Truncated Congruence Sieves and Erdős Problem 25

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Abstract

Let $1 \le n_1 < n_2 < \cdots$ be integers and let $a_i \pmod{n_i}$ be arbitrary residue classes. Define

$$A := \{n \in \mathbb{N} : \text{for every } i, \text{ either } n < n_i \text{ or } n \not\equiv a_i \pmod{n_i}\}.$$

This is the truncated congruence sieve attached to the sequence (n_i, a_i). The basic question is whether the logarithmic density of A must always exist. We prove two unconditional positive results: if $\sum_i 1/n_i < \infty$, then A has natural density; and if the moduli are pairwise coprime, then A has natural density in all cases. We then introduce the first-kill decomposition

$$\mathbb{N} \setminus A = \bigsqcup_{i \ge 1} E_i, \qquad E_i := A^{(i-1)} \cap B_i,$$

and show that each first-kill set is a translate-dilate of a finite quotient sieve. A conditional reduction theorem shows that a uniform harmonic estimate for these quotient sieves, together with a sublogarithmic global charge, would imply a positive solution of the full problem. Finally, we prove two obstructions: the ambient tail union cannot satisfy a vanishing logarithmic bound, and a prime-power tower gadget produces transient harmonic spikes much larger than the entropy-sized contribution of a single congruence. The note therefore isolates a concrete missing lemma rather than a complete proof.

1 Introduction

Let $1 \le n_1 < n_2 < \cdots$ and let $a_i \pmod{n_i}$ be arbitrary residue classes. For each i define

$$B_i := \{n \in \mathbb{N} : n \ge n_i \text{ and } n \equiv a_i \pmod{n_i}\}, \qquad A := \mathbb{N} \setminus \bigcup_{i \ge 1} B_i.$$

The question is whether the logarithmic density

$$\operatorname{ld}(A) := \lim_{X \to \infty} \frac{1}{\log X} \sum_{\substack{n \le X \\ n \in A}} \frac{1}{n}$$

must always exist.

For finite truncations

$$A^{(k)} := \mathbb{N} \setminus \bigcup_{i \le k} B_i,$$

the set $A^{(k)}$ is eventually periodic, so its natural and logarithmic densities both exist. Writing this common density as δ_k, one immediately gets a monotone sequence $\delta_1 \geq \delta_2 \geq \cdots$, hence a limit

$$\delta := \lim_{k\to\infty} \delta_k \in [0, 1].$$

The problem is to decide whether the actual logarithmic density of A exists and equals δ.

This note has three aims.

(i) Prove the two easiest positive cases in a form strong enough for later use: the summable case and the pairwise-coprime case.

(ii) Recast the general problem in terms of *first-kill sets* and the associated *quotient sieves*. This reduces the infinite problem to a family of finite local sieve problems with a common harmonic shape.

(iii) Isolate the explicit obstruction that remains. The ambient tail union cannot be small, and prime-power towers can create large transient harmonic spikes. Any successful proof has to neutralize those spikes globally.

The viewpoint here is informed by several modern sieve ideas. Maynard's multidimensional refinement of the Selberg sieve provides flexible finite-interval weights [7]. Banks, Ford, and Tao formulate a probabilistic model whose extremal behavior is governed by an interval-sieve problem [2], while Granville shows that exceptional zeros can force extremal behavior in sieving intervals [4]. Helfgott and Radziwiłłdevelop a combinatorial sieve for composite moduli in which redundancy among congruence conditions is handled by a form of Rota's cross-cut theorem [5]. On the covering-system side, deep work of Filaseta, Ford, Konyagin, Pomerance, and Yu, and later Hough, shows how subtle distinct-moduli congruence problems can be [3, 6]. Finally, Araújo's recent work on Erdős sieves isolates light-tail conditions in a much more general dynamical setting [1]. The present note does not derive the answer from those papers; rather, it puts the truncated problem into a form that seems compatible with them.

2 Setup and finite truncations

Throughout, $\mathbb{N} = \{1, 2, 3, \ldots\}$. We normalize each residue class by choosing $0 \leq a_i < n_i$. With this convention,

$$B_i = \{a_i + n_i t : t \in \mathbb{N}\}.$$

If $a_i = 0$, this means $B_i = \{n_i, 2n_i, 3n_i, \ldots\}$.

