Monochromatic union-closed and lattice subfamilies of the Boolean lattice

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Abstract

Let $\mathcal{B}_n = 2^{[n]}$. Write $F(n)$ for the largest integer with the property that every two-colouring of $\mathcal{B}_n$ contains a monochromatic union-closed family of size at least $F(n)$, and write $f(n)$ for the analogous parameter when one requires closure under both union and intersection.

We prove

$$\left\lceil \frac{n+1}{2} \right\rceil \leq F(n) \leq n^{\log_2 n + O(\log\log n)}$$

and

$$\left\lceil \frac{n+1}{2} \right\rceil \leq f(n) \leq (3+o(1))n\log_2 n.$$

In particular,

$$F(n) = \exp\bigl(O((\log n)^2)\bigr) = (1+o(1))^n,$$

so the subexponential part of the Erdős–Ulam problem has an affirmative answer.

The bound for $F(n)$ comes from identifying the free rank of a union-closed family with the VC-dimension of a complementary set system. The bound for $f(n)$ comes from a quotient-coordinate representation of lattice families, a width-versus-cube lemma for distributive lattices, and a count of bounded-width posets.

1 Introduction

Write $[n] = \{1, \ldots, n\}$ and $\mathcal{B}_n = 2^{[n]}$. We consider the following two Ramsey-type parameters.

Definition 1.1. Let $f(n)$ be the largest integer such that every two-colouring of $\mathcal{B}_n$ contains a monochromatic family $\mathcal{L} \subseteq \mathcal{B}_n$ with $|\mathcal{L}| \geq f(n)$ and

$$A, B \in \mathcal{L} \Longrightarrow A \cup B \in \mathcal{L} \text{ and } A \cap B \in \mathcal{L}.$$

Let $F(n)$ be defined similarly, but requiring only union-closure.

These quantities appear in a problem of Erdős and Ulam recorded by Erdős in 1978 [7], see also [3]. The question for $F(n)$ asks, in particular, whether every two-colouring forces a monochromatic union-closed family that is subexponential in 2^n but still superpolynomial in n.

The problem sits between two established themes. On the extremal side, recent work of Bouchard bounds the size of a union-closed family in terms of its chain length [4]. On the Ramsey side, Boolean Ramsey theory studies monochromatic copies of fixed posets or Boolean algebras inside larger Boolean lattices; see, for example, Axenovich and Walzer [1] and Johnston, Lu, and Milans [8]. Our setting is different: the ambient dimension is n, while the monochromatic closed family is itself allowed to vary with n.

Our main results are the following.

Theorem 1.2. *For every n,*

$$\left\lceil \frac{n+1}{2} \right\rceil \le F(n) \le n^{\log_2 n + O(\log\log n)}.$$

In particular,

$$F(n) = \exp\big(O((\log n)^2)\big) = (1+o(1))^n.$$

Theorem 1.3. *For every n,*

$$\left\lceil \frac{n+1}{2} \right\rceil \le f(n) \le (3+o(1))n\log_2 n.$$

Theorem 1.2 answers affirmatively the subexponential half of the Erdős–Ulam question. Theorem 1.3 places the lattice parameter $f(n)$ within a logarithmic factor of the trivial chain lower bound.

The proofs are probabilistic, but the structural inputs are quite different. For $F(n)$ the key observation is that the free rank of a union-closed family is exactly the VC-dimension of a complementary set system. For $f(n)$ we use the fact that every family closed under union and intersection is a finite distributive lattice, hence isomorphic to an ideal lattice $J(P)$ by Birkhoff's representation theorem [2]. A large antichain in P yields a Boolean cube inside $J(P)$, and after ruling out large cubes by random colouring one is left with counting bounded-width posets.

We are not aware of prior asymptotic estimates for these two specific parameters.

