The large-prime-divisor route to Erdős Problem #1201

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Abstract

Let $P^+(m)$ denote the largest prime factor of m. Erdős Problem #1201 asks whether, for every $\varepsilon, \eta > 0$, there is a k for which the integers n satisfying

$$P^+(n(n+1)\cdots(n+k)) > n^{1-\varepsilon}$$

have natural density at least $1-\eta$. The known short-interval argument gives the corresponding lower-natural-density statement, but does not prove that the density exists.

This manuscript develops the large-prime-divisor method in the range $1/2 < \alpha < 1$, where each integer $n+j \asymp X$ has at most one prime factor exceeding X^α. We isolate a precise large-prime-divisor correlation estimate, denoted $\mathrm{LPD}(\alpha, h)$, and prove that it implies the full natural-density asymptotic

$$\#\{n \leq N : P^+((n)\cdots(n+h-1)) \leq n^\alpha\} \sim \rho(1/\alpha)^h N.$$

We also prove that already the two-point case of LPD is equivalent to the all-scales dyadic asymptotic for pairs of consecutive smooth numbers in the same range. Thus an unconditional completion of LPD would in particular settle the natural-density pair problem of Erdős and Pomerance in this range.

The paper therefore gives a complete conditional proof from LPD, proves the exact equivalence with a prime-CRT discrepancy estimate, and records the precise obstruction. It does not claim an unconditional proof of the discrepancy estimate; that estimate is exactly the presently missing fixed-shift, all-scales input.

1 Introduction

For $m \geq 2$, let $P^+(m)$ be the largest prime divisor of m, and put $P^+(1) = 1$. For $0 < \alpha < 1$ and $h \geq 1$, define the shifted friable block set

$$\mathcal{S}_{\alpha,h} := \left\{ n \in \mathbb{N} : P^+\left(\prod_{j=0}^{h-1} (n+j) \right) \leq n^\alpha \right\}.$$

Since the largest prime divisor of a product is the maximum of the largest prime divisors of the factors, this is equivalently

$$\mathcal{S}_{\alpha,h} = \{ n \in \mathbb{N} : P^+(n+j) \leq n^\alpha \text{ for every } 0 \leq j < h \}.$$

For Erdős Problem #1201 one takes $\alpha = 1 - \varepsilon$ and considers the complementary good set

$$\mathcal{G}_{\varepsilon,h} := \mathbb{N} \setminus \mathcal{S}_{1-\varepsilon,h} = \left\{ n \in \mathbb{N} : P^+\left(\prod_{j=0}^{h-1} (n+j) \right) > n^{1-\varepsilon} \right\}.$$

Here the strict or non-strict inequality is immaterial for density questions; it changes only boundary conventions.

The Matomäki–Radziwiłłtheorem on multiplicative functions in short intervals gives the following unconditional result.

Theorem 1.1 (Unconditional lower-density form)**.** *For every fixed* $\varepsilon > 0$*,*

$$\lim_{h\to\infty} \overline{d}(\mathcal{S}_{1-\varepsilon,h}) = 0.$$

Equivalently,

$$\lim_{h\to\infty} \underline{d}(\mathcal{G}_{\varepsilon,h}) = 1.$$

Thus, for every $\varepsilon, \eta > 0$*, there exists* k *such that*

$$\underline{d}\left\{n \in \mathbb{N} : P^+(n(n+1)\cdots(n+k)) > n^{1-\varepsilon}\right\} \ge 1-\eta.$$

A proof is included in Section 8. The issue is that lower density close to one does not imply the existence of a natural density. To prove the natural-density version, one needs some all-scales information about shifted friable blocks.

The purpose of this paper is to push the large-prime-divisor approach as far as it currently goes. We work first in the range

$$\frac{1}{2} < \alpha < 1,$$

because then any integer of size $\asymp X$ has at most one prime factor exceeding X^α. This turns the smoothness condition into a finite inclusion–exclusion over large prime divisors.

