An unconditional spread-theoretic solution of Erdős Problem #1190

Draft note

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Abstract

Let

$$\epsilon_m = \sup \sum_i \frac{1}{n_i},$$

where the supremum is over all finite families of distinct moduli $m < n_1 < \cdots < n_k$ for which one can choose residue classes $a_i \pmod{n_i}$ such that no integer lies in two of the classes. We prove

$$\epsilon_m = L(m)^{-1+o(1)}, \qquad L(x) := \exp\{\sqrt{\log x \log\log x}\}.$$

The proof first establishes the sharp upper bound

$$f(x) \le xL(x)^{-1+o(1)}$$

for Erdős Problem #202, where $f(x)$ is the largest size of an admissible family of moduli at most x. The lower bound is the construction of de la Bretèche–Ford–Vandehey. The new ingredient is that the dense-core conjecture for intersecting set systems used conditionally by Bretèche–Ford–Vandehey is replaced by the now-standard spread lemma from sunflower/threshold theory. Thus the argument is unconditional, modulo only quoted established theorems.

1 Definitions and statement

Throughout,

$$L(\alpha, x) := \exp\{\alpha\sqrt{\log x \log_2 x}\}, \qquad \log_2 x := \log\log x,$$

and $L(x) := L(1, x)$. Put

$$X := \log x, \qquad Y := \log_2 x, \qquad M := \sqrt{X/Y}, \qquad U := \sqrt{XY}.$$

Thus $L(\alpha, x) = e^{\alpha U}$ and $MY = U$.

For an integer n, write

$$\omega(n) := \#\{p : p \mid n\}, \qquad \ker(n) := \prod_{p \mid n} p,$$

and, as in [3],

$$h(n) := \prod_{p^\nu \| n} \nu.$$

For a set C of primes and an integer n, denote the exact part of n supported on C by

$$n_C := \prod_{p \in C} p^{v_p(n)}.$$

Call a finite set $Q \subset \mathbb{N}$ *admissible* if there exist residues $a_q \pmod{q}$, $q \in Q$, such that the residue classes are pairwise disjoint. Equivalently, for all distinct $q, q' \in Q$,

$$a_q \not\equiv a_{q'} \pmod{(q, q')}.$$

Let

$$f(x) := \max\{|Q| : Q \subset [1, x] \cap \mathbb{N} \text{ is admissible}\}.$$

For Problem #1190 we use

$$\epsilon_m := \sup \left\{ \sum_{q \in Q} \frac{1}{q} : Q \subset \{m+1, m+2, \dots\} \text{ is finite and admissible} \right\}.$$

If the supremum in the original formulation is attained, this is the same quantity; the proof below gives matching upper and lower bounds for finite families, so the distinction is immaterial for the asymptotic.

Theorem 1. *As $x, m \to \infty$,*

$$f(x) = xL(x)^{-1+o(1)} \qquad \text{and} \qquad \epsilon_m = L(m)^{-1+o(1)}.$$

The lower bound $f(x) \geq xL(x)^{-1+o(1)}$ is exactly the lower-bound construction of de la Bretèche, Ford and Vandehey [3]. The rest of the note proves the matching upper bound and then transfers it to ϵ_m.

2 The spread core lemma

If $\mathcal{F}$ is a family of sets and T is a set, put

$$\mathcal{F}_T := \{F \in \mathcal{F} : T \subseteq F\}.$$

A k-uniform family $\mathcal{F}$ is called κ-spread if

$$|\mathcal{F}_T| \leq |\mathcal{F}| \kappa^{-|T|} \qquad (T \neq \varnothing).$$

We use the following standard consequence of the spread lemma. In this form it follows, for example, from the high-probability spread lemma in modern proofs of the sunflower theorem, itself a consequence of the Park–Pham proof of the Kahn–Kalai conjecture; see [4, 7] and the spread formulation in [5, §2]. Earlier ALWZ–Rao–BCW spread machinery also suffices for the $r = 2$ case used here [1, 6, 2].

Theorem 2 (Spread families contain disjoint members)**.** *There is an absolute constant C_{sp} such that, for all $r \geq 2$ and $k \geq 2$, every k-uniform κ-spread family with*

$$\kappa \geq C_{\mathrm{sp}} r \log(ek)$$

contains r pairwise disjoint members.

For completeness we recall the usual derivation from the random-set spread lemma. The high-probability spread lemma says that if $W \sim \mathrm{Bin}(\Omega, p)$ and $\mathcal{F}$ is k-uniform and κ-spread with $\kappa \geq Cp^{-1}\log(ek/\eta)$, then

$$\mathbb{P}[\exists F \in \mathcal{F} : F \subset W] \geq 1 - \eta.$$

Take a random partition of the ground set Ω into $2r$ cells, each cell having marginal distribution $\mathrm{Bin}(\Omega, 1/(2r))$, and take $\eta = 1/2$. The expected number of cells containing a member of $\mathcal{F}$ is at least r, so some partition has at least r successful cells; the corresponding members of $\mathcal{F}$ are pairwise disjoint.