2 The chain lower bound

Proposition 2.1. *For every n,*

$$f(n), F(n) \ge \left\lceil \frac{n+1}{2} \right\rceil.$$

Proof. Consider the chain

$$\emptyset \subset \{1\} \subset \{1,2\} \subset \cdots \subset [n].$$

It has $n+1$ members, so one colour appears on at least $\lceil (n+1)/2 \rceil$ of them. Any subfamily of a chain is closed under union and intersection, because for any two members the union is the larger one and the intersection is the smaller one. ☐

3 Union-closed families: free rank and VC-dimension

Definition 3.1. A tuple $A_1, \ldots, A_d \subseteq [n]$ is *free* if all nonempty unions

$$\bigcup_{i\in I} A_i, \qquad \emptyset \ne I \subseteq [d],$$

are distinct. For a union-closed family $\mathcal{U} \subseteq \mathcal{B}_n$, let $r(\mathcal{U})$ be the largest d such that $\mathcal{U}$ contains a free d-tuple.

Lemma 3.2. *Let $\mathcal{U} \subseteq \mathcal{B}_n$ be union-closed, and define*

$$\widehat{\mathcal{U}} := \mathcal{U} \cup \{\emptyset\}, \qquad \mathcal{C}(\mathcal{U}) := \{[n] \setminus A : A \in \widehat{\mathcal{U}}\}.$$

Then

$$r(\mathcal{U}) = \mathrm{VCdim}(\mathcal{C}(\mathcal{U})).$$

Proof. Suppose first that $A_1, \dots, A_d \in \mathcal{U}$ are free. Then for each i we have

$$A_i \not\subseteq \bigcup_{j \neq i} A_j,$$

so we may choose a private point

$$x_i \in A_i \setminus \bigcup_{j \neq i} A_j.$$

Put $X = \{x_1, \dots, x_d\}$. For any $S \subseteq [d]$, let

$$C_S := [n] \setminus \bigcup_{i \notin S} A_i \in \mathcal{C}(\mathcal{U}).$$

Because the x_i are private,

$$x_i \in C_S \iff i \in S.$$

Thus $\mathcal{C}(\mathcal{U})$ shatters X, and so $\mathrm{VCdim}(\mathcal{C}(\mathcal{U})) \geq d$.

Conversely, suppose $\mathcal{C}(\mathcal{U})$ shatters $X = \{x_1, \dots, x_d\}$. For each i, choose $D_i \in \mathcal{C}(\mathcal{U})$ such that

$$D_i \cap X = X \setminus \{x_i\}.$$

Then

$$B_i := [n] \setminus D_i \in \widehat{\mathcal{U}}$$

satisfies $B_i \cap X = \{x_i\}$. In particular $B_i \neq \emptyset$, so $B_i \in \mathcal{U}$. Since $\mathcal{U}$ is union-closed, every union of the B_i lies in $\mathcal{U}$. Distinct index sets give distinct traces on X, hence distinct unions. Therefore $B_1, \dots, B_d$ are free, and $r(\mathcal{U}) \geq d$. ☐

Corollary 3.3. *For every union-closed family $\mathcal{U} \subseteq \mathcal{B}_n$,*

$$|\mathcal{U}| \leq |\widehat{\mathcal{U}}| = |\mathcal{C}(\mathcal{U})| \leq \sum_{i=0}^{r(\mathcal{U})} \binom{n}{i}.$$

Proof. By Lemma 3.2, the family $\mathcal{C}(\mathcal{U})$ has VC-dimension $r(\mathcal{U})$. The claim follows from the Sauer–Shelah lemma. ☐

Theorem 3.4. *There is a two-colouring of $\mathcal{B}_n$ in which every monochromatic union-closed family has size at most*

$$\sum_{i=0}^{d-1} \binom{n}{i}, \qquad d := \lceil \log_2 n + \log_2 \log_2 n + 5 \rceil .$$

Consequently,

$$F(n) \leq \sum_{i=0}^{d-1} \binom{n}{i} = n^{\log_2 n + O(\log \log n)}.$$

Proof. Fix d. There are at most 2^{nd} ordered d-tuples $(A_1, \dots, A_d)$ of subsets of $[n]$. If such a tuple is free, then the $2^d - 1$ nonempty unions are all distinct, and the probability that they are monochromatic in a uniformly random two-colouring is

$$2^{1-(2^d-1)} = 2^{2-2^d}.$$

Therefore the expected number of monochromatic free d-tuples is at most

$$2^{nd+2-2^d}.$$

With the stated choice of d we have $2^d \geq 32n \log_2 n$, so this expectation is less than 1. Hence some colouring has no monochromatic free d-tuple.

