A note on Erdős Problem #1196: primitive sets, divisibility chains, and an invariant zeta-weight

15 April 2026

Abstract

This note organizes the main mathematical ideas from the April 2026 discussion thread for Erdős Problem #1196. The thread began with a short proof based on a downward divisibility Markov chain and its adjoint sub-Markov chain; Terence Tao, Jared Duker Lichtman, Will Sawin, Kevin Barreto, and others then reformulated the argument in terms of a canonical invariant weight ν governed by $1/\zeta$, leading to a cleaner hitting-probability proof. We give a self-contained presentation of that framework, prove

$$\sum_{a\in A}\frac{1}{a\log a}\le 1+O\left(\frac{1}{\log x}\right)\qquad (A\subset[x,\infty)\text{ primitive}),$$

and record the sharper consequence

$$\sum_{a\in A}\frac{1}{a\log a}\le 1+\frac{2\gamma}{\log x}+O\left(\frac{1}{\log^2 x}\right).$$

We also summarize the conceptual content of the discussion and list potential applications suggested there.

1 The problem and the discussion

A set $A\subset\mathbb{N}\setminus\{1\}$ is *primitive* if no two distinct elements of A divide one another. Erdős proved in 1935 that

$$f(A):=\sum_{a\in A}\frac{1}{a\log a}$$

is uniformly bounded over all primitive sets [2]. Erdős Problem #1196 asks whether primitive sets far out in the integers satisfy

$$\sup_{A\subset[x,\infty)\text{ primitive}} f(A)=1+o(1)\qquad (x\to\infty).$$

Lichtman earlier proved the weaker upper bound $e^{\gamma}\pi/4$ for the limiting supremum [8]. The problem page and forum thread record the stronger bound

$$\sup_{A\subset[x,\infty)\text{ primitive}} f(A)\le 1+O\left(\frac{1}{\log x}\right)$$

as obtained in April 2026, together with several refinements and conceptual remarks [4, 3].

The discussion developed in three steps.

First, Liam Price described the original short proof as a certificate built from the downward chain $n \mapsto n/q$ with transition weight $\Lambda(q)/\log n$, where Λ is the von Mangoldt function and $\sum_{q|n} \Lambda(q) = \log n$. The adjoint chain becomes sub-Markov after truncation, and a primitive set can intersect such a random divisibility chain at most once [3].

Second, Tao stressed the conceptual significance of working directly on the integers rather than first smoothing onto the reals. In that formulation, the arithmetic identity $\sum_{q|n} \Lambda(q) = \log n$ replaces much of the Mertens-type smoothing used in previous approaches. Jared Duker Lichtman added that earlier papers already contained a probabilistic viewpoint, but the genuinely new step is the *von Mangoldt-weighted arithmetic formulation* of that viewpoint [3].

Third, Tao, Sawin, and Barreto observed that the argument becomes cleaner when one introduces a canonical positive weight ν satisfying

$$\nu(n) = \sum_{q \geq 2} \frac{\Lambda(q)}{\log(nq)} \, \nu(nq),$$

and interprets $\nu(n)$ as the hitting probability of a canonical increasing divisibility chain. The main theorem then follows from the simple fact that a primitive set is a blocking set for such a chain [3]. The rest of the note develops this viewpoint.

2 The canonical zeta-weight

Write $\Lambda(q)$ for the von Mangoldt function, so $\Lambda(q) = \log p$ if $q = p^m$ is a prime power and $\Lambda(q) = 0$ otherwise. All sums $\sum_q$ below are over integers $q \geq 2$.

Definition 2.1. For $x \geq 1$, define

$$\nu(x) := \int_1^\infty x^{-s} \left(\frac{1}{\zeta(s)} \right)' ds.$$

The integral converges absolutely: as $s \downarrow 1$, one has $\zeta(s)^{-1} \sim s - 1$, so $(\zeta^{-1})'(s) = O(1)$; as $s \to \infty$, one has $(\zeta^{-1})'(s) = O(2^{-s})$.

Proposition 2.2 (Basic identities for ν)**.** *For $x > 1$,*

$$\nu(x) = \frac{1}{x} \int_0^\infty \frac{e^{-t}}{\zeta(1 + t/\log x)} \, dt. \tag{1}$$

Moreover,

$$\nu(x) = \sum_{q \geq 2} \frac{\Lambda(q)}{\log(xq)} \, \nu(xq) \qquad (x \geq 1). \tag{2}$$

Finally, $\nu(1) = 1$.

Proof. Integrating by parts gives

$$\nu(x) = \int_1^\infty x^{-s} (\zeta^{-1})'(s) \, ds = \log x \int_1^\infty x^{-s} \zeta(s)^{-1} \, ds,$$

because $x^{-s}\zeta(s)^{-1} \to 0$ both as $s \to \infty$ and as $s \downarrow 1$. Now set $u = s - 1$ and then $t = u \log x$; this yields (1).