The large-prime-divisor correlation

Let X be large, put

$$y = X^\alpha, \qquad I_X = (X, 2X],$$

and for $j \ge 0$ define

$$L_{j,X}(n) := \sum_{\substack{p>y \\ p\mid n+j}} 1.$$

When $1/2 < \alpha < 1$ and $0 \le j < h$ is fixed, $L_{j,X}(n) \in \{0,1\}$ for all $n \in I_X$ and all sufficiently large X. For a finite set $J \subseteq \{0,\dots,h-1\}$ define

$$T_J(X) := \sum_{X<n\le 2X} \prod_{j\in J} L_{j,X}(n),$$

with the convention $T_\varnothing(X) = X + O(1)$. Finally set

$$\lambda_\alpha := \log(1/\alpha).$$

Since $1 < 1/\alpha < 2$, the Dickman–de Bruijn function satisfies

$$\rho(1/\alpha) = 1 - \log(1/\alpha) = 1 - \lambda_\alpha.$$

Definition 1.2 (Large-prime-divisor correlation, LPD(α, h))**.** Let $1/2 < \alpha < 1$ and $h \geq 1$ be fixed. We say that LPD(α, h) holds if, for every set $J \subseteq \{0, \dots, h-1\}$,

$$T_J(X) = \lambda_\alpha^{|J|} X + o_{\alpha,h}(X) \qquad (X \to \infty).$$

For $J = \varnothing$ this means $T_\varnothing(X) = X + o(X)$.

The first main point is that this correlation statement is exactly strong enough to provide the missing natural-density input.

Theorem 1.3 (LPD implies the shifted-friable block asymptotic)**.** *Let* $1/2 < \alpha < 1$ *and* $h \geq 1$ *be fixed. Assume* LPD(α, h)*. Then*

$$\#\left\{ n \leq N : P^+\left(\prod_{j=0}^{h-1} (n+j) \right) \leq n^\alpha \right\} = \rho(1/\alpha)^h N + o_{\alpha,h}(N).$$

In particular, the natural density of $\mathcal{S}_{\alpha,h}$ *exists and equals* $\rho(1/\alpha)^h$*.*

Consequently, if LPD$(1-\varepsilon, h)$ is available for some h with $1/2 < 1-\varepsilon < 1$, then

$$d(\mathcal{G}_{\varepsilon,h}) = 1 - \rho(1/(1-\varepsilon))^h.$$

Choosing h so large that the right-hand error term is less than η would prove the natural-density form of Erdős Problem #1201 for $0 < \varepsilon < 1/2$.

The second main point is that LPD is not a routine completion. Already its two-point case is equivalent to a central open shifted-smooth-number problem.

Theorem 1.4 (The two-point obstruction)**.** *Let* $1/2 < \alpha < 1$*, let* $i < j$ *be fixed integers, and define*

$$B_{i,j}(X) := \#\{X < n \leq 2X : P^+(n+i) \leq X^\alpha,\ P^+(n+j) \leq X^\alpha\}.$$

Then

$$T_{\{i,j\}}(X) = \lambda_\alpha^2 X + o(X)$$

if and only if

$$B_{i,j}(X) = \rho(1/\alpha)^2 X + o(X).$$

Thus, in particular, the case $i = 0, j = 1$ *of two-point* LPD *is equivalent to the all-scales dyadic asymptotic for pairs of consecutive* X^α*-smooth numbers.*

This explains why the last step cannot be supplied by the currently available logarithmic-density, almost-all-scale, or conditional pair results. Section 7 discusses this comparison.

2 Elementary reductions in the range $1/2 < \alpha < 1$

Throughout this section $1/2 < \alpha < 1$, $h \geq 1$, and $0 \leq j < h$ are fixed. All o-terms may depend on α and h.

Lemma 2.1 (Freezing the threshold). *Let*

$$B_h(X;n^\alpha) := \#\{X < n \le 2X : P^+(n+j) \le n^\alpha \text{ for every } 0 \le j < h\}$$

and

$$B_h(X;X^\alpha) := \#\{X < n \le 2X : P^+(n+j) \le X^\alpha \text{ for every } 0 \le j < h\}.$$

Then

$$B_h(X;n^\alpha) = B_h(X;X^\alpha) + o(X).$$

The same assertion remains true if X^α is replaced by $(2X)^\alpha$ or by $(X+O_h(1))^\alpha$.