Fix such a colouring, and let $\mathcal{U}$ be a monochromatic union-closed family. Then $r(\mathcal{U}) \leq d-1$, and so Corollary 3.3 gives

$$|\mathcal{U}| \leq \sum_{i=0}^{d-1} \binom{n}{i}.$$

Finally, since $d = O(\log n)$,

$$\sum_{i=0}^{d-1} \binom{n}{i} \leq \left(\frac{en}{d-1}\right)^{d-1} = n^{\log_2 n + O(\log\log n)}.$$

□

Remark 3.5. A recent theorem of Bouchard shows that if a union-closed family on $[n]$ has longest chain of size $\ell+1$, then its size is at most $\sum_{i=0}^{\ell} \binom{n}{i}$, with equality for the top $\ell+1$ layers [4]. Since every two-colouring of $\mathcal{B}_n$ already forces a monochromatic chain of length roughly $n/2$, chain information alone still permits exponentially large union-closed families. The proof of Theorem 3.4 therefore needs a different obstruction, namely free rank.

4 Lattice families

4.1 A concrete representation

A family closed under union and intersection is a finite distributive lattice. For our purposes it is useful to write such a family directly as an ideal lattice on quotient coordinates.

Lemma 4.1. *Let $\mathcal{L} \subseteq \mathcal{B}_n$ be closed under union and intersection. Then there exist:*

- *a base set $B \subseteq [n]$,*
- *an integer $t \leq n$,*
- *a finite poset P on a t-element set V,*
- *and a map $\phi : [n] \setminus B \to V \cup \{\star\}$ such that $\phi^{-1}(v) \neq \emptyset$ for each $v \in V$,*

with the property that

$$\mathcal{L} = \{B \cup \phi^{-1}(I) : I \in J(P)\}.$$

Here $\phi^{-1}(I) := \{x \in [n] \setminus B : \phi(x) \in I\}$, and the fibre $\phi^{-1}(\star)$ is an optional unused block.

Proof. Let

$$B := \bigcap_{X \in \mathcal{L}} X$$

and put

$$\mathcal{L}' := \{X \setminus B : X \in \mathcal{L}\}.$$

Then $\mathcal{L}'$ is closed under union and intersection and contains $\emptyset$.

Define an equivalence relation on $[n] \setminus B$ by

$$i \sim j \iff (\forall X \in \mathcal{L}')\,(i \in X \Leftrightarrow j \in X).$$

Let V be the set of equivalence classes that occur in at least one member of $\mathcal{L}'$. Classes that never occur will be absorbed into the unused block. Every $X \in \mathcal{L}'$ is a union of classes in V.

Let $\psi(X) \subseteq V$ denote the set of classes contained in X, and let

$$\mathcal{I} := \{\psi(X) : X \in \mathcal{L}'\} \subseteq 2^V.$$

Then $\mathcal{I}$ is closed under union and intersection and contains $\emptyset$.

Define a relation on V by

$$C \preceq D \iff (\forall I \in \mathcal{I})\,(D \in I \Rightarrow C \in I).$$

This is a partial order: reflexivity and transitivity are immediate, and antisymmetry follows from the fact that distinct classes of V have different membership patterns across $\mathcal{L}'$.

Every $I \in \mathcal{I}$ is an ideal of $(V, \preceq)$, because if $D \in I$ and $C \preceq D$, then the defining implication gives $C \in I$.

Conversely, let $J \subseteq V$ be an ideal. For each $D \in V$, define

$$M_D := \bigcap\{I \in \mathcal{I} : D \in I\} \in \mathcal{I}.$$

By definition,

$$C \in M_D \iff C \preceq D,$$

so $M_D = \downarrow D$. Since $\mathcal{I}$ is union-closed and every ideal is the union of the principal ideals of its elements, we obtain

$$J = \bigcup_{D \in J} \downarrow D \in \mathcal{I}.$$

Thus $\mathcal{I} = J(P)$ for the poset $P = (V, \preceq)$.