For a set $S \subseteq \mathbb{N}$, write

$$\mu_X(S) := \frac{1}{\log X} \sum_{\substack{n \leq X \\ n \in S}} \frac{1}{n}$$

for its logarithmic mass up to X. We also use the upper and lower natural densities

$$\overline{d}(S) := \limsup_{X\to\infty} \frac{1}{X} \#(S \cap [1, X]), \qquad \underline{d}(S) := \liminf_{X\to\infty} \frac{1}{X} \#(S \cap [1, X]).$$

When the limit exists we denote it by $d(S)$ or $\mathrm{dens}(S)$.

Lemma 2.1. *For each fixed $k \geq 1$, the set $A^{(k)}$ is eventually periodic. In particular, both $d(A^{(k)})$ and $\mathrm{ld}(A^{(k)})$ exist, and they are equal.*

Proof. Let $L_k := \mathrm{lcm}(n_1, \ldots, n_k)$. For every $n \geq n_k$, membership in $A^{(k)}$ depends only on the congruence class of $n \pmod{L_k}$. Hence $A^{(k)}$ is eventually periodic with period dividing L_k, and the statement follows. □

We therefore write

$$\delta_k := d(A^{(k)}) = \mathrm{ld}(A^{(k)}).$$

Since $A^{(k+1)} \subseteq A^{(k)}$, the sequence (δ_k) is decreasing. Its limit $\delta = \lim_k \delta_k$ will serve as the candidate value for the density of A.

3 Two unconditional positive results

3.1 The summable case

Theorem 3.1. *If*

$$\sum_{i \geq 1} \frac{1}{n_i} < \infty,$$

then A has natural density. In particular, A has logarithmic density.

Proof. By Lemma 2.1, each $A^{(k)}$ has natural density δ_k. If $\ell > k$, then

$$A^{(k)} \setminus A^{(\ell)} \subseteq \bigcup_{k < i \leq \ell} B_i.$$

Each B_i has natural density $1/n_i$, so

$$0 \leq \delta_k - \delta_\ell \leq \sum_{k < i \leq \ell} \frac{1}{n_i}.$$

Thus (δ_k) is Cauchy and converges to some $\delta \in [0, 1]$.

Now fix k. Since $A \subseteq A^{(k)}$, we have

$$\overline{d}(A) \leq \delta_k.$$

Also,

$$A^{(k)} \setminus A \subseteq \bigcup_{i > k} B_i,$$

so

$$\overline{d}(A^{(k)} \setminus A) \leq \sum_{i > k} \frac{1}{n_i}.$$

Because

$$A = A^{(k)} \setminus \left(A^{(k)} \setminus A\right),$$

we get

$$\underline{d}(A) \geq \delta_k - \overline{d}(A^{(k)} \setminus A) \geq \delta_k - \sum_{i > k} \frac{1}{n_i}.$$

Letting $k \to \infty$ gives

$$\underline{d}(A) \geq \delta \geq \overline{d}(A).$$

Therefore $d(A) = \delta$ exists.

Natural density implies logarithmic density with the same value, so A also has logarithmic density. □

3.2 The pairwise-coprime case

Theorem 3.2. *If the moduli n_i are pairwise coprime, then A has natural density.*

Proof. For every fixed k, the Chinese remainder theorem gives

$$d(A^{(k)}) = \prod_{i \leq k} \left(1 - \frac{1}{n_i}\right).$$

If $\sum_i 1/n_i < \infty$, Theorem 3.1 applies.

Assume now that $\sum_i 1/n_i = \infty$. Then

$$\prod_{i \leq k} \left(1 - \frac{1}{n_i}\right) \xrightarrow[k \to \infty]{} 0.$$

Since $A \subseteq A^{(k)}$ for every k,

$$\overline{d}(A) \leq d(A^{(k)})$$

for every k, hence $\overline{d}(A) = 0$. Therefore $d(A) = 0$. □

Remark 3.3. In the divergent pairwise-coprime case the same argument also gives $\mathrm{ld}(A) = 0$.

