1)$ and $V(G_2)$. We shall use two elementary facts: $\chi(G_1 \vee G_2) = \chi(G_1) + \chi(G_2)$, and a subgraph of a clique on a vertices can be coloured using at most a singleton colour classes.

Lemma 1 (lifting from 2 parts to r parts). *Let $r \geq 2$, let X be a graph, and let*

$$G := X \vee K_{r-2},$$

where K_0 is interpreted as the empty graph. If every subgraph $J \subseteq X$ satisfies

$$d_2(J) \leq \varepsilon |V(J)|,$$

then every subgraph $H \subseteq G$ satisfies

$$d_r(H) \leq \varepsilon |V(H) \cap V(X)| \leq \varepsilon |V(H)|.$$

Moreover,

$$\chi(G) = \chi(X) + r - 2.$$

Proof. Let $H \subseteq G$ be any subgraph. Put

$$S := V(H) \cap V(X), \qquad T := V(H) \cap V(K_{r-2}).$$

The subgraph of H induced by S is a subgraph $J \subseteq X$. By the hypothesis on X, there is a set $F \subseteq E(J)$ with

$$|F| \leq \varepsilon |S|$$

such that $J - F$ is bipartite. Let $A \cup B$ be a bipartition of $J - F$.

Now colour $H - F$ as follows. Use the two colour classes A and B for the vertices in S, and give each vertex of T its own singleton colour class. Since

$$2 + |T| \leq 2 + (r-2) = r,$$

this is an r-colouring of $H - F$. Therefore

$$d_r(H) \le |F| \le \varepsilon|S| \le \varepsilon|V(H)|.$$

The chromatic-number identity for joins gives

$$\chi(G) = \chi(X) + \chi(K_{r-2}) = \chi(X) + r - 2.$$

□

4 Counterexamples at every linear scale

Theorem 2. *Fix $r \ge 2$. For every $c > 0$ and every integer $N \ge 1$, there exists a finite graph G such that*

$$\chi(G) > r + 1,$$

every subgraph $H \subseteq G$ satisfies

$$d_r(H) \le c|V(H)|,$$

and every subgraph $H \subseteq G$ with $|V(H)| < N$ is already r-colourable.

Proof. Choose

$$0 < \varepsilon < \min\{c, 1/N\}.$$

By Rödl's theorem, choose a graph X with

$$\chi(X) > 3$$

such that every subgraph $J \subseteq X$ satisfies

$$d_2(J) \le \varepsilon|V(J)|.$$

Set

$$G := X \vee K_{r-2}.$$

By Lemma 1, every subgraph $H \subseteq G$ satisfies

$$d_r(H) \le \varepsilon|V(H)| < c|V(H)|.$$

Also

$$\chi(G) = \chi(X) + r - 2 > 3 + r - 2 = r + 1.$$

Finally, if $|V(H)| < N$, then

$$d_r(H) \le \varepsilon|V(H)| < 1.$$

Since $d_r(H)$ is an integer, this implies $d_r(H) = 0$, so H is already r-colourable. □

Corollary 1. *For every fixed $r \ge 2$,*

$$f_r(n) \not\ll_r n.$$

Equivalently, there is no constant $c_r > 0$ such that the local hypothesis with error budget at least $c_r m$ for all sufficiently large m forces $(r + 1)$-colourability.

Proof. Suppose, to the contrary, that for some fixed $r \geq 2$ there are constants $c > 0$ and $N \geq 1$ such that

$$f_r(m) \geq cm \qquad \text{for all } m \geq N.$$

Apply Theorem 2 with $c/2$ in place of c and with this N. We obtain a graph G with $\chi(G) > r + 1$ such that every m-vertex subgraph $H \subseteq G$ satisfies

$$d_r(H) \leq (c/2)m \qquad (m \geq N),$$

while for $m < N$ every such H is already r-colourable, so $d_r(H) = 0$. The threshold values $f_r(m)$ are nonnegative, since they count allowable edges; equivalently, the zero error function is trivially admissible. Hence $d_r(H) \leq f_r(m)$ also for $m < N$. Thus the hypothesis defining f_r is satisfied for every subgraph of G. The defining conclusion would therefore give $\chi(G) \leq r + 1$, contradicting the choice of G. □

5 Remarks

Remark 1. *The proof uses arbitrary subgraphs, not only induced subgraphs. This matches the formulation of Erdős Problem 1092. Passing to arbitrary subgraphs causes no difficulty in Lemma 1: the part inside X is still a subgraph of X, and the vertices lying in the added clique may always be given singleton colour classes.*

Remark 2. *The argument settles the linear-growth question asked in Problem 1092 for all fixed $r \geq 2$. It does not identify the optimal sublinear order of the maximal threshold $f_r(n)$; it only shows that a positive linear lower bound is impossible.*

Remark 3 (suggested update to the problem page)**.** *The existing comment already records that Rödl's construction disproves the case $r = 2$. The following sentence would incorporate the general fixed-r consequence: "Joining Rödl's graph with K_{r-2} gives the same negative answer for every fixed $r \geq 2$, and hence $f_r(n) \not\gg_r n$ for all $r \geq 2$."*

References

[1] T. F. Bloom, *Erdős Problem #1092*, https://www.erdosproblems.com/1092, accessed 28 April 2026.

[2] V. Rödl, *Nearly bipartite graphs with large chromatic number*, Combinatorica **2** (1982), 377–383. https://doi.org/10.1007/BF02579434.