Chebyshev–Lagrange limit sets and the variants of an Erdős problem

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Abstract

We resolve the scalar fixed-point versions of the Erdős problem on cluster sets of Lagrange interpolation at Chebyshev roots, and we separate the other natural readings of the ambiguous statement. Fix a point $x_0 \in [-1, 1]$ and let $C_{x_0}(f)$ be the set of finite cluster values of $L_n f(x_0)$, where $L_n f$ interpolates f at the n roots of T_n. We prove that every non-empty closed set $A \subset [-1, 1]$ is equal to $C_{x_0}(f)$ for some $f \in C[-1, 1]$, for every fixed x_0. If $A = \varnothing$, then such an f exists exactly at the exceptional points $x_0 = \cos(\pi\alpha)$ with α rational of odd denominator; the positive direction is Erdős's corrected strong-divergence theorem, and the negative direction follows from forced subsequences. We also show that the domain-set and "same cluster set for every x" readings are not the historical Erdős problem and are false in their natural universal forms.

1 The problem and the main classification

We use the Chebyshev roots

$$x_{k,n} = \cos\theta_{k,n}, \qquad \theta_{k,n} = \frac{(2k-1)\pi}{2n}, \qquad 1 \le k \le n.$$

For $f \in C[-1, 1]$, let $L_n f$ be the polynomial of degree at most $n - 1$ satisfying

$$L_n f(x_{k,n}) = f(x_{k,n}), \qquad 1 \le k \le n.$$

For a fixed $x_0 \in [-1, 1]$, define

$$\mathcal{T}_n f := L_n f(x_0), \qquad C_{x_0}(f) := \mathrm{Cl}_{\mathbb{R}}\big((\mathcal{T}_n f)_{n\ge1}\big),$$

where $\mathrm{Cl}_{\mathbb{R}}$ denotes the set of finite real cluster values. Thus $C_{x_0}(f) = \varnothing$ means $|\mathcal{T}_n f| \to \infty$.

Write $x_0 = \cos\theta_0$, $0 \le \theta_0 \le \pi$, and $\alpha_0 = \theta_0/\pi$. We say that x_0 is *odd rational* if α_0 is rational and, in lowest terms, has odd denominator. This includes the endpoints $x_0 = \pm1$, corresponding to denominators 1.

The fixed-point scalar problem is completely classified as follows.

Theorem 1.1 (Fixed-point classification). *Let $x_0 \in [-1, 1]$ and let $A \subset [-1, 1]$ be closed.*

(a) *If $A \ne \varnothing$, then there exists $f \in C[-1, 1]$ such that*

$$C_{x_0}(f) = A.$$

(b) *If $A = \varnothing$, then such an f exists if and only if x_0 is odd rational.*

The proof of part (a) is self-contained and is the main construction of the paper. The positive direction of part (b) uses Erdős's corrected strong-divergence theorem: if $x_0 = \cos(\pi p/q)$ with p, q odd, then there exists $f \in C[-1,1]$ such that $|L_n f(x_0)| \to \infty$ [2, 3]. This is exactly the assertion that the finite cluster set is empty. If the numerator in a lowest-term odd-denominator representation is even, or if $x_0 = 1$, the result follows by the symmetry $x \mapsto -x$: the Chebyshev-root sets are invariant under sign change, and hence $L_n(g \circ (-\operatorname{id}))(x) = L_n g(-x)$. The negative direction of part (b) is proved in Section 6.

The rest of the paper also settles the other common readings of the original ambiguous sentence. Section 7 shows that the domain-set reading is false for every proper closed subset of $[-1,1]$, and that the requirement $C_x(f) = A$ for every x is already false for $A = \{-1,1\}$.