Proof. The two conditions can differ only if, for some $0 \le j < h$, the integer $n+j$ has a prime divisor in an interval of the form

$$X^\alpha < p \le (2X+h)^\alpha.$$

The number of such $n \in (X,2X]$ is at most

$$\sum_{j<h} \sum_{X^\alpha < p \le (2X+h)^\alpha} \left(\frac{X}{p} + 1\right).$$

By Mertens' theorem,

$$\sum_{X^\alpha < p \le (2X+h)^\alpha} \frac{1}{p} = \log \frac{\log((2X+h)^\alpha)}{\log(X^\alpha)} + o(1) = O\left(\frac{1}{\log X}\right) + o(1) = o(1),$$

and the number of primes in the same range is $O(X^\alpha)$, which is $o(X)$. Hence the total number of exceptional n is $o_h(X)$. ☐

Lemma 2.2 (Uniqueness of a large prime factor). *For all sufficiently large X, every integer $m \le 2X+h$ has at most one prime divisor exceeding X^α.*

Proof. If m had two prime divisors exceeding X^α, counted with multiplicity, then

$$m > X^{2\alpha}.$$

Since $2\alpha > 1$, we have $X^{2\alpha} > 2X+h$ for all sufficiently large X, a contradiction. ☐

Lemma 2.3 (One-point large-prime-divisor average). *For each fixed $j \ge 0$,*

$$T_{\{j\}}(X) = \lambda_\alpha X + o(X), \qquad \lambda_\alpha = \log(1/\alpha).$$

Proof. By definition,

$$T_{\{j\}}(X) = \sum_{X<n\le 2X} \sum_{\substack{p > X^\alpha \\ p \mid n+j}} 1 = \sum_{X^\alpha < p \le 2X+h} \#\{X < n \le 2X : n \equiv -j \pmod{p}\}.$$

For each prime p the inner count is $X/p + O(1)$. Hence

$$T_{\{j\}}(X) = X \sum_{X^\alpha < p \le 2X+h} \frac{1}{p} + O(\pi(2X+h)).$$

The error term is $o(X)$, and Mertens' theorem gives

$$\sum_{X^\alpha < p \le 2X+h} \frac{1}{p} = \log\log(2X+h) - \log\log(X^\alpha) + o(1) = \log(1/\alpha) + o(1).$$

This proves the claim. ☐

3 LPD implies the natural-density asymptotic

We now prove Theorem 1.3. First we prove its dyadic form.

Proposition 3.1 (Dyadic block asymptotic from LPD)**.** *Let* $1/2 < \alpha < 1$ *and* $h \geq 1$ *be fixed. Assume* LPD(α, h)*. Then*

$$\#\{X < n \leq 2X : P^+(n+j) \leq X^\alpha \textit{ for every } 0 \leq j < h\} = \rho(1/\alpha)^h X + o(X).$$

Proof. By Lemma 2.2, for all sufficiently large X and all $n \in (X, 2X]$,

$$\mathbf{1}_{P^+(n+j) \leq X^\alpha} = 1 - L_{j,X}(n).$$

Therefore

$$\begin{aligned} &\#\{X < n \leq 2X : P^+(n+j) \leq X^\alpha \text{ for every } 0 \leq j < h\} \\ &= \sum_{X < n \leq 2X} \prod_{j=0}^{h-1} (1 - L_{j,X}(n)) \\ &= \sum_{J \subseteq \{0,\ldots,h-1\}} (-1)^{|J|} T_J(X). \end{aligned}$$

Applying LPD(α, h) gives

$$\sum_{J \subseteq \{0,\ldots,h-1\}} (-1)^{|J|} T_J(X) = X \sum_{r=0}^{h} \binom{h}{r} (-1)^r \lambda_\alpha^r + o(X).$$