4 First-kill sets and quotient sieves

4.1 First-kill sets

Define

$$E_i := A^{(i-1)} \cap B_i \qquad (i \geq 1).$$

Thus E_i consists of the integers removed for the first time by the ith congruence.

Proposition 4.1. *The sets E_i are pairwise disjoint and*

$$\mathbb{N} \setminus A = \bigsqcup_{i \geq 1} E_i.$$

Each E_i is eventually periodic. If

$$e_i := \mathrm{ld}(E_i),$$

then

$$e_i = \delta_{i-1} - \delta_i \qquad (i \geq 1),$$

and hence

$$\sum_{i \geq 1} e_i = 1 - \delta.$$

Proof. By definition, $E_i \subseteq B_i$ and $E_i \subseteq A^{(i-1)}$, so no element of E_i lies in any B_j with $j < i$. Therefore the sets E_i are pairwise disjoint, and every element of $\bigcup_i B_i$ belongs to the first B_i that contains it. This proves the disjoint union.

Since $A^{(i-1)}$ is eventually periodic and B_i is an arithmetic progression, the intersection $E_i = A^{(i-1)} \cap B_i$ is eventually periodic.

Also,

$$A^{(i-1)} = A^{(i)} \sqcup E_i,$$

so taking logarithmic densities gives

$$e_i = \delta_{i-1} - \delta_i.$$

Summing from $i = 1$ to m yields

$$\sum_{i \le m} e_i = 1 - \delta_m.$$

Letting $m \to \infty$ gives the last claim. ☐

4.2 Quotient-sieve representation

Fix i. For $j < i$, let

$$g_{ij} := (n_i, n_j).$$

If $a_i \not\equiv a_j \pmod{g_{ij}}$, then $B_i \cap B_j = \varnothing$. If instead

$$a_i \equiv a_j \pmod{g_{ij}},$$

set

$$q_{ij} := \frac{n_j}{g_{ij}}.$$

Then there is a unique residue class $b_{ij} \pmod{q_{ij}}$ such that

$$a_i + n_i t \equiv a_j \pmod{n_j} \iff t \equiv b_{ij} \pmod{q_{ij}}.$$

Define the associated *quotient sieve*

$$S_i := \Big\{ t \in \mathbb{N} : t \not\equiv b_{ij} \pmod{q_{ij}} \text{ for every } j < i \text{ with } a_i \equiv a_j \pmod{g_{ij}} \Big\}.$$

Proposition 4.2. *For every i,*

$$E_i = \{a_i + n_i t : t \in S_i\}.$$

In particular, S_i is periodic, so its natural density

$$d_i := \operatorname{dens}(S_i)$$

exists, and

$$d_i = n_i e_i \in [0, 1].$$

Proof. Every element of B_i has the form $a_i + n_i t$ with $t \in \mathbb{N}$. Such an element belongs to E_i if and only if it avoids every earlier set B_j. For incompatible j there is no restriction. For compatible j, the condition of lying in B_j is exactly $t \equiv b_{ij} \pmod{q_{ij}}$. Hence

$$a_i + n_i t \in E_i \iff t \in S_i,$$

which proves the representation.

Because only finitely many congruence conditions are involved, S_i is periodic. Finally,

$$\#\{n \le X : n \in E_i\} = \#\Big\{t \le \frac{X - a_i}{n_i} : t \in S_i\Big\},$$

so dividing by X and letting $X \to \infty$ gives $e_i = d_i/n_i$. □

Remark 4.3. The original infinite problem is now localized. Everything difficult is concentrated in the harmonic behavior of the finite quotient sieves S_i.

5 A conditional reduction theorem

Put

$$\alpha_i := \frac{a_i}{n_i} \in [0, 1).$$

By Proposition 4.2,

$$\sum_{\substack{n \le X \\ n \in E_i}} \frac{1}{n} = \frac{1}{n_i} \sum_{\substack{t \le (X - a_i)/n_i \\ t \in S_i}} \frac{1}{t + \alpha_i}.$$

This motivates the following finite quotient-sieve hypothesis.

