# Erdős Problem #1143 for arbitrary fixed $\alpha$ and arithmetic Kakeya

Draft note

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**Abstract**

Erdős Problem #1143 asks for estimates for the minimum number of integers in every interval of $k$ positive integers which are divisible by at least one of prescribed primes $p_1 < \cdots < p_u$, especially when $k = \lfloor \alpha p_u \rfloor$ and $\alpha > 2$ is fixed. We interpret multiples as distinct covered integers.

Let $r = \lfloor \alpha \rfloor$. This note proves that for every fixed $\alpha > 2$, the worst-case version of the problem is equivalent up to the factor $r$ to the inverse arithmetic Kakeya problem for $r$-term arithmetic progressions:

$$K_r(u) \leq G_\alpha(u) \leq rK_r(u),$$

where $K_r(u)$ is the smallest size of an integer set containing $r$-term arithmetic progressions with $u$ distinct common differences, and $G_\alpha(u)$ is the infimum over prime tuples of $F_{\lfloor \alpha p_u \rfloor}(p_1, \ldots, p_u)$. The lower bound is pointwise:

$$F_{\lfloor \alpha p_u \rfloor}(p_1, \ldots, p_u) \geq K_r(u)$$

for every prime tuple. The upper bound is a worst-case construction using long arithmetic progressions of primes.

Consequences include the unconditional pointwise bounds

$$F_{\lfloor \alpha p_u \rfloor}(p_1, \ldots, p_u) \gg u^{6/11} \qquad (\alpha \geq 3)$$

from the Katz–Tao sums-differences estimate, and

$$F_{\lfloor \alpha p_u \rfloor}(p_1, \ldots, p_u) \gg u^{4/7} \qquad (\alpha \geq 7)$$

from a four-slope Katz–Tao estimate. More generally, every finite-slope sum-difference inequality transfers into a pointwise lower bound for all sufficiently large $\alpha$.

## 1 Definitions

For primes $p_1 < \cdots < p_u$ and an integer $k \geq 1$, define

$$F_k(p_1, \ldots, p_u) := \min_I \# \left( I \cap \bigcup_{i=1}^{u} p_i\mathbb{Z} \right),$$

where $I$ ranges over intervals of $k$ consecutive positive integers. This is the distinct-integer interpretation of "multiples of at least one of the $p_i$".

For fixed $\alpha > 2$ define the worst-case quantity

$$G_\alpha(u) := \inf_{p_1 < \cdots < p_u} F_{\lfloor \alpha p_u \rfloor}(p_1, \ldots, p_u),$$

where the infimum is over increasing $u$-tuples of primes. The thread for Erdős Problem #1143 is used here only for the problem statement and for the comment that no partial solution is claimed there [1].

For a finite set $A \subset \mathbb{Z}$ and an integer $r \geq 2$, let

$$\Delta_r(A) := \{d \in \mathbb{Z}_{\geq 1} : \exists a \in \mathbb{Z} \text{ such that } a, a+d, \ldots, a+(r-1)d \in A\}.$$

Define

$$D_r(m) := \max_{|A|=m} |\Delta_r(A)|, \qquad K_r(u) := \min\{|A| : |\Delta_r(A)| \geq u\}.$$

Thus $K_r$ is the inverse arithmetic Kakeya function for $r$-term arithmetic progressions. For $r = 3$, $D_3$ is the common-difference problem in Erdős Problem #1097 [2].

## 2 The pointwise lower bound

**Theorem 1** (Pointwise arithmetic-Kakeya lower bound)**.** *Let $r \geq 2$ be an integer and let $\alpha \geq r$. If $p_1 < \cdots < p_u$ are primes and $P = p_u$, then*

$$F_{\lfloor \alpha P \rfloor}(p_1, \ldots, p_u) \geq K_r(u).$$

*Proof.* Let $L = \lfloor \alpha P \rfloor$. Since $\alpha \geq r$ and $P$ is an integer, $L \geq rP \geq rp_i$ for every $i$. Every interval of $L$ consecutive integers therefore contains at least $r$ multiples of $p_i$; the first $r$ such multiples form an $r$-term arithmetic progression with common difference $p_i$.

For any such interval $I$, put

$$A_I := I \cap \bigcup_{i=1}^{u} p_i\mathbb{Z}.$$

Then

$$\{p_1, \ldots, p_u\} \subseteq \Delta_r(A_I),$$

so $|A_I| \geq K_r(u)$. Taking the minimum over intervals $I$ gives the result. □

The theorem immediately gives

$$G_\alpha(u) \geq K_{\lfloor \alpha \rfloor}(u) \qquad (\alpha > 2).$$

## 3 The reverse transfer in the worst case

The reverse direction is not pointwise: a fixed tuple of primes may force many more covered integers than the worst possible tuple. The reverse direction is a worst-case construction. The only input about primes is the Green–Tao theorem that the primes contain arbitrarily long arithmetic progressions [4].