2 Angle variables and elementary estimates

For $\varphi \in C[0,\pi]$ write $\varphi(\theta) = f(\cos\theta)$ and

$$F_n\varphi := L_n f(x_0).$$

If x_0 is not a row-n node, then

$$F_n\varphi = \sum_{k=1}^{n} \lambda_{k,n}\varphi(\theta_{k,n}), \qquad \lambda_{k,n} = \frac{\cos(n\theta_0)}{n}\,\frac{(-1)^{k-1}\sin\theta_{k,n}}{\cos\theta_0 - \cos\theta_{k,n}}. \tag{1}$$

This follows from

$$\ell_{k,n}(t) = \frac{T_n(t)}{T_n'(x_{k,n})(t - x_{k,n})}, \qquad T_n'(x) = nU_{n-1}(x),$$

and

$$U_{n-1}(\cos\theta_{k,n}) = \frac{\sin(n\theta_{k,n})}{\sin\theta_{k,n}} = \frac{(-1)^{k-1}}{\sin\theta_{k,n}}.$$

If x_0 is a row-n node, then $F_n\varphi = \varphi(\theta_0)$.

Lemma 2.1 (Off-point decay). *If $\varphi \in C[0,\pi]$ vanishes on a neighbourhood of θ_0, then*

$$F_n\varphi \longrightarrow 0.$$

Proof. For rows in which θ_0 is a node, $F_n\varphi = \varphi(\theta_0) = 0$. For all other rows, put

$$H(\theta) = \frac{\varphi(\theta)\sin\theta}{\cos\theta_0 - \cos\theta}.$$

Because φ is zero near θ_0, H is continuous on $[0,\pi]$, including the possible endpoint cases. Formula (1) gives

$$F_n\varphi = \frac{\cos(n\theta_0)}{n}\sum_{k=1}^{n}(-1)^{k-1}H(\theta_{k,n}).$$

Pairing consecutive terms yields

$$\left|\sum_{k=1}^{n}(-1)^{k-1}H(\theta_{k,n})\right| \le \frac{n}{2}\omega_H(\pi/n) + \|H\|_\infty\,,$$

where ω_H is the modulus of continuity of H. Dividing by n gives the claim. □

We shall need rows along which the leading factor $\cos(n\theta_0)$ is not small.

Lemma 2.2 (Good dyadic rows)**.** *For every $\theta_0 \in [0, \pi]$ and every odd positive integer R, there are arbitrarily large powers of two $n = 2^r$ such that*

$$|\cos(n\theta_0)| \geq \kappa(\theta_0) > 0 \tag{2}$$

and

$$\cos(ns\theta_0) \neq 0 \qquad (1 \leq s \leq R,\ s \text{ odd}). \tag{3}$$

Proof. Let $\alpha = \theta_0/\pi$. If α is irrational, then (3) is automatic. Also, for $y \in \mathbb{R}/\mathbb{Z}$, if $|\cos \pi y| < 1/2$, then $y \in (1/3, 2/3)$ modulo 1, and therefore $2y \in (0, 1/3) \cup (2/3, 1)$ modulo 1, so $|\cos(2\pi y)| > 1/2$. Hence infinitely many r satisfy $|\cos(2^r\theta_0)| \geq 1/2$.

If $\alpha = a/b$ in lowest terms, write $b = 2^h b_0$ with b_0 odd. For every $r \geq h$ and every odd s, the rational number $2^r sa/b$ has odd denominator after reduction, and so cannot be congruent to $1/2$ modulo an integer. Thus (3) holds for all $r \geq h$. The values $\cos(2^r\theta_0)$, $r \geq h$, belong to a finite set not containing 0, so (2) holds along all sufficiently large powers of two. □

3 Primitive rows and the algebraic spike

A node

$$\theta = \frac{(2k-1)\pi}{2d}$$

is called *primitive of order d* if $(2k-1, d) = 1$. Let P_d be the set of primitive nodes of order d. Whenever x_0 is not a row-d node, define

$$\nu_d := \sum_{\theta \in P_d} \lambda_d(\theta)\delta_\theta, \qquad V_d := \|\nu_d\| = \sum_{\theta \in P_d} |\lambda_d(\theta)|,$$

where $\lambda_d(\theta)$ is the row-d weight in (1).