Next,

$$\nu(xq) = \int_1^\infty (xq)^{-s} (\zeta^{-1})'(s)\, ds = \log(xq) \int_1^\infty (xq)^{-s} \zeta(s)^{-1}\, ds.$$

Hence

$$\frac{\Lambda(q)}{\log(xq)} \nu(xq) = \Lambda(q) \int_1^\infty (xq)^{-s} \zeta(s)^{-1}\, ds.$$

All terms are nonnegative, so Tonelli gives

$$\sum_q \frac{\Lambda(q)}{\log(xq)} \nu(xq) = \int_1^\infty x^{-s} \zeta(s)^{-1} \sum_q \Lambda(q) q^{-s}\, ds.$$

For $s > 1$, the Dirichlet series identity

$$\sum_q \Lambda(q) q^{-s} = -\frac{\zeta'(s)}{\zeta(s)}$$

holds absolutely, so the right-hand side is

$$\int_1^\infty x^{-s} \left(\frac{1}{\zeta(s)} \right)' ds = \nu(x).$$

This proves (2).

Finally,

$$\nu(1) = \int_1^\infty (\zeta^{-1})'(s)\, ds = \lim_{T \to \infty} \zeta(T)^{-1} - \lim_{s \downarrow 1} \zeta(s)^{-1} = 1.$$

□

Remark 2.3. Formula (2) makes ν an exact invariant measure for the downward divisibility process. This is the exact version of the approximate invariance of $1/(n \log n)$ emphasized in the forum discussion [3].

3 The canonical upward divisibility chain

Definition 3.1. For $n \in \mathbb{N}$, define transition probabilities

$$\mathbf{P}(n \to nq) := \frac{\Lambda(q)}{\log(nq)} \frac{\nu(nq)}{\nu(n)} \qquad (q \geq 2).$$

By (2), these probabilities sum to 1. The corresponding Markov chain $(X_j)_{j \geq 0}$ with $X_0 = 1$ is the *canonical upward divisibility chain.*

Because every transition multiplies the current state by an integer $q \geq 2$, the chain is strictly increasing and every sample path is a divisibility chain

$$1 = X_0 \mid X_1 \mid X_2 \mid \cdots .$$

Proposition 3.2 (Hitting probabilities)**.** *For each $n \in \mathbb{N}$, the probability that the canonical upward chain ever visits n is exactly $\nu(n)$.*

Proof. Let $h(n) := \mathbf{P}(X_j = n \text{ for some } j \geq 0)$. We prove by strong induction on n that $h(n) = \nu(n)$. The case $n = 1$ is immediate: $h(1) = 1 = \nu(1)$.

Now fix $n > 1$ and assume the claim for all smaller positive integers. Since the chain is increasing, the only way to hit n is to arrive at n in one step from some predecessor n/q, where $q \mid n$ and $q \geq 2$. These alternatives are disjoint, so

$$h(n) = \sum_{q|n,\ q\geq 2} h(n/q)\,\mathbf{P}(n/q \to n).$$

Using the induction hypothesis and the definition of the transition probabilities,

$$h(n) = \sum_{q|n,\ q\geq 2} \nu(n/q)\frac{\Lambda(q)}{\log n}\frac{\nu(n)}{\nu(n/q)} = \frac{\nu(n)}{\log n}\sum_{q|n}\Lambda(q).$$

Now $\sum_{q|n}\Lambda(q) = \log n$, hence $h(n) = \nu(n)$. □

Corollary 3.3 (Primitive sets are bounded by ν)**.** *If $A \subset \mathbb{N} \setminus \{1\}$ is primitive, then*

$$\sum_{a\in A}\nu(a) \leq 1.$$

Proof. For each $a \in A$, let E_a be the event that the upward chain ever visits a. By Proposition 3.2, $\mathbf{P}(E_a) = \nu(a)$. Since the chain is increasing by divisibility, if both E_a and E_b occur then one of a, b must divide the other. Because A is primitive, the events $(E_a)_{a\in A}$ are pairwise disjoint. Therefore

$$\sum_{a\in A}\nu(a) = \sum_{a\in A}\mathbf{P}(E_a) = \mathbf{P}\left(\bigcup_{a\in A}E_a\right) \leq 1.$$