**Lemma 2** (Prime compression)**.** *Let $D, M \geq 1$ and $\varepsilon > 0$. There exist positive integers $h, v$ such that*

$$v + hn \quad (0 \leq n \leq D)$$

*are all prime. If $P := v + hD$, then*

$$v + hn \geq (1-\varepsilon)P \quad (0 \leq n \leq D), \qquad hM \leq \varepsilon P, \qquad P > \varepsilon^{-1}.$$

*Proof.* Choose an integer

$$R > \max\{D/\varepsilon, M/\varepsilon, \varepsilon^{-1}\}.$$

By the Green–Tao theorem, there is an arithmetic progression of primes

$$q_0, q_0 + h, q_0 + 2h, \ldots, q_0 + Rh.$$

Set

$$v := q_0 + (R - D)h, \qquad P := v + hD = q_0 + Rh.$$

Then $v + hn$ is prime for all $0 \le n \le D$. Moreover,

$$P - (v + hn) = h(D - n) \le hD \le \varepsilon Rh \le \varepsilon P,$$

and

$$hM/P \le hM/(Rh) < \varepsilon.$$

Finally $P \ge Rh \ge R > \varepsilon^{-1}$. $\square$

**Theorem 3** (Worst-case transfer for every fixed $\alpha$)**.** *Let $r \ge 2$ be an integer and let*

$$r \le \alpha < r + 1.$$

*Then, for every $u \ge 1$,*

$$K_r(u) \le G_\alpha(u) \le rK_r(u).$$

*Equivalently, for every fixed $\alpha > 2$ and $r = \lfloor \alpha \rfloor$,*

$$K_r(u) \le G_\alpha(u) \le rK_r(u).$$

*Proof.* The lower bound $K_r(u) \le G_\alpha(u)$ is Theorem 1. It remains to prove the upper bound.

Let $m := K_r(u)$. Choose a finite set $A \subset \mathbb{Z}$ with $|A| = m$ and choose distinct positive integers

$$d_1 < \cdots < d_u$$

lying in $\Delta_r(A)$. Translating $A$ if necessary, assume $A \subset \mathbb{Z}_{\ge 1}$. For each $i$, choose $a_i$ such that

$$a_i, a_i + d_i, \ldots, a_i + (r-1)d_i \in A.$$

Let $M := \max A$ and $D := d_u$.

Choose a real number $s$ with

$$\max(0, \alpha - r) < s < 1.$$

Then

$$\mu_1 := 1 - s, \qquad \mu_2 := \alpha - s - (r-1), \qquad \mu_3 := r - \alpha + s$$

are positive. Choose $\varepsilon > 0$ so small that

$$2\varepsilon < \mu_1, \qquad 2\varepsilon < \mu_2, \qquad 2r\varepsilon < \mu_3.$$

By Lemma 2, choose $h, v$ such that

$$p_i := v + hd_i \qquad (1 \le i \le u)$$

are primes and, with $P := p_u = v + hD$,

$$p_i \ge (1 - \varepsilon)P, \qquad hM \le \varepsilon P, \qquad P > \varepsilon^{-1}. \tag{1}$$

The $p_i$ are increasing because the $d_i$ are increasing.

By the Chinese remainder theorem, choose an integer $w$ satisfying

$$w + ha_i \equiv 0 \pmod{p_i} \qquad (1 \le i \le u).$$

Replacing $w$ by a sufficiently large congruent integer modulo $p_1 \cdots p_u$, assume the interval below consists of positive integers. Put

$$t := \lfloor sP \rfloor, \qquad L := \lfloor \alpha P \rfloor,$$

and

$$I := \{w - t + 1, w - t + 2, \dots, w - t + L\}.$$

We claim that, for every $i$,

$$I \cap p_i\mathbb{Z} = \{w + ha_i + jp_i : 0 \le j \le r - 1\}. \tag{2}$$

Indeed, the multiples of $p_i$ are exactly the integers $w + ha_i + jp_i$ with $j \in \mathbb{Z}$.