Corollary 3 (Dense core for an intersecting uniform family)**.** *There is an absolute constant A with the following property. Let $\mathcal{A}$ be an intersecting family of distinct k-element sets, where $1 \leq k \leq K$. Put*

$$\Lambda := A\log(eK).$$

Then there is a nonempty set C such that

$$|\mathcal{A}_C| > |\mathcal{A}|\Lambda^{-|C|}.$$

Proof. For $k = 1$ the family has only one member, so the claim is immediate. Assume $k \geq 2$ and choose $A \geq 2C_{\mathrm{sp}}$. If the conclusion failed, then for every nonempty C we would have $|\mathcal{A}_C| \leq |\mathcal{A}|\Lambda^{-|C|}$, so $\mathcal{A}$ would be Λ-spread. Since $\Lambda \geq C_{\mathrm{sp}} \cdot 2\log(ek)$, Theorem 2 with $r = 2$ gives two disjoint members of $\mathcal{A}$, contradicting that $\mathcal{A}$ is intersecting. $\square$

Remark 4. *BFV formulated a dense-core conjecture with denominator $t(|C|)$ independent of the ambient uniformity. The weaker denominator $(A\log(eK))^{|C|}$ is already enough here, because in the number-theoretic application $K \ll M = \sqrt{\log x/\log_2 x}$ and the accumulated logarithmic loss is $K\log\log K = o(\sqrt{\log x \log_2 x})$.*

3 BFV inputs

We quote two estimates from [3]. The first is their Rankin-type counting lemma.

Lemma 5 (BFV counting lemma)**.** *Fix $B > 0$. Uniformly for $2 \leq y \leq x$ and $0 \leq \alpha \leq B$,*

$$\#\{n \leq y : \omega(n) \geq \alpha M\} \leq yL(-\alpha/2 + o(1), x).$$

Consequently, uniformly for $0 \leq W \leq K \leq BM$ and $y \leq x$,

$$\#\{n \leq y : \omega(n) = K - W\} \leq yL(-K/(2M) + o(1), x)(\log x)^{W/2}.$$

Proof of the consequence. The set with $\omega(n) = K - W$ is contained in the set with $\omega(n) \geq K - W$. Applying the first estimate with $\alpha = (K - W)/M$ gives

$$yL(-(K-W)/(2M) + o(1), x) = yL(-K/(2M) + o(1), x)L(W/(2M), x).$$

Since $L(W/(2M), x) = \exp\{WY/2\} = (\log x)^{W/2}$, the claimed form follows. $\square$

The second input is the pruning step from BFV's Section 4.1.

Lemma 6 (BFV pruning)**.** *Let $(Q, (a_q))$ be an extremal admissible system with $Q \subset [1, x]$ and $|Q| = f(x) = S$. Then there is a subfamily $Q' \subset Q$, with $S' := |Q'| \geq SL(o(1), x)$, and an integer $K \in [1, 3M]$ such that*

(i) *$q \in [xL(-2, x), x]$ for every $q \in Q'$;*

(ii) *$h(q) \leq e^{\sqrt{X}}$ for every $q \in Q'$;*

(iii) *$\omega(q) = K$ for every $q \in Q'$;*

(iv) *the squarefree kernels* $\ker(q)$*, $q \in Q'$, are distinct.*

This is obtained in [3] by deleting negligible classes of moduli, pigeonholing the value of $\omega(q)$, and using their bound for the number of integers with prescribed squarefree kernel and bounded $h(q)$. The already-known lower bound $f(x) \geq xL(-1 + o(1), x)$ ensures that all losses are $L(o(1), x)$ relative to S.

4 The descending chain with spread cores

Fix an extremal admissible system and choose Q' as in Lemma 6. Set $Q'_0 := Q'$ and $S'_0 := S' := |Q'|$.

We construct nested families

$$Q'_{r-1} \supset Q_r \supset Q'_r \qquad (r = 1, 2, \ldots, R),$$

pairwise coprime integers $P_1, \ldots, P_R$, and residues $m_r \pmod{P_r}$. Let

$$w_r := \omega(P_r), \qquad W_r := w_1 + \cdots + w_r, \qquad V_r := h(P_1 \cdots P_r).$$

The construction maintains:

(a) $P_1, \ldots, P_r$ are pairwise coprime and $W_r \leq K$;

(b) for $q \in Q_r$, $P_1 \cdots P_r \mid q$ and $\gcd(P_j, q/P_j) = 1$ for each $j \leq r$;

(c) for $q \in Q'_r$, $a_q \equiv m_j \pmod{P_j}$ for each $j \leq r$;

(d) after removing the primes in $P_1 \cdots P_r$, the remaining squarefree kernels are distinct as q varies in Q'_r.