Conjecture 5.1 (Quotient-sieve harmonic estimate)**.** There exist nonnegative quantities τ_i such that for every i and every $Y \ge 1$,

$$\sum_{\substack{t \le Y \\ t \in S_i}} \frac{1}{t + \alpha_i} = d_i \log Y + O\Big(d_i \log \frac{2}{d_i} + \tau_i\Big),$$

with an absolute implied constant, and such that

$$\sum_{n_i \le X} \frac{\tau_i}{n_i} = o(\log X).$$

Here terms of the form $d_i \log(2/d_i)$ are understood to be 0 when $d_i = 0$.

The next theorem shows that this local harmonic estimate, together with the sublogarithmic global charge condition, would settle the original problem.

Lemma 5.2 (Weighted entropy bound)**.** *Let (x_m) be nonnegative and (u_m) be positive. Set*

$$S := \sum_m x_m, \qquad U := \sum_m \frac{1}{u_m}.$$

Then

$$\sum_m x_m \log \frac{1}{u_m x_m} \le S \log \frac{U}{S},$$

with the convention that $0 \log(1/0) = 0$.

Proof. If $S = 0$ there is nothing to prove. Otherwise define

$$p_m := \frac{x_m}{S}, \qquad q_m := \frac{(1/u_m)}{U}.$$

Then $\sum_m p_m = \sum_m q_m = 1$, so nonnegativity of relative entropy gives

$$0 \le \sum_m p_m \log \frac{p_m}{q_m} = \frac{1}{S} \sum_m x_m \log \frac{x_m U u_m}{S}.$$

Rearranging yields the claim. □

Corollary 5.3. *For every $X \ge 2$,*

$$\sum_{n_i \le X} e_i \log \frac{1}{n_i e_i} \ll \log\log X,$$

where terms with $e_i = 0$ are interpreted as 0.

Proof. Apply Lemma 5.2 with $x_i = e_i$ and $u_i = n_i$, summing over indices with $n_i \le X$. Write

$$S_X := \sum_{n_i \le X} e_i, \qquad U_X := \sum_{n_i \le X} \frac{1}{n_i}.$$

Since $E_i \subseteq B_i$, we have $e_i \le 1/n_i$, hence $S_X \le U_X$. Also the E_i are disjoint, so $S_X \le 1$. Lemma 5.2 gives

$$\sum_{n_i \le X} e_i \log \frac{1}{n_i e_i} \le S_X \log \frac{U_X}{S_X}.$$

If $U_X \ge 1$, then

$$S_X \log \frac{U_X}{S_X} = S_X \log U_X - S_X \log S_X \le \log U_X + \frac{1}{e}.$$

If $U_X < 1$, then $S_X < 1$ as well, and

$$S_X \log \frac{U_X}{S_X} \le -S_X \log S_X \le \frac{1}{e}.$$

Therefore

$$\sum_{n_i \le X} e_i \log \frac{1}{n_i e_i} \le \log^+ U_X + O(1).$$

Finally,

$$U_X = \sum_{n_i \le X} \frac{1}{n_i} \le \sum_{m \le X} \frac{1}{m} \ll \log X,$$

which proves the result. □

Theorem 5.4 (Conditional reduction)**.** *Assume Conjecture 5.1. Then the logarithmic density of A exists and equals*

$$\delta = \lim_{k \to \infty} \delta_k.$$

Proof. Let

$$C(X) := \sum_{\substack{n \le X \\ n \notin A}} \frac{1}{n}.$$

By Proposition 4.1 and the disjoint union $\mathbb{N} \setminus A = \bigsqcup_i E_i$,

$$C(X) = \sum_{n_i \le X} \sum_{\substack{n \le X \\ n \in E_i}} \frac{1}{n}.$$

Split the indices into $n_i \le X/2$ and $X/2 < n_i \le X$.

If $X/2 < n_i \le X$, then there is at most one quotient value $t \ge 1$ such that $a_i + n_i t \le X$, namely $t = 1$. Hence each such i contributes $O(1/X)$, and therefore the total contribution of this near-top range is

$$\ll \#\{i : X/2 < n_i \le X\} \cdot \frac{1}{X} \ll 1.$$

Also,

$$\sum_{X/2 < n_i \le X} e_i \log \frac{X}{n_i} \le (\log 2) \sum_i e_i \ll 1.$$

Thus the near-top range is harmless.