For $0 \le j \le r - 1$, the lower endpoint condition is immediate from $ha_i \ge 1$ and $t \ge 0$. For the upper endpoint, it suffices to check $j = r - 1$:

$$ha_i + (r - 1)p_i \le hM + (r - 1)P \le (r - 1 + \varepsilon)P.$$

On the other hand,

$$L - t \ge \alpha P - 1 - sP = (\alpha - s)P - 1.$$

Since $2\varepsilon < \mu_2 = \alpha - s - (r - 1)$ and $P > \varepsilon^{-1}$,

$$(r - 1 + \varepsilon)P \le (\alpha - s)P - 1.$$

Thus the displayed $r$ multiples all lie in $I$.

Now exclude the neighboring multiples. For $j = -1$,

$$ha_i - p_i \le hM - (1 - \varepsilon)P \le -(1 - 2\varepsilon)P \le -sP \le -t,$$

using $2\varepsilon < \mu_1 = 1 - s$. Hence this multiple is below $I$. For $j = r$,

$$ha_i + rp_i \ge r(1 - \varepsilon)P.$$

Also

$$L - t \le \alpha P - (sP - 1) = (\alpha - s)P + 1.$$

Since $2r\varepsilon < \mu_3 = r - \alpha + s$ and $P > \varepsilon^{-1}$,

$$r(1 - \varepsilon)P > (\alpha - s)P + 1,$$

so the $j = r$ multiple is above $I$. This proves (2).

Define

$$A^* := hA + \{0, v, 2v, \dots, (r - 1)v\}.$$

Then $|A^*| \le r|A| = rm$. For each $i$ and $0 \le j \le r - 1$,

$$ha_i + jp_i = h(a_i + jd_i) + jv \in A^*,$$

because $a_i + jd_i \in A$. Therefore

$$I \cap \bigcup_{i=1}^{u} p_i\mathbb{Z} \subseteq w + A^*,$$

and hence

$$F_L(p_1, \dots, p_u) \le \#\left(I \cap \bigcup_{i=1}^{u} p_i\mathbb{Z}\right) \le |A^*| \le rK_r(u).$$

Since $L = \lfloor \alpha P \rfloor$ and $P = p_u$, this proves $G_\alpha(u) \le rK_r(u)$. □

*Remark* 4. Green and Ruzsa proved the integer-length transfer $F'_k(N) \le G_k(N) \le kF'_k(N)$, where $F'_k(N)$ is the smallest size of a set containing $k$-term arithmetic progressions with $N$ distinct common differences [3, Proposition 4.1]. Theorem 3 is the same mechanism with the endpoint bookkeeping needed for arbitrary real $\alpha$.

# 4 Sum-difference exponents and $D_r$

For $j \in \mathbb{Q} \cup \{\infty\}$ write

$$\pi_j(x,y) = x + jy \quad (j \in \mathbb{Q}), \qquad \pi_\infty(x,y) = y.$$

For a finite set $J \subset \mathbb{Z}$ and $\beta \geq 1$, consider the inequality

$$|\pi_\infty(E)| \leq C_{J,\beta} \left( \max_{j \in J} |\pi_j(E)| \right)^\beta \qquad (E \subset \mathbb{Z}^2 \text{ finite}). \tag{3}$$

This is a finite-slope sum-difference estimate in cardinal form. Tao's notation $\mathrm{SD}(R; s)$ is the corresponding exponent for rational slopes, and it is projectively invariant [6].

**Proposition 5** (Equivalence with common differences)**.** *Let $r \geq 2$ and let $J \subset \{0, 1, \dots, r-1\}$. If (3) holds for $J$ with exponent $\beta$, then*

$$D_r(m) \ll_{r,J,\beta} m^\beta, \qquad K_r(u) \gg_{r,J,\beta} u^{1/\beta}.$$

*Conversely, if $D_r(m) \ll m^\beta$, then (3) holds for $J = \{0, 1, \dots, r-1\}$ with exponent $\beta$ up to changing the implicit constant by a factor depending on $r$.*

*Proof.* Suppose first that (3) holds and let $A \subset \mathbb{Z}$, $|A| = m$. For each $d \in \Delta_r(A)$ choose one $a_d$ such that $a_d, a_d + d, \dots, a_d + (r-1)d \in A$, and put

$$E := \{(a_d, d) : d \in \Delta_r(A)\}.$$

Then $\pi_\infty(E) = \Delta_r(A)$, while $\pi_j(E) \subseteq A$ for every $j \in \{0, 1, \dots, r-1\}$ and hence for every $j \in J$. Therefore $|\Delta_r(A)| \ll m^\beta$. Taking the maximum over $A$ gives $D_r(m) \ll m^\beta$, and inversion gives $K_r(u) \gg u^{1/\beta}$.