A primitive node of order d occurs in row n if and only if $n = ds$ with s odd. In that case, provided neither row has x_0 as a node,

$$\lambda_n(\theta) = \gamma(n, d)\lambda_d(\theta), \qquad \gamma(n, d) = \frac{d}{n}(-1)^{(n/d-1)/2}\frac{\cos(n\theta_0)}{\cos(d\theta_0)}. \tag{4}$$

Indeed, if $n = ds$ and s is odd, then the sign changes by $(-1)^{(s-1)/2}$ and all other factors in (1) are common.

Lemma 3.1 (Primitive mass)**.** *Fix $\theta_0 \in [0, \pi]$ and an odd integer R. There is $c_R > 0$ such that, whenever n is a power of two and $s \leq R$ is odd with $\cos(ns\theta_0) \neq 0$,*

$$V_{ns} \geq c_R|\cos(ns\theta_0)|\log n \tag{5}$$

for all sufficiently large n.

Proof. Put $d = ns$. Since n is a power of two and s is odd, a row-d node with odd numerator $m = 2k-1$ is primitive exactly when $(m, s) = 1$.

First assume $0 < \theta_0 < \pi$. Choose a small closed interval $J \subset (0, \pi)$ centered at θ_0. For $\theta \in J$,

$$\frac{\sin\theta}{|\cos\theta_0 - \cos\theta|} \geq \frac{c}{|\theta - \theta_0|}.$$

Thus, for primitive nodes of order d in J,

$$|\lambda_d(\theta)| \geq \frac{c|\cos(d\theta_0)|}{d|\theta - \theta_0|}.$$

Writing $\theta = m\pi/(2d)$, the denominator is comparable to $|m - 2d\theta_0/\pi|/d$. Among odd integers, the condition $(m, s) = 1$ has gaps bounded by a constant depending only on R. Therefore the sum over primitive m with $|m - 2d\theta_0/\pi| \le c_J d$ dominates

$$c_R|\cos(d\theta_0)| \sum_{r \le cd} \frac{1}{r} \ge c'_R|\cos(d\theta_0)|\log n.$$

If $\theta_0 = 0$, then

$$|\lambda_d(\theta)| = \frac{|\cos(d\theta_0)|}{d} \frac{\sin\theta}{1-\cos\theta} \asymp \frac{|\cos(d\theta_0)|}{d\theta} \asymp \frac{|\cos(d\theta_0)|}{m}$$

for nodes $\theta = m\pi/(2d)$ near 0. Summing over primitive odd $m \le cd$ gives the same logarithmic lower bound. The case $\theta_0 = \pi$ is identical after replacing m by $2d - m$. □

For odd s let

$$\chi(s) = (-1)^{(s-1)/2}.$$

This is the non-principal real character modulo 4 on the odd integers.

Lemma 3.2 (Finite algebraic spike)**.** *Fix θ_0 and an odd integer R. Let n be a sufficiently large power of two satisfying Lemma 2.2 for this R. For odd $s \le R$ define*

$$a_s := \frac{\mu(s)\chi(s)\cos(ns\theta_0)}{s\cos(n\theta_0)}, \tag{6}$$

where μ is the number-theoretic Mobius function. On primitive nodes of orders ns, $s \le R$ odd, set

$$v(\theta) = \frac{a_s}{V_{ns}} \operatorname{sgn} \lambda_{ns}(\theta), \qquad \theta \in P_{ns}, \tag{7}$$

and set $v = 0$ at all other nodes belonging to rows $1, \dots, Rn$.