Finally, if $\phi : [n] \setminus B \to V \cup \{\star\}$ sends each coordinate to its equivalence class in V and sends every never-used coordinate to $\star$, then the members of $\mathcal{L}'$ are exactly the sets $\phi^{-1}(I)$ with $I \in J(P)$. Adding back B gives the claim. □

4.2 Width forces Boolean cubes

Lemma 4.2. *Let P be a finite poset of width w. Then $J(P)$ contains an embedded Boolean cube of dimension w.*

Proof. Choose an antichain $A = \{a_1, \ldots, a_w\} \subseteq P$ and put

$$K := \downarrow A \setminus A.$$

For each $S \subseteq A$, the set $K \cup S$ is an ideal of P: every element below a point of S already lies in K, and K itself is an ideal. Distinct subsets S give distinct ideals, and

$$(K \cup S) \cup (K \cup T) = K \cup (S \cup T), \qquad (K \cup S) \cap (K \cup T) = K \cup (S \cap T).$$

Hence

$$\{K \cup S : S \subseteq A\} \cong B_w.$$

□

Lemma 4.3. *The number of embedded Boolean d-cubes in $\mathcal{B}_n$ is at most $(d+2)^n$.*

Proof. Any embedded Boolean d-cube has the form

$$\left\{ K \cup \bigcup_{i \in I} B_i : I \subseteq [d] \right\},$$

where $K, B_1, \ldots, B_d$ are pairwise disjoint and each B_i is nonempty. Each element of $[n]$ chooses one of the d variable blocks, the base block K, or an unused block, so there are at most $(d+2)^n$ possibilities. □

Lemma 4.4. *Let d be an integer such that*

$$2^d > n \log_2(d+2) + 5.$$

Then there exists a two-colouring of $\mathcal{B}_n$ with no monochromatic embedded Boolean d-cube.

Proof. By Lemma 4.3, the expected number of monochromatic d-cubes in a uniformly random two-colouring is at most

$$(d+2)^n \, 2^{1-2^d} \le 2^{-4}.$$

So some colouring contains none. □

4.3 Counting bounded-width posets

Lemma 4.5. *Let $C = (x_1 < \cdots < x_r)$ and $D = (y_1 < \cdots < y_s)$ be two chains. The number of partial orders on $C \sqcup D$ extending these chain orders is at most*

$$\binom{r+s}{r}^2 \le 2^{2(r+s)}.$$

Proof. For each $p \in [r]$, define

$$a_p := \max\{q : y_q < x_p\}, \qquad b_p := \max\{q : y_q \not> x_p\},$$

with the convention that the maximum of the empty set is 0. Then

$$0 \le a_p \le b_p \le s,$$

and both (a_p) and (b_p) are nondecreasing in p. Moreover these two sequences determine all cross-relations:

$$y_q < x_p \iff q \le a_p, \qquad x_p < y_q \iff q > b_p,$$

and otherwise x_p and y_q are incomparable.

The number of nondecreasing r-tuples in $\{0, 1, \ldots, s\}$ is $\binom{r+s}{r}$, so the total number of possible relation patterns is at most $\binom{r+s}{r}^2$. □

Lemma 4.6. *Fix integers $t \ge 0$ and $d \ge 2$. The number of unlabeled posets on t vertices of width less than d is at most*

$$\binom{t+d-2}{d-2} 2^{2(d-2)t}.$$

Proof. By Dilworth's theorem, every such poset admits a partition into $d-1$ chains; we allow some of them to be empty and regard the chains as ordered. Choosing the chain lengths is therefore equivalent to choosing a composition

$$\ell_1 + \cdots + \ell_{d-1} = t,$$

which can be done in $\binom{t+d-2}{d-2}$ ways.

Once the lengths are fixed, the relation between chain i and chain j can be chosen in at most

$$\binom{\ell_i + \ell_j}{\ell_i}^2 \le 2^{2(\ell_i+\ell_j)}$$

ways by Lemma 4.5. Multiplying over all pairs $i < j$ gives at most

$$2^{2\sum_{i<j}(\ell_i+\ell_j)}.$$

Each ℓ_i appears in exactly $d-2$ pairs, so

$$\sum_{i<j}(\ell_i + \ell_j) = (d-2)t.$$

This proves the claim. □

Theorem 4.7. *There exists a two-colouring of $\mathcal{B}_n$ in which every monochromatic family closed under union and intersection has size at most*

$$(3 + o(1))n \log_2 n.$$

Consequently,

$$f(n) \le (3 + o(1))n \log_2 n.$$

Proof. Let d be the least integer such that

$$2^d > n \log_2(d + 2) + 5.$$

Then

$$d = \log_2 n + \log_2 \log_2 \log_2 n + O(1), \qquad d = (1 + o(1)) \log_2 n.$$

By Lemma 4.4, there is a two-colouring of $\mathcal{B}_n$ with no monochromatic embedded Boolean d-cube.