Conversely, let $E \subset \mathbb{Z}^2$ be finite and put

$$A := \bigcup_{j=0}^{r-1} \pi_j(E).$$

Then $|A| \leq r \max_j |\pi_j(E)|$. For every $d \in \pi_\infty(E)$, choose $(a, d) \in E$. Then $a, a+d, \dots, a+(r-1)d \in A$, so $d \in \Delta_r(A)$. Thus

$$|\pi_\infty(E)| \leq D_r(|A|) \ll_r \left( \max_{0 \leq j < r} |\pi_j(E)| \right)^\beta.$$

□

The useful general rule is therefore:

$$\boxed{\text{sum-difference bound with exponent } \beta \implies F_{\lfloor \alpha p_u \rfloor} \gg u^{1/\beta}}$$

whenever the slopes in the inequality can be embedded among the positions $0, 1, \dots, \lfloor \alpha \rfloor - 1$ of an arithmetic progression.

# 5 Concrete unconditional consequences

## 5.1 The range $\alpha \geq 3$

Katz and Tao proved the partial sums-differences estimate

$$|A \overset{G}{-} B| \ll \max(|A|, |B|, |A \overset{G}{+} B|)^{11/6}$$

for finite subsets $A, B$ of a torsion-free abelian group and $G \subseteq A \times B$ [5]. Equivalently, in the notation above,

$$\mathrm{SD}(\{0, 1, \infty\}; -1) \leq 11/6.$$

After the projective change $t \mapsto 2/(t+1)$, this becomes a bound with output slope $\infty$ and input slopes $\{0, 1, 2\}$. Proposition 5 therefore gives

$$D_3(m) \ll m^{11/6}, \qquad K_3(u) \gg u^{6/11}.$$

Since every $r$-term progression with $r \geq 3$ contains a 3-term progression with the same common difference,

$$\Delta_r(A) \subseteq \Delta_3(A), \qquad D_r(m) \leq D_3(m), \qquad K_r(u) \geq K_3(u) \quad (r \geq 3).$$

Combining this with Theorem 1 gives the following.

**Corollary 6.** *For every fixed $\alpha \geq 3$ and every increasing $u$-tuple of primes $p_1 < \cdots < p_u$,*

$$F_{\lfloor \alpha p_u \rfloor}(p_1, \ldots, p_u) \gg u^{6/11}.$$

*The implied constant is absolute.*

## 5.2 The range $\alpha \geq 7$

A stronger finite-slope estimate is also known:

$$\mathrm{SD}(\{0, 1, 2, \infty\}; -1) \leq 2 - \frac{1}{4} = \frac{7}{4}.$$

Tao records this four-slope bound, with the upper bound due to Katz and Tao [6]. Applying the projective change $t \mapsto 6/(t+1)$ sends the output slope $-1$ to $\infty$ and sends the input slopes $\{0, 1, 2, \infty\}$ to

$$\{6, 3, 2, 0\} = \{0, 2, 3, 6\}.$$

Thus Proposition 5 gives

$$D_7(m) \ll m^{7/4}, \qquad K_7(u) \gg u^{4/7}.$$

Since $K_r(u) \geq K_7(u)$ for $r \geq 7$, we obtain:

**Corollary 7.** *For every fixed $\alpha \geq 7$ and every increasing $u$-tuple of primes $p_1 < \cdots < p_u$,*

$$F_{\lfloor \alpha p_u \rfloor}(p_1, \ldots, p_u) \gg u^{4/7}.$$

*The implied constant is absolute.*

### 5.3 Large $\alpha$ and the current global sum-difference exponent

The arithmetic Kakeya conjecture predicts that the best possible exponent $\beta_r$ in $D_r(m) \ll m^{\beta_r+o(1)}$ satisfies

$$\beta_r \to 1 \qquad (r \to \infty).$$

Green and Ruzsa prove that this formulation is equivalent to several standard forms of arithmetic Kakeya, including the Erdős–Selfridge prime-multiple form [3].

Unconditionally, the current best global finite-slope sum-difference exponent recorded by Tao is

$$\inf_R \mathrm{SD}(R; -1) \le 1.67513\ldots.$$

Because SD is projectively invariant, any finite rational-slope inequality with output slope $-1$ can be transformed to one with output slope $\infty$; after clearing denominators and translating, its input slopes lie in $\{0, 1, \ldots, r_0 - 1\}$ for some finite $r_0$. Proposition 5 then gives:

**Corollary 8.** *For every $\varepsilon > 0$ there exists an integer $r_0(\varepsilon)$ such that, whenever $\lfloor \alpha \rfloor \ge r_0(\varepsilon)$ and $p_1 < \cdots < p_u$ are primes,*

$$F_{\lfloor \alpha p_u \rfloor}(p_1, \ldots, p_u) \gg_{\alpha,\varepsilon} u^{1/(1.67513\ldots+\varepsilon)}.$$

*Numerically,* $1/1.67513\ldots = 0.59697\ldots$.