Assume Q'_{r-1} has been defined and $W_{r-1} < K$. Define

$$B_r := \{q/(P_1 \cdots P_{r-1}) : q \in Q'_{r-1}\}.$$

Every $B \in B_r$ has $K - W_{r-1}$ distinct prime factors, and the supports

$$\mathcal{A}_r := \{\mathrm{supp}(B) : B \in B_r\}, \qquad \mathrm{supp}(B) := \{p : p \mid B\},$$

are distinct and uniform.

The family $\mathcal{A}_r$ is intersecting. Indeed, if $\mathrm{supp}(B) \cap \mathrm{supp}(B') = \varnothing$, then for $q = (P_1 \cdots P_{r-1})B$ and $q' = (P_1 \cdots P_{r-1})B'$ the greatest common divisor is exactly $P_1 \cdots P_{r-1}$. By the induction hypothesis the residues a_q and $a_{q'}$ agree modulo this product. The two congruences would therefore be compatible, contradicting admissibility.

Apply Corollary 3 to $\mathcal{A}_r$, using the original K as the ambient bound. With

$$\Lambda := A\log(eK),$$

there is a nonempty set C_r of primes, say $|C_r| = w_r$, such that at least $S'_{r-1}\Lambda^{-w_r}$ members $B \in B_r$ satisfy $C_r \subseteq \operatorname{supp}(B)$.

For these B, group according to the exact C_r-part

$$P = B_{C_r} = \prod_{p\in C_r} p^{v_p(B)}.$$

Since

$$\sum_{\operatorname{supp}(P)=C_r} \frac{1}{h(P)^2} = \left(\sum_{\nu\geq 1}\frac{1}{\nu^2}\right)^{w_r} = \left(\frac{\pi^2}{6}\right)^{w_r},$$

there is a choice P_r with $\operatorname{supp}(P_r) = C_r$ such that, if

$$Q_r := \{q = (P_1\cdots P_{r-1})B \in Q'_{r-1} : B_{C_r} = P_r\},$$

then

$$S_r := |Q_r| \geq \frac{S'_{r-1}}{\Gamma^{w_r} h(P_r)^2}, \qquad \Gamma := \frac{\pi^2}{6}\Lambda \ll \log(eK). \tag{4.1}$$

Among the P_r residue classes modulo P_r, choose one, call it $m_r \pmod{P_r}$, for which

$$Q'_r := \{q \in Q_r : a_q \equiv m_r \pmod{P_r}\}$$

has size

$$S'_r := |Q'_r| \geq S_r/P_r. \tag{4.2}$$

This completes the step. Because each step removes at least one previously unfixed prime from the common support, the process terminates after some $R \leq K$ steps with $W_R = K$. At that point Q'_R has a single element and that element is $P_1\cdots P_R$. Hence, by Lemma 6,

$$P_1\cdots P_R \geq xL(-2,x). \tag{4.3}$$

5 The upper bound for $f(x)$

Iterating (4.1) and (4.2) gives, for every $1 \leq r \leq R$,

$$S_r \geq \frac{S'}{\Gamma^{W_r} V_r^2 \prod_{j<r} P_j}. \tag{5.1}$$

On the other hand, every $q \in Q_r$ has the form

$$q = (P_1\cdots P_r)n, \qquad n \leq \frac{x}{P_1\cdots P_r}, \qquad \omega(n) = K - W_r.$$

Lemma 5, with $d := K/M$, gives uniformly in r,

$$S_r \le \frac{x}{P_1 \cdots P_r} L(-d/2 + o(1), x)(\log x)^{W_r/2}. \tag{5.2}$$

Comparing (5.1) and (5.2), and cancelling $\prod_{j<r} P_j$, yields

$$P_r \le \frac{x}{S'} L(-d/2 + o(1), x)(\log x)^{W_r/2} \Gamma^{W_r} V_r^2. \tag{5.3}$$

Multiplying (5.3) over $r = 1, \dots, R$ and using (4.3), we obtain

$$\begin{aligned} xL(-2, x) \le \Big(\frac{x}{S'} L(-d/2 + o(1), x)\Big)^R \exp\left\{ \frac{Y}{2} \sum_{r=1}^{R} W_r \right\} \\ \times \exp\left\{ \sum_{r=1}^{R} W_r \log \Gamma + 2 \sum_{r=1}^{R} \log V_r \right\}. \end{aligned} \tag{5.4}$$