The binomial sum is $(1 - \lambda_\alpha)^h$. Since $1 < 1/\alpha < 2$, the Dickman function is

$$\rho(1/\alpha) = 1 - \log(1/\alpha) = 1 - \lambda_\alpha.$$

This gives the asserted dyadic asymptotic. □

Proof of Theorem 1.3. By Proposition 3.1 and Lemma 2.1, for every large X,

$$\#\{X < n \leq 2X : P^+(n+j) \leq n^\alpha \text{ for every } 0 \leq j < h\} = c_{\alpha,h} X + o(X),$$

where

$$c_{\alpha,h} := \rho(1/\alpha)^h.$$

Let

$$A(N) := \#\{n \leq N : P^+(n+j) \leq n^\alpha \text{ for every } 0 \leq j < h\}.$$

Fix $\delta > 0$. There is X_0 such that, for all $X \geq X_0$,

$$\left| \#\{X < n \leq 2X : P^+(n+j) \leq n^\alpha \ \forall j < h\} - c_{\alpha,h} X \right| \leq \delta X.$$

Decompose $(1, N]$ into dyadic intervals

$$(N/2, N], \ (N/4, N/2], \ldots$$

until the remaining initial segment has length $< X_0$. Summing the previous estimate over all dyadic pieces of length at least X_0 gives

$$|A(N) - c_{\alpha,h}N| \le \delta N + O_{\alpha,h,\delta}(X_0).$$

Dividing by N, letting $N \to \infty$, and then letting $\delta \to 0$ proves

$$A(N) = c_{\alpha,h}N + o(N).$$

□

Corollary 3.2 (Natural-density Erdős conclusion from LPD)**.** *Let* $0 < \varepsilon < 1/2$*. Suppose that for every fixed* h *the assertion* LPD$(1-\varepsilon, h)$ *holds. Then, for every* $\eta > 0$*, there exists* k *such that the natural density*

$$d\left\{n \in \mathbb{N} : P^+(n(n+1)\cdots(n+k)) > n^{1-\varepsilon}\right\}$$

exists and is at least $1 - \eta$*.*

Proof. Take $\alpha = 1 - \varepsilon$. Since $0 < \varepsilon < 1/2$, we have $1/2 < \alpha < 1$. By Theorem 1.3,

$$d(\mathcal{S}_{\alpha,h}) = \rho(1/\alpha)^h.$$

Because $0 < \rho(1/\alpha) < 1$, choose h so that $\rho(1/\alpha)^h < \eta$, and put $k = h - 1$. The complement has natural density

$$1 - \rho(1/\alpha)^h > 1 - \eta.$$

□

4 The two-point case and consecutive smooth pairs

This section proves Theorem 1.4. It is the most important diagnostic calculation in the paper: completing even the first nontrivial case of LPD gives the dyadic natural-density pair theorem for consecutive smooth numbers.

Proof of Theorem 1.4. For large X, Lemma 2.2 gives

$$\mathbf{1}_{P^+(n+i)\le X^\alpha} = 1 - L_{i,X}(n), \qquad \mathbf{1}_{P^+(n+j)\le X^\alpha} = 1 - L_{j,X}(n)$$

for $X < n \le 2X$. Therefore

$$\begin{aligned} B_{i,j}(X) &= \sum_{X<n\le 2X} (1 - L_{i,X}(n))(1 - L_{j,X}(n)) \\ &= X - T_{\{i\}}(X) - T_{\{j\}}(X) + T_{\{i,j\}}(X) + O(1). \end{aligned}$$

By Lemma 2.3,

$$T_{\{i\}}(X) = T_{\{j\}}(X) = \lambda_\alpha X + o(X).$$

Hence

$$B_{i,j}(X) = (1 - 2\lambda_\alpha)X + T_{\{i,j\}}(X) + o(X).$$

Since $\rho(1/\alpha) = 1 - \lambda_\alpha$, the relation

$$B_{i,j}(X) = \rho(1/\alpha)^2 X + o(X)$$

is equivalent to

$$(1 - 2\lambda_\alpha)X + T_{\{i,j\}}(X) = (1 - \lambda_\alpha)^2 X + o(X),$$

which is equivalent to

$$T_{\{i,j\}}(X) = \lambda_\alpha^2 X + o(X).$$

□

Remark 4.1. The theorem uses the frozen threshold X^α. By Lemma 2.1, the equivalent formulation with thresholds $(n+i)^\alpha$ and $(n+j)^\alpha$ has the same asymptotic.