We now count all lattice families coming from posets of width less than d. For each $t \le n$, the number of choices for the unlabeled poset P is at most

$$\binom{t+d-2}{d-2} 2^{2(d-2)t}$$

by Lemma 4.6. Once P is fixed, Lemma 4.1 shows that a realization of $J(P)$ inside $\mathcal{B}_n$ is determined by a choice, for each element of $[n]$, of one of the t active vertices, the base block, or the unused block. Thus the number of realizations on $[n]$ is at most $(t+2)^n$. Hence the total number N_d of lattice families in $\mathcal{B}_n$ arising from posets of width less than d satisfies

$$N_d \le \sum_{t=0}^{n} (t+2)^n \binom{t+d-2}{d-2} 2^{2(d-2)t}.$$

Therefore

$$N_d \le n(n+2)^n \binom{n+d-2}{d-2} 2^{2(d-2)n}.$$

Taking base-2 logarithms and using $d = O(\log n)$ gives

$$\log_2 N_d \le n \log_2(n+2) + 2(d-2)n + O((\log n)^2) = (3+o(1))n \log_2 n.$$

Set

$$M := \lceil \log_2 N_d \rceil + 5.$$

In a uniformly random two-colouring, let X be the number of monochromatic embedded Boolean d-cubes, and let Y be the number of monochromatic lattice families of width less than d and size greater than M.

By construction,

$$\mathbb{E}X \le (d+2)^n 2^{1-2^d} \le 2^{-4}.$$

Also, a fixed family of size m is monochromatic with probability 2^{1-m}, so

$$\mathbb{E}Y \le N_d \, 2^{1-M} \le 2^{-4}.$$

Hence $\mathbb{E}(X+Y) < 1$, and therefore some colouring satisfies $X = Y = 0$.

Fix such a colouring, and let $\mathcal{L}$ be a monochromatic family closed under union and intersection. By Lemma 4.1, we may write

$$\mathcal{L} = \{B \cup \phi^{-1}(I) : I \in J(P)\}$$

for some finite poset P on at most n vertices. If P had width at least d, then by Lemma 4.2 the family $J(P)$, and hence $\mathcal{L}$, would contain an embedded Boolean d-cube, contradicting $X = 0$. Thus P has width less than d. Since $Y = 0$, no monochromatic width-$< d$ lattice family has size greater than M. Therefore $|\mathcal{L}| \le M$, and so

$$f(n) \le M = (3+o(1))n \log_2 n.$$

□

Combining Proposition 2.1 and theorem 4.7 yields Theorem 1.3.

5 Remarks and open questions

Remark 5.1. The proof of Theorem 4.7 counts all bounded-width posets very crudely. For fixed width k, Brightwell and Goodall proved that the number of labeled width-k posets on $[n]$ is within a polynomial factor of $n!4^{n(k-1)}$ [5]. This suggests that the counting step in our proof is already on roughly the right exponential scale when the width is fixed, although their theorem is not uniform in k and does not by itself improve the bound above.

Question 5.2. Determine the correct order of growth of $f(n)$.

At present we know only that

$$\frac{n+1}{2} \le f(n) \le (3+o(1))n \log_2 n.$$

Thus the main remaining issue in the lattice case is whether the logarithmic gap can be closed, or whether $n \log n$ is in fact the right scale.

Question 5.3. Determine the correct order of growth of $F(n)$.

The present bounds are

$$\frac{n+1}{2} \le F(n) \le n^{\log_2 n + O(\log\log n)}.$$

Thus the subexponential question is settled, but it remains open whether $F(n)$ is polynomial, quasipolynomial, or something in between.

Question 5.4. Can one classify large union-closed families with bounded free rank?

This seems to be the structural bottleneck for improving Theorem 3.4. Bouchard's chain-length theorem shows that long chains alone do not force small size, while Theorem 3.4 shows that large free rank can be suppressed by a random colouring. What is missing is a useful description of families that have long chains but bounded free rank.

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