Then, for every $1 \le j \le Rn$,

$$F_j v = \begin{cases} 1, & j = n, \\ 0, & j \ne n, \end{cases} \tag{8}$$

where $F_j v$ means evaluation of the row-j interpolation functional on the finite node values prescribed by v. Moreover

$$\|v\|_\infty \le \frac{C_R}{\log n}. \tag{9}$$

Proof. If row j has x_0 as a node, then $F_j v = v(\theta_0) = 0$, because θ_0 is not in any of the primitive sets P_{ns} with $s \le R$ by the choice of n. Hence assume row j is not a node row.

If j is not an odd multiple of n, no primitive row ns, s odd, can occur in row j, so $F_j v = 0$. Let $j = nu$ with $u \le R$ odd. Then, by (4), (6), and (7),

$$F_{nu} v = \sum_{s|u} \gamma(nu, ns) a_s.$$

For each divisor $s \mid u$,

$$\begin{aligned} \gamma(nu, ns)a_s &= \frac{s}{u}\chi(u/s)\frac{\cos(nu\theta_0)}{\cos(ns\theta_0)} \frac{\mu(s)\chi(s)\cos(ns\theta_0)}{s\cos(n\theta_0)} \\ &= \frac{\chi(u)\cos(nu\theta_0)}{u\cos(n\theta_0)}\mu(s). \end{aligned}$$

Therefore

$$F_{nu} v = \frac{\chi(u)\cos(nu\theta_0)}{u\cos(n\theta_0)} \sum_{s|u} \mu(s),$$

which is 1 if $u = 1$ and 0 if $u > 1$.

For the norm estimate, Lemma 3.1 gives $V_{ns} \ge c_R|\cos(ns\theta_0)|\log n$, while (2) gives $|a_s| \le C|\cos(ns\theta_0)|/s$. Hence $|a_s|/V_{ns} \le C_R/\log n$. □

4 Continuous block spikes

Lemma 4.1 (Block spike)**.** *For each fixed $\theta_0 \in [0, \pi]$ there is a sequence $\eta_R \downarrow 0$, indexed by odd R, with the following property. Let R be odd and let n be a sufficiently large good dyadic row for R. For every $\delta > 0$ there is $\psi = \psi_{n,R,\delta} \in C[0, \pi]$ such that*

(i) *ψ vanishes on a neighbourhood of θ_0;*

(ii) *$F_n\psi = 1$;*

(iii) *$F_j\psi = 0$ for $1 \le j \le Rn$, $j \ne n$;*

(iv) *$|F_j\psi| \le \eta_R + \delta$ for every $j > Rn$;*

(v) *$\|\psi\|_\infty \le C_R/\log n$.*

Proof. Let v be the finite spike from Lemma 3.2, and let P be the finite set where $v \ne 0$. Since P is disjoint from θ_0, put $\rho = \operatorname{dist}(P, \theta_0) > 0$.

First consider exact atoms from P in future rows. If j is a node row for x_0, then a function vanishing near θ_0 contributes zero. If j is not a node row and is not an odd multiple of n, no point of P occurs in row j. If $j = nu$ with $u > R$ odd, the exact-atom contribution is

$$\sum_{\substack{s|u \\ s\le R}} \gamma(nu, ns)a_s.$$

For the terms that actually occur, $s \le R$, Lemma 2.2 gives $\cos(ns\theta_0) \ne 0$, so the calculation in Lemma 3.2 gives

$$\sum_{\substack{s|u \\ s\le R}} \gamma(nu, ns)a_s = \frac{\chi(u)\cos(nu\theta_0)}{u\cos(n\theta_0)} \sum_{\substack{s|u \\ s\le R}} \mu(s).$$

Since $u > 1$, $\sum_{s|u} \mu(s) = 0$, and therefore the last divisor sum is $-\sum_{s|u,\, s>R} \mu(s)$. This avoids any division by $\cos(ns\theta_0)$ for the omitted divisors $s > R$. Because $|\cos(n\theta_0)|$ is bounded below along the chosen dyadic rows,

$$\left|\sum_{\substack{s|u \\ s\le R}} \gamma(nu, ns)a_s\right| \le C\frac{\tau(u)}{u} \le \frac{C}{\sqrt{R}}, \tag{10}$$

where $\tau(u) \le 2\sqrt{u}$. Set $\eta_R = C/\sqrt{R}$.