5 The exact prime-CRT discrepancy behind LPD

The large-prime-divisor correlation has a simple formal main term. The hard part is proving cancellation in a prime-indexed CRT discrepancy.

Let $J \subseteq \{0, \dots, h-1\}$ be nonempty, and write

$$\mathcal{P}_J(X) := \{(p_j)_{j \in J} : p_j > X^\alpha \text{ is prime for every } j \in J\}.$$

Only primes $p_j \leq 2X + h$ can occur in $T_J(X)$, so this range is implicit. For $\mathbf{p} = (p_j)_{j \in J}$ define

$$R_J(X; \mathbf{p}) := \#\{X < n \leq 2X : n \equiv -j \pmod{p_j} \text{ for all } j \in J\}.$$

If $p_i = p_j$ for distinct $i, j \in J$ and X is large, then $p_i > |i - j|$, so the congruences

$$n \equiv -i \pmod{p_i}, \qquad n \equiv -j \pmod{p_i}$$

are inconsistent. Therefore only tuples of distinct primes contribute. For distinct primes the Chinese remainder theorem predicts the main term

$$\frac{X}{\prod_{j \in J} p_j}.$$

Define the discrepancy

$$\Delta_J(X) := \sum_{\mathbf{p} \in \mathcal{P}_J(X)} \left(R_J(X; \mathbf{p}) - \frac{X}{\prod_{j \in J} p_j} \right),$$

where inconsistent tuples with repeated primes are included with their actual value $R_J(X; \mathbf{p}) = 0$.

Proposition 5.1 (LPD as a prime-CRT discrepancy estimate)**.** *Let* $1/2 < \alpha < 1$ *and* $J \subseteq \{0, \dots, h-1\}$ *be fixed and nonempty. Then*

$$T_J(X) = \lambda_\alpha^{|J|} X + o(X)$$

if and only if

$$\Delta_J(X) = o(X).$$

Proof. Expanding the definition of $T_J(X)$ gives

$$T_J(X) = \sum_{\mathbf{p} \in \mathcal{P}_J(X)} R_J(X; \mathbf{p}).$$

Therefore

$$T_J(X) = X \sum_{\mathbf{p} \in \mathcal{P}_J(X)} \frac{1}{\prod_{j \in J} p_j} + \Delta_J(X).$$

It remains to evaluate the reciprocal-prime sum. Since repeated-prime tuples contribute only

$$O_J\left(\sum_{p > X^\alpha} \frac{1}{p^2} \right) = o(1)$$

to the reciprocal sum, we have

$$\sum_{\mathbf{p} \in \mathcal{P}_J(X)} \frac{1}{\prod_{j \in J} p_j} = \left(\sum_{X^\alpha < p \le 2X+h} \frac{1}{p} \right)^{|J|} + o(1).$$

By Mertens' theorem this is

$$\lambda_\alpha^{|J|} + o(1).$$

Thus

$$T_J(X) = \lambda_\alpha^{|J|} X + \Delta_J(X) + o(X),$$

and the equivalence follows. $\square$

Remark 5.2 (Why the elementary CRT proof fails). For each fixed tuple $\mathbf{p}$, the trivial estimate

$$R_J(X; \mathbf{p}) = \frac{X}{\prod_{j \in J} p_j} + O(1)$$

is true. Summing this $O(1)$ over prime tuples is hopeless: for $|J| = 2$, the number of pairs of primes in the relevant range is $\asymp X^2/\log^2 X$, much larger than X. The required statement is not pointwise CRT, but cancellation of the accumulated endpoint errors over the prime tuples. In the pair case this cancellation is equivalent, by Theorem 1.4, to the dyadic consecutive-smooth-pair asymptotic.

