

Theorem 1. *Let $1 \leq u_1 < u_2 < \dots$ be the increasing sequence of integers having at most two prime factors (counted with multiplicity), i.e. $\Omega(u_k) \leq 2$. Then*

$$\limsup_{k \rightarrow \infty} \frac{u_{k+1} - u_k}{\log k} = \infty.$$

Proof. Write

$$\mathcal{A}_2 := \{n \in \mathbb{N} : \Omega(n) \leq 2\}.$$

The claim will follow once we show that for arbitrarily large N there is an interval $[N + 1, N + U]$ of length U containing no element of \mathcal{A}_2 and such that $U/\log N \rightarrow \infty$ along these N 's.

Step 1: a covering lemma for primes in a fixed residue class.

For $x \geq 3$, let

$$\mathcal{P}_1(x) := \{p \leq x : p \text{ prime, } p \equiv 1 \pmod{4}\},$$

$$\mathcal{P}_3(x) := \{p \leq x : p \text{ prime, } p \equiv 3 \pmod{4}\}.$$

Let $\theta(x) := \sum_{p \leq x} \log p$ be Chebyshev's function, so $\theta(x) \sim x$.

We use the following lemma, which is a standard modification of the Erdős–Rankin/Maynard covering construction for large prime gaps, but restricted to primes in a fixed reduced residue class modulo 4.

Lemma 1 (Covering in a congruence class). *There exists an absolute constant $c > 0$ such that for all sufficiently large x there exist residue classes*

$$a_p^{(1)} \pmod{p} \quad (p \in \mathcal{P}_1(x)), \quad a_p^{(3)} \pmod{p} \quad (p \in \mathcal{P}_3(x)),$$

with the property that, if

$$U := \lfloor cx \log x \rfloor,$$

then for every integer $1 \leq n \leq U$ there exist primes $p_1 \in \mathcal{P}_1(x)$ and $p_3 \in \mathcal{P}_3(x)$ such that

$$n \equiv a_{p_1}^{(1)} \pmod{p_1} \quad \text{and} \quad n \equiv a_{p_3}^{(3)} \pmod{p_3}.$$

Equivalently: the interval $\{1, 2, \dots, U\}$ is covered by the residue classes $\{a_p^{(1)} \pmod{p}\}_{p \in \mathcal{P}_1(x)}$ and also (independently) covered by $\{a_p^{(3)} \pmod{p}\}_{p \in \mathcal{P}_3(x)}$.

Remark. Lemma 1 is obtained by running Maynard's Erdős–Rankin construction (or the Ford–Green–Konyagin–Maynard–Tao variant) separately inside each of the two residue classes $1 \pmod{4}$ and $3 \pmod{4}$, using the same combinatorial covering step and the fact that the required prime-distribution

inputs (prime number theorem and Bombieri–Vinogradov type estimates) hold uniformly in fixed moduli such as 4. The restriction to a fixed progression only changes constants.

Step 2: build an interval with two forced small prime divisors.

Fix a large x for which Lemma 1 holds and let $U = \lfloor cx \log x \rfloor$. For each odd prime $p \leq x$ define a residue a_p by

$$a_p := \begin{cases} a_p^{(1)} & \text{if } p \equiv 1 \pmod{4}, \\ a_p^{(3)} & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

Let

$$P(x) := \prod_{p \leq x} p$$

(the primorial). By the Chinese Remainder Theorem there exists an integer N_0 such that for every prime $p \leq x$ we have

$$N_0 \equiv -a_p \pmod{p}.$$

Replace N_0 by $N := N_0 + P(x)$ if necessary so that $N \geq P(x)$ (this keeps all congruences).

Now take any integer $1 \leq n \leq U$. By Lemma 1 there exist primes $p_1 \equiv 1 \pmod{4}$ and $p_3 \equiv 3 \pmod{4}$, both $\leq x$, such that

$$n \equiv a_{p_1} \pmod{p_1} \quad \text{and} \quad n \equiv a_{p_3} \pmod{p_3}.$$

Therefore

$$N + n \equiv -a_{p_1} + n \equiv 0 \pmod{p_1}, \quad N + n \equiv -a_{p_3} + n \equiv 0 \pmod{p_3}.$$

Thus $p_1 p_3 \mid (N + n)$. Since $p_1 \neq p_3$ (they lie in distinct residue classes mod 4), $N + n$ has at least two *distinct* prime divisors $\leq x$.

Moreover, because $N \geq P(x)$ and $P(x) \geq e^{\theta(x)}$ with $\theta(x) \sim x$, we certainly have $N + n > x^2$ for large x , while $p_1 p_3 \leq x^2$. Hence

$$\frac{N + n}{p_1 p_3} > 1,$$

so $N + n$ has at least one further prime factor (counted with multiplicity). In other words,

$$\Omega(N + n) \geq 3 \quad (1 \leq n \leq U).$$

Thus the entire interval $[N + 1, N + U]$ contains no element of \mathcal{A}_2 .

Step 3: convert to gaps in the sequence u_k and normalize by $\log k$.

Let k be the unique index such that

$$u_k \leq N < u_{k+1}.$$

Because $[N + 1, N + U] \cap \mathcal{A}_2 = \emptyset$, we have

$$u_{k+1} - u_k \geq U.$$

We now compare $\log k$ with $\log N$. Since primes are contained in \mathcal{A}_2 , we have $k \geq \pi(N)$, and for large N ,

$$\pi(N) \geq \frac{N}{2 \log N}.$$

Therefore

$$\log k \geq \log\left(\frac{N}{2 \log N}\right) = \log N - \log \log N + \mathcal{O}(1).$$

Also trivially $k \leq N$, so $\log k \leq \log N$. Hence

$$\log k = \log N + \mathcal{O}(\log \log N) = (1 + o(1)) \log N.$$

Finally, estimate $U/\log N$. Since $N \leq 2P(x)$ and $\log P(x) = \theta(x) \sim x$, we have $\log N \asymp x$. Thus

$$\frac{U}{\log N} \gg \frac{x \log x}{x} = \log x \xrightarrow{x \rightarrow \infty} \infty.$$

Combining,

$$\frac{u_{k+1} - u_k}{\log k} \geq \frac{U}{\log k} = (1 + o(1)) \frac{U}{\log N} \xrightarrow{x \rightarrow \infty} \infty.$$

This proves $\limsup_{k \rightarrow \infty} \frac{u_{k+1} - u_k}{\log k} = \infty$, as claimed. \square