Now suppose $n_i \le X/2$, and set

$$Y_i := \frac{X - a_i}{n_i}.$$

Then $Y_i \ge 1$, and Conjecture 5.1 gives

$$\sum_{\substack{n \le X \\ n \in E_i}} \frac{1}{n} = \frac{1}{n_i} \left(d_i \log Y_i + O\Big(d_i \log \frac{2}{d_i} + \tau_i \Big) \right).$$

Since $d_i = n_i e_i$, this becomes

$$\sum_{\substack{n \le X \\ n \in E_i}} \frac{1}{n} = e_i \log Y_i + O\Big(e_i \log \frac{2}{d_i} + \frac{\tau_i}{n_i} \Big).$$

Because $Y_i = X/n_i - \alpha_i$ and $X/n_i \ge 2$, we have

$$\log Y_i = \log \frac{X}{n_i} + O(1).$$

Summing over $n_i \le X/2$ yields

$$C(X) = \sum_{n_i \le X} e_i \log \frac{X}{n_i} + O\left(\sum_{n_i \le X} e_i \log \frac{2}{d_i} + \sum_{n_i \le X} \frac{\tau_i}{n_i} + 1 \right).$$

Using $d_i = n_i e_i$ and Corollary 5.3, we get

$$\sum_{n_i \le X} e_i \log \frac{2}{d_i} \ll 1 + \sum_{n_i \le X} e_i \log \frac{1}{n_i e_i} \ll \log \log X.$$

Thus

$$C(X) = \sum_{n_i \le X} e_i \log \frac{X}{n_i} + o(\log X). \tag{1}$$

Set $\delta_0 := 1$, and define a step function $\Delta : [1, \infty) \to [0, 1]$ by

$$\Delta(t) := \begin{cases} 1, & 1 \le t < n_1, \\ \delta_k, & n_k \le t < n_{k+1}. \end{cases}$$

Since $e_i = \delta_{i-1} - \delta_i$, we have

$$\sum_{n_i \le t} e_i = 1 - \Delta(t) \qquad (t \ge 1).$$

Therefore

$$\sum_{n_i \le X} e_i \log \frac{X}{n_i} = \int_1^X \left(\sum_{n_i \le t} e_i \right) \frac{dt}{t} = \int_1^X (1 - \Delta(t)) \frac{dt}{t}.$$

Fix $\varepsilon > 0$. Since $\Delta(t) \to \delta$ as $t \to \infty$, there exists $T = T(\varepsilon)$ such that

$$|\Delta(t) - \delta| < \varepsilon \qquad (t \ge T).$$

Hence for $X \ge T$,

$$\left| \int_1^X \frac{\Delta(t) - \delta}{t} \, dt \right| \le \int_1^T \frac{dt}{t} + \varepsilon \int_T^X \frac{dt}{t} = O_\varepsilon(1) + \varepsilon \log X.$$

Dividing by $\log X$ and letting $X \to \infty$ gives

$$\frac{1}{\log X} \int_1^X \Delta(t) \frac{dt}{t} \longrightarrow \delta.$$

Consequently,

$$\frac{1}{\log X} \sum_{n_i \le X} e_i \log \frac{X}{n_i} \longrightarrow 1 - \delta.$$

Combining this with (1), we get

$$\frac{C(X)}{\log X} \longrightarrow 1 - \delta.$$

Finally,

$$\mu_X(A) = \frac{1}{\log X} \sum_{\substack{n \le X \\ n \in A}} \frac{1}{n} = \frac{H_X}{\log X} - \frac{C(X)}{\log X} = 1 - \frac{C(X)}{\log X} + o(1),$$

where $H_X = \sum_{n \le X} 1/n = \log X + O(1)$. Therefore

$$\mu_X(A) \to \delta,$$

which proves the theorem. □

Remark 5.5. The reduction theorem isolates two pieces. The entropy-scale term

$$d_i \log \frac{2}{d_i}$$

is globally harmless. The genuinely new input is a sublogarithmic bound on the global charge

$$\sum_{n_i \le X} \frac{\tau_i}{n_i}.$$

6 Two explicit obstructions

6.1 Why the ambient tail union cannot be small

A natural first guess is to control the tail union itself. This is impossible in general.