5.1 The pair discrepancy in small-cofactor variables

For $J = \{i, j\}$ with $i < j$, the discrepancy can also be written as an averaged shifted-prime equation. If $p > X^\alpha$ divides $n + i$ and $q > X^\alpha$ divides $n + j$, then

$$n + i = ap, \qquad n + j = bq,$$

where

$$1 \le a, b \le 3X^{1-\alpha}$$

for all large X. The two equations imply

$$ap - bq = i - j. \tag{1}$$

Thus

$$T_{\{i,j\}}(X) = \sum_{a,b \leq 3X^{1-\alpha}} \# \left\{ \begin{array}{l} p, q > X^{\alpha} \text{ primes :} \\ X < ap - i \leq 2X, \\ ap - bq = i - j \end{array} \right\} + O_h(X^{1-\alpha}). \tag{1}$$

The harmless error accounts for endpoint conventions. For fixed a, b, this is a prime-pair problem in a one-dimensional linear family. The individual fixed-(a, b) asymptotics are beyond present methods in general; the hope is to prove the needed cancellation after averaging over a, b. This is the natural dispersion-theoretic form of two-point LPD.

6 A clean conditional theorem

The exact analytic input isolated in Proposition 5.1 can be stated as a hypothesis. This is not meant to disguise the difficulty; it is the precise fixed-shift all-scales estimate that a dispersion, circle-method, or friable-Elliott–Halberstam approach would have to prove.

Hypothesis 6.1 (Fixed-shift prime-CRT discrepancy, $\mathrm{PCD}(\alpha, h)$)**.** *Let* $1/2 < \alpha < 1$ *and* $h \geq 1$ *be fixed. For every nonempty set* $J \subseteq \{0, \ldots, h-1\}$,

$$\Delta_J(X) = o_{\alpha,h,J}(X),$$

where $\Delta_J(X)$ *is defined in Section 5.*

Theorem 6.2 (Conditional completion of the LPD route)**.** *Let* $1/2 < \alpha < 1$ *and* $h \geq 1$ *be fixed. If* $\mathrm{PCD}(\alpha, h)$ *holds, then*

$$\# \left\{ n \leq N : P^{+}\left(\prod_{j=0}^{h-1} (n+j) \right) \leq n^{\alpha} \right\} = \rho(1/\alpha)^h N + o(N).$$

Consequently,

$$d \left\{ n \in \mathbb{N} : P^{+}\left(\prod_{j=0}^{h-1} (n+j) \right) > n^{\alpha} \right\} = 1 - \rho(1/\alpha)^h.$$

Proof. By Proposition 5.1, $\mathrm{PCD}(\alpha, h)$ implies $\mathrm{LPD}(\alpha, h)$. The first assertion follows from Theorem 1.3. The second assertion follows by taking complements. ☐

Corollary 6.3 (Conditional natural-density form of Erdős #1201)**.** *Let* $0 < \varepsilon < 1/2$. *Suppose that for every fixed* h *the hypothesis* $\mathrm{PCD}(1-\varepsilon, h)$ *holds. Then for every* $\eta > 0$ *there exists* k *such that*

$$d \left\{ n \in \mathbb{N} : P^{+}(n(n+1)\cdots(n+k)) > n^{1-\varepsilon} \right\}$$

exists and is at least $1 - \eta$.

Proof. This is Corollary 3.2 plus Proposition 5.1. ☐

Remark 6.4 (Why this is the right missing theorem). In the pair case, $\mathrm{PCD}(\alpha, 2)$ is equivalent to the dyadic natural-density theorem for pairs of consecutive smooth numbers, by Theorem 1.4. In higher rank it is the analogous fixed-shift all-scales correlation statement for the large prime divisors of $n, n+1, \ldots, n+h-1$.

7 Comparison with available results

The literature contains several powerful theorems that are close to the required input but do not currently imply $\mathrm{PCD}(\alpha, h)$ or $\mathrm{LPD}(\alpha, h)$ for all scales.

