A note on Erdős Problem #1201

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Abstract

We prove, using the Matomäki–Radziwiłł theorem on multiplicative functions in short intervals, that for every $\varepsilon, \eta > 0$ there is a k such that the set of integers n for which

$$P^+(n(n+1)\cdots(n+k)) > n^{1-\varepsilon}$$

has lower asymptotic density at least $1-\eta$. In fact, the exceptional upper density tends to zero as $k \to \infty$. This settles Erdős Problem #1201 as stated on the Erdős Problems website. The argument is an immediate but apparently not explicitly recorded consequence of the quantitative short-interval theorem of Matomäki and Radziwiłł.

1 Statement

For $m \geq 2$, let $P^+(m)$ denote the largest prime divisor of m, and put $P^+(1) = 1$. For a set $A \subseteq \mathbb{N}$, write

$$\overline{d}(A) = \limsup_{N\to\infty} \frac{|A \cap [1,N]|}{N}, \qquad \underline{d}(A) = \liminf_{N\to\infty} \frac{|A \cap [1,N]|}{N}.$$

The problem asks whether, for every $\varepsilon, \eta > 0$, one can choose k so that the density of

$$\mathcal{G}_{\varepsilon,k} := \{n \in \mathbb{N} : P^+(n(n+1)\cdots(n+k)) > n^{1-\varepsilon}\}$$

is at least $1-\eta$. We prove the following stronger form.

Theorem 1. *For every $\varepsilon > 0$,*

$$\lim_{h\to\infty} \overline{d}\Big\{n \in \mathbb{N} : P^+\Big(\prod_{j=0}^{h-1}(n+j)\Big) \leq n^{1-\varepsilon}\Big\} = 0.$$

Consequently, for every $\varepsilon, \eta > 0$ there is a k such that

$$\underline{d}(\mathcal{G}_{\varepsilon,k}) \geq 1 - \eta.$$

2 The short-interval input

We use the following form of the Matomäki–Radziwiłłtheorem [5, Theorem 1]. The original statement uses intervals with slightly different endpoint conventions; the half-open version below follows by changing only harmless absolute constants.

Theorem 2 (Matomäki–Radziwiłł)**.** *There are absolute constants $C, C_0 > 0$ such that the following holds. Let $f : \mathbb{N} \to [-1, 1]$ be multiplicative. Let $2 \le h \le X$ and let $\delta > 0$. Then, for all but at most*

$$CX\left(\frac{(\log h)^{1/3}}{\delta^2 h^{\delta/25}} + \frac{1}{\delta^2 (\log X)^{1/50}}\right)$$

integers $x \in [X, 2X]$, one has

$$\left|\frac{1}{h}\sum_{x\le m<x+h} f(m) - \frac{1}{X}\sum_{X\le m<2X} f(m)\right| \le \delta + C_0\frac{\log\log h}{\log h}.$$

The constants are uniform in f, h, X and δ.

We also use the standard Dickman–de Bruijn estimate for smooth numbers; see, for example, Tenenbaum [6, Chapter III.5]. Let

$$\Psi(x, y) := |\{m \le x : P^+(m) \le y\}|.$$

If $0 < \beta < 1$ is fixed, then, uniformly for $t \in [1, 2]$,

$$\Psi(tX, X^\beta) = tX\,\rho(1/\beta) + o(X) \qquad (X \to \infty), \tag{1}$$

where ρ is the Dickman–de Bruijn function. In particular, if $u > 1$, then $0 < \rho(u) < 1$.

3 Proof of Theorem 1

If $\varepsilon \ge 1$, the result is trivial: for $n \ge 2$ the quantity $n^{1-\varepsilon}$ is at most 1, whereas the product $n(n+1)\cdots(n+k)$ has a prime divisor. Henceforth assume $0 < \varepsilon < 1$.

Choose

$$\beta = 1 - \frac{\varepsilon}{2} \in (0, 1), \qquad r = \rho(1/\beta).$$

Since $1/\beta > 1$, we have $r < 1$. Put

$$\gamma = 1 - r > 0, \qquad \delta = \frac{\gamma}{4}.$$

Choose h large enough that

$$C_0\frac{\log\log h}{\log h} \le \delta. \tag{2}$$

We shall estimate the upper density of the bad set

$$\mathcal{B}_{\varepsilon,h} := \left\{ n \in \mathbb{N} : P^+\Big(\prod_{j=0}^{h-1}(n+j)\Big) \le n^{1-\varepsilon} \right\}.$$

Fix a large dyadic scale X and set $Y = X^\beta$. Define

$$f_X(m) := 1_{P^+(m)\le Y}.$$

This is a completely multiplicative function taking values in $\{0,1\}$: equivalently, $f_X(p^a)=1$ for primes $p \le Y$ and $f_X(p^a)=0$ for primes $p>Y$ and $a \ge 1$.