Proposition 6.1. *There is no general estimate of the form*

$$\mu_X\left(\bigcup_{i>k} B_i\right) \le \Phi_k + o(1) \qquad (X \to \infty),$$

with $\Phi_k \to 0$.

Proof. Take $n_i = p_i$ (the ith prime) and $a_i = 0$. Then

$$\bigcup_{i>k} B_i$$

is the set of integers divisible by some prime $> p_k$. Its complement is the set of positive integers all of whose prime factors are at most p_k. The harmonic sum of that complement is bounded by the Euler product

$$\sum_{\substack{n\ge 1\\ p\mid n \Rightarrow p\le p_k}} \frac{1}{n} = \prod_{p\le p_k}\left(1-\frac{1}{p}\right)^{-1} < \infty.$$

Therefore the complement has logarithmic density 0, and the tail union has logarithmic density 1. □

Remark 6.2. So any useful blocking lemma must control the *conditioned* tail

$$A^{(k)} \cap \bigcup_{i>k} B_i,$$

not the ambient tail union.

6.2 A prime-power tower gadget

The next construction shows that a single first-kill set can have a transient harmonic contribution far larger than its entropy-sized eventual density loss.

Proposition 6.3 (Tower gadget)**.** *Let $u \in \mathbb{N}$ be coprime to 6, and let $r \ge 1$. For $0 \le j \le r$ set*

$$m_j := u\,2^{r-j}3^j.$$

Attach residues by

$$a_r := 0, \qquad a_j \equiv u3^r(1+2^{r-j-1}) \pmod{m_j} \quad (0 \le j < r).$$

Consider only this finite block of congruences. Then the first-kill set attached to the top modulus $m_r = u3^r$ is exactly

$$E_{m_r} = \{u3^r(1+2^r t) : t \ge 0\}.$$

Equivalently, the quotient sieve attached to m_r is

$$S_{m_r} = \{t \in \mathbb{N} : t \equiv 1 \pmod{2^r}\}.$$

Proof. A point of B_{m_r} has the form

$$n = u3^r t, \qquad t \in \mathbb{N}.$$

Fix $j < r$. Then

$$m_j = u2^{r-j}3^j, \qquad (u3^r, m_j) = u3^j.$$

Hence the congruence $u3^r t \equiv a_j \pmod{m_j}$ reduces, after dividing by $u3^j$, to a congruence modulo 2^{r-j}. With the chosen residue a_j this becomes

$$t \equiv 1 + 2^{r-j-1} \pmod{2^{r-j}}.$$

Thus the earlier moduli remove precisely the quotient classes

$$t \equiv 0 \pmod 2, \quad t \equiv 3 \pmod 4, \quad t \equiv 5 \pmod 8, \quad \dots, \quad t \equiv 1 + 2^{r-1} \pmod{2^r}.$$

These are exactly all residue classes modulo 2^r except $1 \pmod{2^r}$. Indeed, if $t \not\equiv 1 \pmod{2^r}$ and $s = v_2(t-1) < r$, then

$$t \equiv 1 + 2^s \pmod{2^{s+1}},$$

which corresponds to $j = r - s - 1$. Therefore the surviving quotient values are exactly $t \equiv 1 \pmod{2^r}$, and the claimed description of E_{m_r} follows. □

Remark 6.4. For the tower gadget,

$$e_{m_r} = \frac{1}{u6^r},$$

while the first surviving point of E_{m_r} is $u3^r$, contributing $1/(u3^r)$ to the harmonic sum. By contrast,

$$e_{m_r} \log \frac{1}{e_{m_r}} \asymp \frac{r}{u6^r}.$$

So an entropy-sized bound on a single E_i cannot control its initial harmonic spike.

7 A transverse-tower program

The conditional reduction theorem and the tower gadget suggest a more structured route to the full problem.