7.1 Logarithmic and almost-all-scale pair results

Teräväinen proved logarithmically averaged binary correlation results for multiplicative functions, including a logarithmic-density form of the Erdős–Pomerance conjecture for consecutive smooth numbers [9]. Logarithmic density, however, does not imply natural density.

Tao and Teräväinen's recent quantitative-correlation work gives asymptotics for pairs of consecutive smooth numbers at logarithmically almost all scales [8]. In the notation of the present paper, this supplies pair information outside a small exceptional set of scales. But an exceptional set of logarithmic density zero can still contain arbitrarily long multiplicative gaps. Such gaps can carry order-X oscillations in ordinary dyadic counts, so almost-all-scale information does not by itself yield natural-density convergence.

7.2 Wang's conditional pair theorem

Wang proved that the Elliott–Halberstam conjecture for friable integers implies several natural-density conjectures concerning $P^+(n)$ and $P^+(n+1)$, including the pair smooth-number density [11]. In the range $1/2 < \alpha < 1$, Theorem 1.4 shows that Wang's conditional pair result implies the two-point LPD statement. It does not, by itself, give the required h-point correlations for every fixed h.

7.3 Finite-complexity smooth-linear-form theorems

Matthiesen and Wang prove asymptotic results for simultaneous smooth values of finite-complexity systems of shifted linear forms [7]. The consecutive one-parameter system

$$n, \quad n+1, \quad \ldots, \quad n+h-1$$

is affinely dependent: all forms have the same linear part. It is therefore outside the finite-complexity setup in which those smooth-linear-form asymptotics apply directly. Averaging over a second variable in progressions

$$n, \quad n+m, \quad \ldots, \quad n+(h-1)m$$

would create a finite-complexity-looking system, but specializing the result back to the single slice $m = 1$ is not justified by an average over m.

7.4 Positive-density lower bounds for largest-prime-factor orderings

There are unconditional lower-bound results for orderings of $P^+(n)$ and $P^+(n+1)$. For instance, Lü and Wang improved the known positive lower density for the event $P^+(n) < P^+(n+1)$ [5]. Such results are important but qualitatively different from PCD: they do not provide the all-scales asymptotic for consecutive smooth pairs, let alone higher shifted smooth blocks.

8 The unconditional short-interval lower-density argument

For completeness, we include the lower-density proof based on Matomäki–Radziwiłł. This is the part of the argument that is currently unconditional.

For a set $A \subseteq \mathbb{N}$, write

$$\overline{d}(A) = \limsup_{N\to\infty} \frac{|A \cap [1, N]|}{N}, \qquad \underline{d}(A) = \liminf_{N\to\infty} \frac{|A \cap [1, N]|}{N}.$$

If the two limits agree, their common value is the natural density $d(A)$.

We use the following form of the Matomäki–Radziwiłłtheorem [6].

Theorem 8.1 (Matomäki–Radziwiłł)**.** *There are absolute constants $C, C_0 > 0$ such that the following holds. Let $f : \mathbb{N} \to [-1, 1]$ be multiplicative. Let $2 \le h \le X$ and $\delta > 0$. Then, for all but at most*

$$CX \left(\frac{(\log h)^{1/3}}{\delta^2 h^{\delta/25}} + \frac{1}{\delta^2 (\log X)^{1/50}} \right)$$

integers $x \in [X, 2X)$, one has

$$\left| \frac{1}{h} \sum_{x\le m<x+h} f(m) - \frac{1}{X} \sum_{X\le m<2X} f(m) \right| \le \delta + C_0 \frac{\log\log h}{\log h}.$$