Choose an integer $K > Rn$ so large that

$$\|v\|_\infty \frac{|P|}{K\rho} < \delta/4.$$

Then choose pairwise disjoint intervals around the points of P, none meeting θ_0, none containing a point of $E_K \setminus P$ where

$$E_K = \{\theta_{k,j} : 1 \le j \le K,\ 1 \le k \le j\},$$

and with total length $|I|$ so small that

$$\operatorname{dist}\Big(\theta_0, \bigcup I\Big) \ge \rho/2, \qquad C\,\|v\|_\infty \frac{|I|}{\rho} < \delta/4.$$

Define a tent function on these intervals taking the prescribed value $v(\theta)$ at each $\theta \in P$ and vanishing at all endpoints; set it equal to zero outside the intervals. Lemma 3.2 gives (i), (ii), (iii), and (v).

For $Rn < j \le K$, the supports contain no row-j nodes except exact atoms from P, so (10) gives the bound. For $j > K$, split the row-j nodes in the supports into exact atoms and all other nodes. The exact atoms are again bounded by (10). For non-exact nodes, if row j is a node row then the weights away from θ_0 are zero; otherwise (1) and the separation from θ_0 give an individual bound $C/(j\rho)$. A row-j grid has at most $C(j|I| + |P|)$ points in the support intervals. Therefore the non-exact part is at most

$$C \|v\|_\infty \left(\frac{|I|}{\rho} + \frac{|P|}{j\rho} \right) < \delta.$$

Together with (10), this proves (iv). □

5 Every non-empty closed set at every fixed point

We now prove Theorem 1.1(a). Fix $x_0 = \cos\theta_0$ and a non-empty closed set $A \subset [-1, 1]$. Choose $c \in A$, and choose a sequence $(a_m)_{m\ge1}$ in A whose finite cluster set is exactly A, for example by concatenating finite $1/r$-nets of A.

Let $B > 1$ be such that $|a - c| \le B$ for every $a \in A$. Choose odd integers $R_m \to \infty$ so fast that

$$\sum_{m=1}^{\infty} \eta_{R_m} < \frac{1}{16},$$

and then choose positive δ_m so that, with

$$\varepsilon_m = \eta_{R_m} + \delta_m,$$

one has

$$\sum_{m=1}^{\infty} \varepsilon_m < \frac{1}{8}. \tag{11}$$

Using Lemma 2.2, choose recursively good dyadic rows n_m such that

$$n_m > R_i n_i \qquad (1 \le i < m) \tag{12}$$

and such that the block spike

$$\psi_m := \psi_{n_m, R_m, \delta_m}$$

satisfies

$$\|\psi_m\|_\infty \le \frac{2^{-m}}{4B}. \tag{13}$$

This is possible because Lemma 4.1 gives $\|\psi_m\|_\infty \le C_{R_m}/\log n_m$.

Define coefficients recursively by

$$b_m := a_m - c - \sum_{i<m} b_i F_{n_m} \psi_i. \tag{14}$$

For $i < m$, (12) gives $n_m > R_i n_i$, so $|F_{n_m}\psi_i| \le \varepsilon_i$. Induction using (11) gives

$$|b_m| \le 2B \qquad (m \ge 1). \tag{15}$$

Indeed, if the bound holds for $i < m$, then

$$|b_m| \le B + 2B \sum_{i<m} \varepsilon_i < B + B/4 < 2B.$$

Set

$$\varphi(\theta) = c + \sum_{m=1}^{\infty} b_m \psi_m(\theta). \tag{16}$$

The series converges uniformly by (13) and (15). Since every ψ_m vanishes near θ_0, in particular $\varphi(\theta_0) = c$. Define

$$f(x) = \varphi(\arccos x), \qquad -1 \le x \le 1.$$

Then $f \in C[-1,1]$.