By (1),

$$M_X := \frac{1}{X}\sum_{X\le m<2X} f_X(m) = \frac{\Psi(2X,X^\beta)-\Psi(X,X^\beta)}{X} = r+o(1).$$

Thus, for all sufficiently large X,

$$M_X \le r+\delta. \tag{3}$$

Apply Theorem 2 to $f=f_X$ with the above h and δ. Since the theorem is uniform in f, the dependence of f_X on X causes no difficulty. Apart from at most

$$CX\left(\frac{(\log h)^{1/3}}{\delta^2 h^{\delta/25}} + \frac{1}{\delta^2(\log X)^{1/50}}\right) \tag{4}$$

integers $n \in [X,2X]$, we have, using (2) and (3),

$$\frac{1}{h}\sum_{j=0}^{h-1} f_X(n+j) \le M_X+\delta+C_0\frac{\log\log h}{\log h} \le r+3\delta = 1-\frac{\gamma}{4} < 1.$$

Therefore, for every non-exceptional $n\in[X,2X]$, at least one of the integers

$$n, n+1, \dots, n+h-1$$

has a prime factor exceeding $Y=X^\beta$.

On the other hand, if $n \in \mathcal{B}_{\varepsilon,h}\cap[X,2X]$, then each factor $n+j$ in the product satisfies

$$P^+(n+j) \le n^{1-\varepsilon} \le (2X)^{1-\varepsilon}.$$

Since $\beta = 1-\varepsilon/2 > 1-\varepsilon$, we have

$$(2X)^{1-\varepsilon} < X^\beta = Y$$

for all sufficiently large X. Hence $f_X(n+j)=1$ for every $0\le j<h$, so that

$$\frac{1}{h}\sum_{j=0}^{h-1} f_X(n+j) = 1.$$

Such an n must therefore be exceptional in Theorem 2. From (4) we get

$$\limsup_{X\to\infty}\frac{|\mathcal{B}_{\varepsilon,h}\cap[X,2X]|}{X} \le C\frac{(\log h)^{1/3}}{\delta^2 h^{\delta/25}}. \tag{5}$$

The right-hand side tends to 0 as $h\to\infty$.

Finally, the dyadic estimate (5) implies the same bound for upper asymptotic density. Indeed, for any fixed h and any N, decompose $[1, N]$ into dyadic intervals

$$(N/2, N],\ (N/4, N/2],\ (N/8, N/4], \ldots,$$

up to a bounded initial segment. Letting $N \to \infty$ and then the harmless error in (5) tend to zero yields

$$\overline{d}(\mathcal{B}_{\varepsilon,h}) \leq C \frac{(\log h)^{1/3}}{\delta^2 h^{\delta/25}}.$$

This upper bound tends to zero with h. This proves the first assertion of Theorem 1. Taking $h = k + 1$ and complementing gives

$$\underline{d}(\mathcal{G}_{\varepsilon,k}) = 1 - \overline{d}(\mathcal{B}_{\varepsilon,k+1}) \geq 1 - \eta$$

for some sufficiently large k. The theorem follows. □

4 Literature note

The Erdős Problems entry for #1201 currently records the problem as open, cites Erdős's 1980 survey [3, p. 107], says that Erdős could prove the case $\varepsilon = 1/2$, and states that no solution or partial solution is claimed in the comments [2].

The proof above is not a new short-interval theorem. It is a direct specialization of Matomäki and Radziwiłł's quantitative theorem on multiplicative functions in short intervals [5]. Their paper explicitly records as Corollary 6 that, for fixed $u > 0$ and any function $\psi(x) \to \infty$, the number of $x^{1/u}$-smooth integers in $[x, x + \psi(x)]$ is asymptotic to $\rho(u)\psi(x)$ for almost all x. The proof here uses the quantitative form of their Theorem 1 with h first fixed and then sent to infinity; this is the small extra bookkeeping needed for Erdős's formulation, where k is fixed after ε and η are chosen.

The author did not find, in this search, an explicit published or posted statement saying that Matomäki–Radziwiłłsettles Erdős Problem #1201; the conclusion should therefore be read as an apparently unrecorded corollary rather than as a claim that no such note exists. There is closely related literature: Balog and Wooley constructed arbitrarily long strings of consecutive smooth integers [1], which is compatible with Theorem 1 because their strings form a very sparse set; and Laishram and Shorey studied deterministic lower bounds for $P^+(n(n+1)\cdots(n+k-1))$ in terms of k [4], which is different from the density question with a lower bound of order $n^{1-\varepsilon}$.

References

[1] A. Balog and T. D. Wooley, *On strings of consecutive integers with no large prime factors*, J. Austral. Math. Soc. Ser. A **64** (1998), 266–276.

[2] T. F. Bloom, *Erdős Problem #1201*, https://www.erdosproblems.com/1201, accessed 30 April 2026.

[3] P. Erdős, *A survey of problems in combinatorial number theory*, Annals of Discrete Mathematics **6** (1980), 89–115.

[4] S. Laishram and T. N. Shorey, *The greatest prime divisor of a product of consecutive integers*, Acta Arith. **120** (2005), no. 3, 299–306.

[5] K. Matomäki and M. Radziwiłł, *Multiplicative functions in short intervals*, Ann. of Math. (2) **183** (2016), no. 3, 1015–1056.

[6] G. Tenenbaum, *Introduction to Analytic and Probabilistic Number Theory*, Cambridge Studies in Advanced Mathematics, vol. 46, Cambridge University Press, Cambridge, 1995.