We also use the standard Dickman–de Bruijn estimate. Let

$$\Psi(x, y) := |\{m \le x : P^+(m) \le y\}|.$$

If $0 < \beta < 1$ is fixed, then uniformly for $t \in [1, 2]$,

$$\Psi(tX, X^\beta) = tX\rho(1/\beta) + o(X) \qquad (X \to \infty).$$

Proof of Theorem 1.1. If $\varepsilon \ge 1$, the assertion is trivial. Assume $0 < \varepsilon < 1$. Choose

$$\beta = 1 - \frac{\varepsilon}{2} \in (0, 1), \qquad r = \rho(1/\beta).$$

Since $1/\beta > 1$, we have $r < 1$. Put

$$\gamma = 1 - r > 0, \qquad \delta = \frac{\gamma}{4}.$$

For all sufficiently large h,

$$C_0 \frac{\log\log h}{\log h} \le \delta.$$

We will show that, for all such h,

$$\overline{d}(\mathcal{S}_{1-\varepsilon,h}) \le C \frac{(\log h)^{1/3}}{\delta^2 h^{\delta/25}},$$

and the right-hand side tends to zero as $h \to \infty$.

Fix such an h and a large dyadic scale X. Put $Y = X^\beta$ and define

$$f_X(m) := \mathbf{1}_{P^+(m)\le Y}.$$

This is a completely multiplicative function taking values in $\{0,1\}$. By the Dickman–de Bruijn estimate,

$$M_X := \frac{1}{X}\sum_{X\le m<2X} f_X(m) = r + o(1).$$

For all sufficiently large X,

$$M_X \le r + \delta.$$

Apply Theorem 8.1 to f_X with the chosen h and δ. Apart from at most

$$CX\left(\frac{(\log h)^{1/3}}{\delta^2 h^{\delta/25}} + \frac{1}{\delta^2(\log X)^{1/50}}\right)$$

integers $n \in [X,2X)$, one has

$$\begin{aligned}\frac{1}{h}\sum_{j=0}^{h-1} f_X(n+j) &\le M_X + \delta + C_0\frac{\log\log h}{\log h}\\ &\le r + 3\delta = 1 - \frac{\gamma}{4} < 1.\end{aligned}$$

Thus every non-exceptional $n \in [X,2X)$ has at least one shift $n+j$ with $P^+(n+j) > Y$.

If $n \in \mathcal{S}_{1-\varepsilon,h} \cap [X,2X)$, then every $n+j$ satisfies

$$P^+(n+j) \le n^{1-\varepsilon} \le (2X)^{1-\varepsilon}.$$

Since $\beta = 1-\varepsilon/2 > 1-\varepsilon$, we have $(2X)^{1-\varepsilon} < X^\beta = Y$ for all sufficiently large X. Therefore $f_X(n+j) = 1$ for every $j < h$, and such an n must be exceptional in the Matomäki–Radziwiłłtheorem. It follows that

$$\limsup_{X\to\infty} \frac{|\mathcal{S}_{1-\varepsilon,h}\cap[X,2X)|}{X} \le C\frac{(\log h)^{1/3}}{\delta^2 h^{\delta/25}}.$$

A dyadic decomposition of $[1,N]$ then gives the same upper bound for $\overline{d}(\mathcal{S}_{1-\varepsilon,h})$. Letting $h\to\infty$ proves the theorem. ☐

9 What remains to be proved

The preceding sections reduce the natural-density problem to the following concrete analytic estimate.

> For each fixed $1/2 < \alpha < 1$, each fixed $h \ge 1$, and each nonempty $J \subseteq \{0,\dots,h-1\}$, prove
>
> $$\sum_{\mathbf{p}\in\mathcal{P}_J(X)}\left(R_J(X;\mathbf{p}) - \frac{X}{\prod_{j\in J} p_j}\right) = o(X).$$

This is a fixed-shift, all-scales cancellation problem over large prime moduli. For $|J| = 1$ it follows from Mertens' theorem. For $|J| = 2$ it is equivalent to the all-scales dyadic asymptotic for shifted smooth pairs. For larger $|J|$ it becomes an averaged prime-pattern problem of increasing rank after the substitution $n + j = a_j p_j$.

Thus the large-prime-divisor route is fully reduced, but not unconditionally closed. A successful proof of the displayed estimate would complete the natural-density argument in the range $0 < \varepsilon < 1/2$. Extending the method to $\varepsilon \geq 1/2$ would require iterated Buchstab decompositions, because an integer $n + j \asymp X$ can then have more than one prime factor exceeding $X^{1-\varepsilon}$.

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