## 6 Worst-case upper constructions for large $\alpha$

The transfer theorem also imports known arithmetic-Kakeya constructions into Erdős Problem #1143. Green and Ruzsa prove that, in the notation of their Conjecture 1,

$$\lim_{N\to\infty} \frac{\log F_k(N)}{\log N} \le 1 - \frac{c}{\log\log k}$$

for an absolute constant $c > 0$ [3, Theorem 1.2]. Since $K_k(N) \le F_k(N)$, Theorem 3 gives, for $r = \lfloor \alpha \rfloor$ sufficiently large,

$$G_\alpha(u) \le r K_r(u) \le r\, u^{1-c/\log\log r + o(1)}.$$

Thus the worst-case exponent in Problem #1143 cannot be 1 for fixed large $r$ by present constructions; the arithmetic Kakeya conjecture predicts that it nevertheless tends to 1 as $r \to \infty$.

## 7 The small interval range $2 < \alpha < 3$

When $2 < \alpha < 3$, the transfer theorem uses $r = 2$. Here

$$D_2(m) = \binom{m}{2}, \qquad K_2(u) = \left\lceil \frac{1 + \sqrt{1 + 8u}}{2} \right\rceil.$$

Indeed, a set of $m$ integers has at most $\binom{m}{2}$ positive pairwise differences, and the set

$$\{1, 2, 4, \ldots, 2^{m-1}\}$$

has all positive pairwise differences distinct. Therefore Theorem 3 gives $G_\alpha(u) \asymp \sqrt{u}$ in this range, but not the sharp constant.

There is a simple pointwise improvement of the lower constant. If $L = \lfloor \alpha p_u \rfloor$ with $\alpha > 2$, then every odd prime $p_i$ has at least two multiples in every interval of length $L$. Choose one pair of such multiples for each odd $p_i$ and draw the corresponding edge on the set of covered integers. A triangle with all edge lengths odd primes is impossible: if its vertices are $x < y < z$, then the three edge lengths are $y - x$, $z - y$ and $z - x = (y - x) + (z - y)$, and the sum of two odd primes is even. Hence, excluding possibly the single prime 2, Mantel's theorem gives

$$u - 1 \leq \frac{m^2}{4},$$

where $m$ is the number of covered integers. Thus

$$F_{\lfloor \alpha p_u \rfloor}(p_1, \dots, p_u) \geq 2\sqrt{u - 1} \qquad (\alpha > 2).$$

This recovers the familiar square-root barrier in the first range; the new content of the arithmetic-Kakeya transfer begins when $\alpha \geq 3$.

## 8 Summary of what is established

Let $r = \lfloor \alpha \rfloor \geq 2$.

- For every prescribed prime tuple,

  $$F_{\lfloor \alpha p_u \rfloor}(p_1, \dots, p_u) \geq K_r(u).$$

  This is pointwise.

- In the worst case over prime tuples,

  $$K_r(u) \leq G_\alpha(u) \leq r K_r(u).$$

  Thus the fixed-$\alpha$ worst-case problem is exactly the inverse arithmetic-Kakeya problem for $r$-term progressions, up to the constant factor $r$.

- For $\alpha \geq 3$,

  $$F_{\lfloor \alpha p_u \rfloor}(p_1, \dots, p_u) \gg u^{6/11}.$$

- For $\alpha \geq 7$,

  $$F_{\lfloor \alpha p_u \rfloor}(p_1, \dots, p_u) \gg u^{4/7}.$$

- For every $\varepsilon > 0$, the current global finite-slope sum-difference exponent gives, for all sufficiently large $\alpha$ depending on $\varepsilon$,

  $$F_{\lfloor \alpha p_u \rfloor}(p_1, \dots, p_u) \gg_{\alpha, \varepsilon} u^{1/(1.67513\ldots + \varepsilon)}.$$

The remaining research problem is therefore clean: improve upper bounds for $D_r(m)$, or equivalently improve finite-slope sum-difference exponents whose slopes can be embedded among $0, 1, \dots, r - 1$. Every such improvement immediately improves Erdős Problem #1143 for all $\alpha$ with $\lfloor \alpha \rfloor \geq r$.

## References

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