□

4 The Erdős–Sárközy–Szemerédi bound

4.1 An elementary two-sided bound for ν

Proposition 4.1. *For every $x > 1$,*

$$\frac{1}{x\log x} - \frac{2}{x\log^2 x} \leq \nu(x) \leq \frac{1}{x\log x}.$$

Proof. Let $L := \log x > 0$. By the integral test,

$$\frac{1}{u} \leq \zeta(1+u) \leq 1 + \frac{1}{u} \qquad (u > 0),$$

so

$$\frac{u}{1+u} \leq \frac{1}{\zeta(1+u)} \leq u.$$

Insert this into (1). For the upper bound,

$$\nu(x) \leq \frac{1}{x}\int_0^\infty e^{-t}\frac{t}{L}\,dt = \frac{1}{xL}.$$

For the lower bound,

$$\nu(x) \geq \frac{1}{x} \int_0^\infty e^{-t} \frac{t}{L+t}\, dt.$$

Since

$$\frac{t}{L+t} = \frac{t}{L} - \frac{t^2}{L(L+t)} \geq \frac{t}{L} - \frac{t^2}{L^2},$$

we obtain

$$\int_0^\infty e^{-t} \frac{t}{L+t}\, dt \geq \frac{1}{L} \int_0^\infty te^{-t}\, dt - \frac{1}{L^2} \int_0^\infty t^2 e^{-t}\, dt = \frac{1}{L} - \frac{2}{L^2},$$

which proves the claim. □

Theorem 4.2 (Main theorem)**.** *Let $A \subset [x, \infty) \cap \mathbb{N}$ be primitive. Then, as $x \to \infty$,*

$$\sum_{a \in A} \frac{1}{a \log a} \leq 1 + O\left(\frac{1}{\log x}\right).$$

More explicitly, for $x > e^2$,

$$\sum_{a \in A} \frac{1}{a \log a} \leq \frac{1}{1 - 2/\log x}.$$

Proof. By Corollary 3.3,

$$1 \geq \sum_{a \in A} \nu(a).$$

For every $a \geq x$, Proposition 4.1 gives

$$\nu(a) \geq \left(1 - \frac{2}{\log x}\right) \frac{1}{a \log a}.$$

Therefore

$$1 \geq \sum_{a \in A} \nu(a) \geq \left(1 - \frac{2}{\log x}\right) \sum_{a \in A} \frac{1}{a \log a},$$

which is equivalent to the stated bound. □

4.2 The first correction term

Proposition 4.3. *As $x \to \infty$,*

$$\nu(x) = \frac{1}{x \log x} - \frac{2\gamma}{x \log^2 x} + O\left(\frac{1}{x \log^3 x}\right).$$

Proof. Let $L = \log x$. The Laurent expansion of the zeta function at 1 gives

$$\zeta(1+u) = \frac{1}{u} + \gamma + O(u) \qquad (u \downarrow 0),$$

whence

$$\frac{1}{\zeta(1+u)} = u - \gamma u^2 + O(u^3).$$

Fix $u_0 \in (0,1)$ so that this expansion holds uniformly for $0 \le u \le u_0$. Split (1) at $t = u_0 L$:

$$\nu(x) = \frac{1}{x}\int_0^{u_0 L} \frac{e^{-t}}{\zeta(1+t/L)}\,dt + \frac{1}{x}\int_{u_0 L}^{\infty} \frac{e^{-t}}{\zeta(1+t/L)}\,dt.$$

The second integral is $O(x^{-1-u_0})$, hence negligible. In the first integral,

$$\frac{1}{\zeta(1+t/L)} = \frac{t}{L} - \gamma\frac{t^2}{L^2} + O\left(\frac{t^3}{L^3}\right) \qquad (0 \le t \le u_0 L).$$

Therefore

$$\nu(x) = \frac{1}{x}\left(\frac{1}{L}\int_0^{u_0 L} te^{-t}\,dt - \frac{\gamma}{L^2}\int_0^{u_0 L} t^2 e^{-t}\,dt + O\left(\frac{1}{L^3}\int_0^{u_0 L} t^3 e^{-t}\,dt\right)\right) + O(x^{-1-u_0}).$$

Replacing the truncated integrals by the full Gamma integrals only changes the value by exponentially small terms. Since

$$\int_0^{\infty} te^{-t}\,dt = 1, \qquad \int_0^{\infty} t^2 e^{-t}\,dt = 2, \qquad \int_0^{\infty} t^3 e^{-t}\,dt = 6,$$

the result follows. □

Corollary 4.4. *If $A \subset [x,\infty) \cap \mathbb{N}$ is primitive, then*

$$\sum_{a\in A} \frac{1}{a\log a} \le 1 + \frac{2\gamma}{\log x} + O\left(\frac{1}{\log^2 x}\right) \qquad (x \to \infty).$$

Proof. By Proposition 4.3, uniformly for $a \ge x$,

$$\nu(a) \ge \frac{1}{a\log a}\left(1 - \frac{2\gamma}{\log x} + O\left(\frac{1}{\log^2 x}\right)\right).$$