At selected rows, all future spikes vanish exactly: if $i > m$, then $n_m < R_i n_i$ and $n_m \ne n_i$, so Lemma 4.1(iii) gives $F_{n_m}\psi_i = 0$. Also $F_{n_m}\psi_m = 1$. Therefore (14) gives

$$F_{n_m}\varphi = a_m.$$

Thus every point of A is a cluster value.

Conversely, let $n \to \infty$ through non-selected rows. Fix M. For $m > M$, if $n \le R_m n_m$, then $F_n\psi_m = 0$ because $n \ne n_m$; if $n > R_m n_m$, then $|F_n\psi_m| \le \varepsilon_m$. Hence

$$\left| \sum_{m>M} b_m F_n \psi_m \right| \le 2B \sum_{m>M} \varepsilon_m.$$

For the finitely many $m \le M$, Lemma 2.1 gives $F_n\psi_m \to 0$. Letting first $n \to \infty$ through non-selected rows and then $M \to \infty$, we get

$$F_n\varphi \to c$$

along all non-selected rows. Therefore

$$C_{x_0}(f) = \{c\} \cup \mathrm{Cl}_{\mathbb{R}}(a_m) = A.$$

This proves Theorem 1.1(a).

6 Forced subsequences and the empty set

It remains to prove the negative direction of Theorem 1.1(b). We need two standard forced-subsequence mechanisms.

The norm of the point-evaluation interpolation functional is the pointwise Lebesgue function

$$\Lambda_n(x_0) := \|f \mapsto L_n f(x_0)\| = \sum_{k=1}^{n} |\ell_{k,n}(x_0)|,$$

with the convention $\Lambda_n(x_0) = 1$ when x_0 is a row-n node.

Lemma 6.1 (Bounded Lebesgue subsequences force $f(x_0)$)**.** *If* $\sup_j \Lambda_{n_j}(x_0) < \infty$*, then, for every* $f \in C[-1,1]$*,*

$$L_{n_j} f(x_0) \to f(x_0).$$

Proof. Let $B = \sup_j \Lambda_{n_j}(x_0)$. Choose a polynomial p with $\|f - p\|_\infty < \varepsilon/(B+1)$. For all sufficiently large j, $L_{n_j} p = p$. Hence

$$|L_{n_j} f(x_0) - f(x_0)| \le |L_{n_j}(f-p)(x_0)| + |p(x_0) - f(x_0)| \le (B+1)\,\|f-p\|_\infty < \varepsilon.$$

□

Lemma 6.2 (Irrational angles have bounded subsequences). *If $0 < \theta_0 < \pi$ and θ_0/π is irrational, then there is a subsequence (n_j) with*

$$\sup_j \Lambda_{n_j}(x_0) < \infty.$$

Consequently $f(x_0) \in C_{x_0}(f)$ for every $f \in C[-1,1]$.

Proof. By the classical inhomogeneous approximation theorem of Minkowski, applied to $\alpha = \theta_0/\pi$ and $\beta = 1/2$ [1], there are infinitely many n such that

$$\|n\alpha - 1/2\| \le C/n.$$

For these n, $|\cos(n\theta_0)| \le C'/n$. Let $\theta_{k,n}$ be a nearest row-n node to θ_0. Its distance from θ_0 is $O(1/n^2)$ along the chosen subsequence, and the nearest-node term in (1) is $O(1)$. All other terms are bounded by

$$\frac{|\cos(n\theta_0)|}{n} \sum_{k \ne k_0} \frac{C}{|\theta_0 - \theta_{k,n}|} \le \frac{C}{n^2} \cdot n \log n = O(\log n/n).$$

Thus $\Lambda_n(x_0) = O(1)$ on this subsequence. Lemma 6.1 completes the proof. □