Step 1: Composite-moduli decomposition

Use an enveloping sieve to construct a controlled representation, or at least a controlled majorant and minorant, for $\mathbf{1}_{S_i}$ as a linear combination of arithmetic progression indicators with composite moduli. The point is not merely that inclusion–exclusion furnishes some formal expansion; what is needed is an expansion with coefficients and moduli under control after redundant congruence information has been collapsed. This is exactly the bookkeeping problem that appears in Helfgott–Radziwiłł's treatment of composite-moduli inputs and cross-cut cancellation [5].

Step 2: Transverse harmonic control

For the part of the quotient sieve not dominated by repeated prime-power structure, one hopes for a Selberg- or Maynard-style harmonic estimate of the form

$$\sum_{\substack{t \le Y \\ t \in S_i^{\mathrm{tr}}}} \frac{1}{t+\alpha_i} = d_i^{\mathrm{tr}} \log Y + O\left(d_i^{\mathrm{tr}} \log \frac{2}{d_i^{\mathrm{tr}}}\right),$$

uniformly in Y. The conditional reduction theorem shows that such an error term would already be globally harmless.

Step 3: Tower charging

The tower part is the real obstruction. The model statement is that every positive transient spike coming from repeated prime-power structure can be charged to earlier tower data, producing nonnegative charges τ_i such that

$$\sum_{\substack{t \le Y \\ t \in S_i}} \frac{1}{t+\alpha_i} = d_i \log Y + O\left(d_i \log \frac{2}{d_i} + \tau_i\right)$$

uniformly in Y, with

$$\sum_{n_i \le X} \frac{\tau_i}{n_i} = o(\log X).$$

Proposition 6.3 shows that some nontrivial charge term really is necessary.

These steps can be bundled into the following structural conjecture.

Conjecture 7.1 (Transverse-tower decomposition)**.** For every first-kill quotient sieve S_i there is a decomposition

$$S_i = S_i^{\mathrm{tr}} \sqcup S_i^{\mathrm{tw}}$$

and a nonnegative charge τ_i such that

$$\sum_{\substack{t \le Y \\ t \in S_i}} \frac{1}{t+\alpha_i} = d_i \log Y + O\left(d_i \log \frac{2}{d_i} + \tau_i\right)$$

uniformly in Y, and

$$\sum_{n_i \le X} \frac{\tau_i}{n_i} = o(\log X).$$

By Theorem 5.4, this conjecture would imply that $\mathrm{ld}(A)$ exists for every truncated congruence sieve.

8 Shortcomings and outlook

The present route is conceptually sharp but still incomplete. The main unresolved points are these.

(1) **Standard Maynard-Tao technology is not a black-box input here.** The multidimensional Selberg sieve works on finite intervals and is designed for lower bounds or existence statements. Erdős Problem 25 asks for a *uniform harmonic asymptotic* for one adversarial quotient sieve as the cutoff Y varies.

(2) **Interval-sieve obstructions are real.** The interval-sieve problem is sensitive to exceptional zeros and related extremal phenomena [2, 4]. So the transverse piece is already subtler than a generic large-sieve estimate.

(3) **Composite-moduli redundancy must be built in from the start.** Distinct subsets of congruence conditions can have the same conjunction. Any serious decomposition has to quotient out that redundancy before estimating coefficients; naive inclusion–exclusion is too unstable.

(4) **Tower spikes do not disappear automatically.** Proposition 6.3 shows that a top first-kill set can be compressed to one residue class modulo a high prime power while preserving a very early survivor. What remains to be proved is that infinitely many such spikes cannot synchronize strongly enough to force oscillation in the global logarithmic mass.

The rigorous content of this note can therefore be summarized as follows.

- The summable case and the pairwise-coprime case are settled positively.
- The full problem has been reduced to local harmonic estimates for finite quotient sieves together with a global charge-summability condition.
- The entropy contribution is globally harmless.
- The explicit obstruction isolated here is the prime-power tower effect.

So the problem no longer looks like an undifferentiated infinite sieve. What remains is a concrete local-to-global gap: prove the quotient-sieve harmonic estimate with a charge term that is globally sublogarithmic. Whether that can be done in full generality is still open.

References

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