Combine this with Corollary 3.3. □

5 Further remarks and potential applications

5.1 Asymptotic extremality of k-almost-primes

Let

$$\mathcal{N}_k := \{n \in \mathbb{N} : \Omega(n) = k\},$$

which is primitive and satisfies $\mathcal{N}_k \subset [2^k, \infty)$. Theorem 4.2 yields

$$\sum_{n\in\mathcal{N}_k} \frac{1}{n\log n} \le 1 + O\left(\frac{1}{k}\right).$$

Thus the k-almost-primes are asymptotically extremal at the main-term scale. This matches Lichtman's lower bound from 2020 and the much sharper asymptotic of Gorodetsky–Lichtman–Wong,

$$\sum_{n\in\mathcal{N}_k} \frac{1}{n\log n} = 1 - (c + o(1))k^2 2^{-k},$$

for an explicit $c > 0$ [7, 6]. In the language of the forum, ν is sharp at the main-term scale but not at the secondary scale because the chain can jump by prime powers and may therefore skip a would-be perfect blocking set [3].

5.2 A function-field model

The discussion also noted a clean analogue over $\mathbb{F}_q[T]$. If f is monic of degree d, define

$$\nu_q(f) := \frac{1}{q^d(d+1)}.$$

If Λ_q denotes the function-field von Mangoldt function, then

$$\nu_q(f) = \sum_{h \text{ monic}} \frac{\Lambda_q(h)}{\deg(fh)} \nu_q(fh).$$

Indeed, the sum of $\Lambda_q(h)$ over monic h of degree e is exactly q^e, so

$$\sum_h \frac{\Lambda_q(h)}{\deg(fh)} \nu_q(fh) = \frac{1}{q^d} \sum_{e \geq 1} \frac{1}{(d+e)(d+e+1)} = \frac{1}{q^d(d+1)} = \nu_q(f).$$

This gives a useful toy model for the number-field theory and aligns with Sawin's suggestion that function fields may be the right laboratory for understanding the invariant measure more explicitly [3]. For broader background on primitive sets in function fields, see [5].

5.3 Prospective applications to the anatomy of integers

Tao suggested that the canonical upward/downward divisibility process may provide a unifying probabilistic framework for classical results in the anatomy of integers, including Billingsley's theorem, the Erdős–Kac theorem, and Dickman-type laws for smooth numbers [3]. The heuristic is compelling: under the downward chain, a large integer typically moves from n to something of size n^U with U approximately uniform on $[0, 1]$, while the weight $d(\log \log n)$ is approximately invariant.

At the same time, Tao also stressed a serious limitation: the weight $\nu(n) \asymp 1/(n \log n)$ is naturally tied to logarithmic or doubly logarithmic averages, whereas the standard statements of Billingsley and Erdős–Kac concern the uniform distribution on $\{1, \ldots, N\}$. To bridge that gap one likely needs to augment the chain with extra state information, such as the location of $\log \log n$, and then prove a local limit theorem or a comparable conditioning statement [3]. Thus the framework already offers a convincing conceptual picture, but converting it into short proofs of the classical natural-density results remains prospective. For a modern direct proof of Billingsley's theorem from a different perspective, see Arratia–Kochman [1].

5.4 Methodological significance

The thread makes a broader methodological point. Earlier work on primitive sets often passed through continuous approximations and Mertens-type products. The 2026 discussion shows that an integer-space Markov-chain picture, once paired with the exact identity $\sum_{q|n} \Lambda(q) = \log n$, can produce a shorter and more transparent certificate. In Tao's words, the argument reveals a tighter connection between primitive sets and the probabilistic anatomy of integers than had previously been made explicit [3]. That connection is arguably the most durable mathematical takeaway from the discussion.

References

[1] R. Arratia and F. Kochman, *A Simple Direct Proof of Billingsley's Theorem*, arXiv:1401.1553, 2014.

[2] P. Erdős, *Note on sequences of integers no one of which is divisible by any other*, J. London Math. Soc. **10** (1935), 126–128.

[3] T. F. Bloom (ed.), *Erdős Problem #1196 – discussion thread*, https://www.erdosproblems.com/forum/thread/1196, accessed 15 April 2026.

[4] T. F. Bloom, *Erdős Problem #1196*, https://www.erdosproblems.com/1196, accessed 15 April 2026.

[5] A. Gómez-Colunga, C. Kavaler, N. McNew, and M. Zhu, *On the Erdős primitive set conjecture in function fields*, J. Number Theory **219** (2021), 412–444.

[6] O. Gorodetsky, J. D. Lichtman, and M. D. Wong, *On Erdős sums of almost primes*, Comptes Rendus. Mathématique **362** (2024), no. G12, 1571–1596.

[7] J. D. Lichtman, *Almost primes and the Banks–Martin conjecture*, J. Number Theory **211** (2020), 513–529.

[8] J. D. Lichtman, *A proof of the Erdős primitive set conjecture*, Forum Math. Pi **11** (2023), e18.