A short note on Erdős Problem #258

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Abstract

We show that a recent theorem of Tao and Teräväinen on prime factors of consecutive integers implies the following conjecture of Erdős and Straus: for every sequence of positive integers $(a_n)_{n\geq 1}$ with $a_n \to \infty$, the series

$$\sum_{n=1}^{\infty} \frac{\tau(n)}{a_1 a_2 \cdots a_n}$$

is irrational. This settles the problem recorded as Erdős Problem #258 in Bloom's online database.

1. Introduction

Write $\tau(n)$ for the number of positive divisors of n and set

$$Q_n := a_1 a_2 \cdots a_n.$$

If $a_n \to \infty$, then $a_n \geq 2$ for all sufficiently large n, so the series

$$\sum_{n=1}^{\infty} \frac{\tau(n)}{Q_n}$$

converges absolutely.

Erdős and Straus proved that this series is irrational when

$$2 \leq a_1 \leq a_2 \leq \cdots,$$

see [5, Theorem 2.23]. In the same paper they conjectured that the monotonicity hypothesis can be removed and that it is enough to assume only $a_n \to \infty$; see [5, Conjecture 2.24]. The same problem is listed later in Erdős and Graham [4, p. 62] and in Erdős [3, p. 103]. In Bloom's online Erdős problem database, this is Problem #258 [1].

The purpose of this note is simply to record that the conjecture is now an immediate consequence of a theorem of Tao and Teräväinen [6, Theorem 1.1].

2. A tail lemma

The key point is that along infinitely many shifts, the divisor function is at most exponential.

Lemma 1. *Let $(a_n)_{n\geq 1}$ be a sequence of positive integers with $a_n \to \infty$, and let*

$$S := \sum_{n=1}^{\infty} \frac{\tau(n)}{Q_n}.$$

Assume that there exists a real number $\Lambda > 1$ and infinitely many integers N such that

$$\tau(N+k) \leq \Lambda^k \qquad (k \geq 1).$$

Then S is irrational.

Proof. Assume for contradiction that $S = A/B \in \mathbb{Q}$ with $A \in \mathbb{Z}$ and $B \in \mathbb{Z}_{>0}$. Fix an integer

$$M > \Lambda(2B + 1).$$

Since $a_n \to \infty$, there exists N_0 such that

$$a_n \geq M \qquad (n > N_0).$$

By hypothesis, there are infinitely many N for which $\tau(N + k) \leq \Lambda^k$ for every $k \geq 1$; choose one such $N \geq N_0$.

Now define

$$I_N := BQ_N \left(S - \sum_{n=1}^{N} \frac{\tau(n)}{Q_n} \right) = B \sum_{k=1}^{\infty} \frac{\tau(N + k)}{a_{N+1} a_{N+2} \cdots a_{N+k}}.$$

Since $BQ_N S = AQ_N \in \mathbb{Z}$ and

$$BQ_N \sum_{n=1}^{N} \frac{\tau(n)}{Q_n} = B \sum_{n=1}^{N} \tau(n) \frac{Q_N}{Q_n} \in \mathbb{Z},$$

we have $I_N \in \mathbb{Z}$. Also $I_N > 0$ because every term in the tail is positive.

On the other hand, using $\tau(N + k) \leq \Lambda^k$ and $a_{N+j} \geq M$ for all $j \geq 1$, we obtain

$$0 < I_N \leq B \sum_{k=1}^{\infty} \left(\frac{\Lambda}{M} \right)^k < B \sum_{k=1}^{\infty} \left(\frac{1}{2B + 1} \right)^k = \frac{B}{2B} = \frac{1}{2}.$$

This contradicts the fact that I_N is a positive integer. Therefore S is irrational. □

3. Application of Tao–Teräväinen

We now invoke the following theorem of Tao and Teräväinen [6, Theorem 1.1].

Theorem 1 (Tao–Teräväinen)**.** *There exists an absolute constant $C > 0$ such that for infinitely many positive integers N, one has*

$$\omega(N + k) \leq \Omega(N + k) \leq Ck \qquad (k \geq 1).$$

In particular,

$$\tau(N + k) \leq 2^{Ck} \qquad (k \geq 1)$$

for those same integers N.

Proof. The first assertion is exactly [6, Theorem 1.1]. For the second, write

$$N + k = \prod_{i=1}^{r} p_i^{\alpha_i}.$$

Then

$$\tau(N + k) = \prod_{i=1}^{r} (\alpha_i + 1) \leq \prod_{i=1}^{r} 2^{\alpha_i} = 2^{\Omega(N+k)} \leq 2^{Ck}.$$

□

Combining Lemma 1 with Theorem 1, we obtain the desired result.

Theorem 2. *Let* $(a_n)_{n\geq 1}$ *be a sequence of positive integers with* $a_n \to \infty$*. Then*

$$\sum_{n=1}^{\infty} \frac{\tau(n)}{a_1 a_2 \cdots a_n}$$

is irrational.

Proof. Apply Lemma 1 with $\Lambda = 2^C$, where C is the constant from Theorem 1. □

Remark 1. *Thus Tao and Teräväinen's theorem settles the conjecture of Erdős and Straus [5, Conjecture 2.24], and hence also Erdős Problem #258 as recorded in [1]. The proof is completely independent of any monotonicity of* (a_n)*: the only input from the sequence* (a_n) *is the fact that its terms eventually dominate any fixed constant.*

References

[1] T. F. Bloom, *Erdős Problem #258*, https://www.erdosproblems.com/258, accessed 2026-04-14.

[2] P. Erdős, *On arithmetical properties of Lambert series*, J. Indian Math. Soc. (N.S.) **12** (1948), 63–66.

[3] P. Erdős, *On the irrationality of certain series: problems and results*, in *New Advances in Transcendence Theory* (Durham, 1986), ed. A. Baker, Cambridge Univ. Press, Cambridge, 1988, pp. 102–109.

[4] P. Erdős and R. L. Graham, *Old and New Problems and Results in Combinatorial Number Theory*, Monographies de L'Enseignement Mathématique, vol. 28, Université de Genève, Geneva, 1980.

[5] P. Erdős and E. G. Straus, *Some number theoretic results*, Pacific J. Math. **36** (1971), 635–646.

[6] T. Tao and J. Teräväinen, *Quantitative correlations and some problems on prime factors of consecutive integers*, arXiv:2512.01739, 2025.