Lemma 6.3 (Even denominators are node subsequences). *If $\theta_0/\pi = a/b$ in lowest terms with b even, then x_0 is a Chebyshev node for infinitely many rows. Hence $f(x_0) \in C_{x_0}(f)$ for every $f \in C[-1,1]$.*

Proof. Write $b = 2s$. Then a is odd. For every odd positive integer m, put $n = sm$. Then

$$\frac{a}{b} = \frac{am}{2n},$$

and am is odd, so x_0 is one of the row-n Chebyshev roots. Along these rows, $L_n f(x_0) = f(x_0)$. □

If x_0 is not odd rational, then either θ_0/π is irrational, or it is rational with even denominator. Lemmas 6.2 and 6.3 show that $f(x_0)$ is a finite cluster value for every continuous f. Thus $C_{x_0}(f)$ can never be empty. This proves the negative direction of Theorem 1.1(b). The positive direction is Erdős's theorem, as explained in Section 1.

7 Other readings of the ambiguous problem

The scalar fixed-point classification above is the meaningful way to read Erdős's claimed prescribed-limit-set assertion. Two other readings are natural but false as universal statements.

7.1 The domain-set reading

One might read the original sentence as asking for a continuous f such that a prescribed closed set $A \subset [-1,1]$ is the set of points x at which the scalar sequence $L_n f(x)$ has a finite cluster value. This cannot produce a proper closed set.

Theorem 7.1 (Domain-set reading). *For every $f \in C[-1,1]$, the set*

$$D(f) := \{x \in [-1,1] : C_x(f) \ne \varnothing\}$$

contains the dense set consisting of irrational angles and rational even-denominator angles. Consequently, if $D(f)$ is closed, then $D(f) = [-1,1]$.

Proof. The inclusion follows from Lemmas 6.2 and 6.3. The stated set is dense in $[-1,1]$, so any closed set containing it is the whole interval. □

Thus the domain-set version of the problem is false for every proper closed subset of $[-1,1]$.

7.2 The same cluster set at every point

Another possible reading is that a single f should satisfy $C_x(f) = A$ for every $x \in [-1, 1]$. This is also not the Erdős statement, and it is false in its natural universal form.

Proposition 7.2. *There is no $f \in C[-1, 1]$ such that*

$$C_x(f) = \{-1, 1\} \qquad \textit{for every } x \in [-1, 1].$$

Proof. For every x in the dense forced set of Theorem 7.1, the value $f(x)$ belongs to $C_x(f)$. If $C_x(f) = \{-1, 1\}$ for all x, then $f(x) \in \{-1, 1\}$ on a dense set. By continuity, $f([-1, 1]) \subset \{-1, 1\}$. Since $[-1, 1]$ is connected, f is constant. Then all interpolants are the same constant, so every cluster set is a singleton, not $\{-1, 1\}$. □

7.3 Function-space limit points

A final possible reading is that $(L_n f)_n$ is considered as a sequence in a function space, such as $C[-1, 1]$, and that A is its set of limit points. This is not type-compatible with the quoted formulation, where $A \subset [-1, 1]$ is a scalar set. After any additional identification, for example identifying real numbers with constant functions, one obtains a different problem not stated by Erdős. The scalar fixed-point theorem above is the version matching Erdős's later comments.

8 Summary

For Chebyshev roots, the fixed scalar problem has the following complete answer.

closed set $A \subset [-1, 1]$	fixed point x_0
$A \neq \varnothing$	realizable at every $x_0 \in [-1, 1]$
$A = \varnothing$	realizable exactly when $x_0 = \cos(\pi\alpha)$, $\alpha \in \mathbb{Q}$ has odd denominator

The arbitrary fixed-point version with empty closed sets is therefore false away from the odd-rational points. The historically meaningful exceptional version is true in the scalar fixed-point sense: at odd-rational points, every closed $A \subset [-1, 1]$, including $\varnothing$, occurs as $C_{x_0}(f)$ for some continuous f.

References

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