The final exponential in (5.4) is negligible on the L-scale after taking the Rth root. Indeed, since $K \le 3M$,

$$\frac{1}{R} \sum_{r=1}^{R} W_r \log \Gamma \le K \log \Gamma \ll M \log\log(eK) = o(U), \tag{5.5}$$

and $V_r \le h(q) \le e^{\sqrt{X}}$ for every $q \in Q_r$, whence

$$\frac{2}{R} \sum_{r=1}^{R} \log V_r \le 2\sqrt{X} = o(U). \tag{5.6}$$

It remains to estimate $\sum_r W_r$. Since each $w_j \ge 1$ and $\sum_{j=1}^{R} w_j = K$,

$$\sum_{r=1}^{R} W_r = \sum_{j=1}^{R} (R - j + 1) w_j \le R(K - R + 1) + \sum_{j=1}^{R} (R - j + 1) = RK - \frac{R^2}{2} + O(R). \tag{5.7}$$

This is the configuration in which all surplus mass $K - R$ is placed at $j = 1$.

Taking logarithms in (5.4), using (5.5), (5.6), and (5.7), and recalling that $dU = KY$, gives

$$\begin{aligned} X - 2U &\le R \log(x/S') - \frac{R}{2} KY + \frac{Y}{2} \sum_{r=1}^{R} W_r + R\, o(U) \\ &\le R \log(x/S') - \frac{R^2 Y}{4} + R\, o(U), \end{aligned}$$

where the $O(RY)$ term from (5.7) becomes $R\, o(U)$ because $Y = o(U)$. Therefore

$$\log(S'/x) \le -\frac{X - 2U}{R} - \frac{RY}{4} + o(U). \tag{5.8}$$

By AM–GM,

$$\frac{X - 2U}{R} + \frac{RY}{4} \ge \sqrt{(X - 2U)Y} = U(1 + o(1)).$$

Thus (5.8) implies

$$S' \leq xL(-1+o(1),x).$$

Since $S' \geq SL(o(1),x)$, we get

$$f(x) = S \leq xL(-1+o(1),x).$$

Combined with the BFV lower bound, this proves

$$f(x) = xL(-1+o(1),x). \tag{5.9}$$

6 Deduction of Erdős Problem #1190

We now prove the reciprocal-sum estimate. Let Q be any finite admissible family with all moduli $> m$, and set

$$N(t) := \#\{q \in Q : q \leq t\}.$$

Then $N(t) \leq f(t)$ for every t. Stieltjes integration by parts gives

$$\sum_{q \in Q} \frac{1}{q} = \int_{(m,\infty)} \frac{dN(t)}{t} = \int_m^\infty \frac{N(t)}{t^2}\,dt \leq \int_m^\infty \frac{f(t)}{t^2}\,dt.$$

From (5.9), for every fixed $\eta > 0$ and all sufficiently large t,

$$f(t) \leq tL(t)^{-1+\eta}.$$

Hence, for m large enough in terms of η,

$$\sum_{q \in Q} \frac{1}{q} \leq \int_m^\infty \frac{L(t)^{-1+\eta}}{t}\,dt. \tag{6.1}$$

We use the elementary estimate

$$\int_m^\infty \frac{\exp\{-a\sqrt{\log t \log\log t}\}}{t}\,dt = L(m)^{-a+o(1)} \qquad (a > 0 \text{ fixed}). \tag{6.2}$$

Indeed, after putting $s = \log t$ and $S = \log m$, the integral is $\int_S^\infty e^{-a\sqrt{s\log s}}ds$. The lower bound follows from integrating over $[S, S+1]$. For the upper bound, split into dyadic intervals $[2^jS, 2^{j+1}S]$; the jth contribution is at most

$$2^jS\exp\{-a\sqrt{2^jS\log(2^jS)}\},$$

and the resulting series is $\exp\{-a\sqrt{S\log S} + O(\log S)\}$, which is $L(m)^{-a+o(1)}$. Applying (6.2) with $a = 1-\eta$ in (6.1) and then letting $\eta \downarrow 0$ gives

$$\epsilon_m \leq L(m)^{-1+o(1)}. \tag{6.3}$$

For the lower bound, choose

$$x := mL(m)^2.$$

Then $L(x) = L(m)^{1+o(1)}$, and the BFV lower bound gives an admissible family $Q_x \subset [1, x]$ with

$$|Q_x| \geq xL(x)^{-1+o(1)}.$$

Discard from Q_x the at most m moduli not exceeding m. Since

$$\frac{xL(x)^{-1+o(1)}}{m} = L(m)^{1+o(1)} \to \infty,$$

the remaining admissible family has $xL(x)^{-1+o(1)}$ elements, all lying in $(m, x]$. Therefore

$$\epsilon_m \geq \frac{xL(x)^{-1+o(1)}}{x} = L(x)^{-1+o(1)} = L(m)^{-1+o(1)}. \tag{6.4}$$

Combining (6.3) and (6.4) proves

$$\epsilon_m = L(m)^{-1+o(1)}.$$

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