A corrected completion note for Erdős Problem #279

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Abstract

We distinguish the strong formulation relevant to Erdős Problem #279 from the weaker density-one formulation that is actually supported by Erdős–Graham. For a set $A \subseteq \mathbb{N}$ and a threshold $K \geq 1$, let $P_K(A)$ denote the statement that one can choose fixed least residues $a_n \pmod{n}$ $(n \in A)$ such that every sufficiently large integer is of the form $a_n + tn$ with $t \geq K$. We show that a universal strong theorem for $K = 1$ would imply $P_K(A)$ for every K, and that a universal strong theorem for $K = 2$ would already settle the prime case $P_3(\mathcal{P})$. This is why no such theorem can be inserted casually as a known fact. We then prove, self-containedly, the correct density-one result: if $\sum_{n \in A} 1/n = \infty$, then for every fixed K there are least residues $a_n \pmod{n}$ such that the exceptional set of integers not representable as $a_n + tn$ with $t \geq K$ has natural density 0.

1 Two formulations

Let $A \subseteq \mathbb{N}$, and for each $n \in A$ choose a *least residue*

$$0 \leq a_n < n.$$

The least-residue normalization is essential: without it, one can replace a_n by $a_n - \ell n$ and thereby change the lower bound on t without changing the congruence class.

For $K \geq 1$, define the strong property $P_K(A)$ by:

there are least residues a_n $(n \in A)$ such that every sufficiently large integer N can be written in the form $N = a_n + tn$ for some $n \in A$ and some integer $t \geq K$.

Also define the density-one property $D_K(A)$ by:

there are least residues a_n $(n \in A)$ such that the exceptional set

$$E_K(A) := \{N \geq 1 : N \neq a_n + tn \text{ for all } n \in A,\ t \geq K\}$$

has natural density 0.

Problem #279 asks whether $P_3(\mathcal{P})$ holds for the set of primes $\mathcal{P}$; this is the first open case on Bloom's webpage [1].

2 Why a strong $K = 1$ or $K = 2$ theorem would be unexpectedly strong

We first record two simple lemmas.

Lemma 1 (partition lemma)**.** *If $A \subseteq \mathbb{N}$ satisfies*

$$\sum_{n \in A} \frac{1}{n} = \infty,$$

then for every integer $m \geq 1$ one can partition A as

$$A = A_0 \sqcup A_1 \sqcup \cdots \sqcup A_{m-1}$$

so that each A_r still satisfies

$$\sum_{n \in A_r} \frac{1}{n} = \infty.$$

Proof. Choose disjoint finite blocks $B_1, B_2, \dots \subseteq A$ with

$$\sum_{n \in B_j} \frac{1}{n} > 1 \qquad (j \geq 1),$$

which is possible because the reciprocal sum over A diverges. Distribute these blocks cyclically by setting

$$A_r := \bigcup_{j \equiv r+1 \pmod{m}} B_j \qquad (0 \leq r < m).$$

Each A_r contains infinitely many blocks, each contributing more than 1, so each reciprocal sum diverges. □

Lemma 2 (multiplication lemma)**.** *Let $m \geq 1$. Suppose*

$$A = A_0 \sqcup A_1 \sqcup \cdots \sqcup A_{m-1}$$

and suppose $P_K(A_r)$ holds for every $r \in \{0, \dots, m-1\}$. Then $P_{mK}(A)$ holds.

Proof. For each r, choose least residues $b_n^{(r)} \in \{0, \dots, n-1\}$ $(n \in A_r)$ witnessing $P_K(A_r)$. Thus every sufficiently large integer M can be written

$$M = b_n^{(r)} + sn$$

for some $n \in A_r$ and some integer $s \geq K$.

For $n \in A_r$, define a least residue a_n by

$$a_n \equiv m b_n^{(r)} + r \pmod{n}, \qquad 0 \leq a_n < n.$$

Let N be sufficiently large, and write

$$N = mM + r, \qquad 0 \leq r < m.$$

For large enough N, the integer M is large enough for $P_K(A_r)$, so there are $n \in A_r$ and $s \geq K$ with

$$M = b_n^{(r)} + sn.$$

Since a_n is the least residue of $mb_n^{(r)} + r$ modulo n, there is an integer $\lambda \geq 0$ such that

$$mb_n^{(r)} + r = a_n + \lambda n.$$

Therefore

$$N = mM + r = m(b_n^{(r)} + sn) + r = a_n + (ms + \lambda)n.$$

Because $s \geq K$ and $\lambda \geq 0$, the coefficient of n is at least mK. Hence $P_{mK}(A)$ holds. ☐

Corollary 3. *If $P_1(A)$ held for every $A \subseteq \mathbb{N}$ with $\sum_{n\in A} 1/n = \infty$, then $P_K(A)$ would hold for every such A and every $K \geq 1$.*

Proof. Fix K. By Lemma 1, partition A into K subsets each with divergent reciprocal sum. Apply the hypothetical P_1 theorem on each piece, then invoke Lemma 2 with $m = K$. ☐

The case $K = 2$ is already dangerous for the primes.

Lemma 4 (parity lift)**.** *Let B be a set of odd integers. If $P_2(B)$ holds, then $P_3(B \cup \{2\})$ holds.*

Proof. Choose least residues $b_q \pmod q$ $(q \in B)$ witnessing $P_2(B)$. Define least residues for $B \cup \{2\}$ by

$$a_2 := 1, \qquad a_q \equiv 2b_q \pmod q \quad (q \in B), \qquad 0 \leq a_q < q.$$

Let N be sufficiently large.

If N is odd, then

$$N = 1 + 2t, \qquad t = \frac{N-1}{2}.$$

For all sufficiently large odd N we have $t \geq 3$, so such N are covered by the modulus 2.

If N is even, write $N = 2M$. Since $P_2(B)$ holds, for all sufficiently large M there exist $q \in B$ and $s \geq 2$ such that

$$M = b_q + sq.$$

Hence

$$N = 2b_q + 2sq.$$

Because $0 \leq a_q < q$ and $a_q \equiv 2b_q \pmod q$, there is an $\varepsilon \in \{0, 1\}$ such that

$$2b_q = a_q + \varepsilon q.$$

Therefore

$$N = a_q + (2s + \varepsilon)q.$$

Since $s \geq 2$, we have $2s + \varepsilon \geq 4 \geq 3$. Thus all sufficiently large even N are also covered, and $P_3(B \cup \{2\})$ follows. ☐

Corollary 5. *If $P_2(A)$ held for every $A \subseteq \mathbb{N}$ with $\sum_{n\in A} 1/n = \infty$, then $P_3(\mathcal{P})$ would hold.*

Proof. Apply the hypothesis to the set B of odd primes. Its reciprocal sum diverges. Then Lemma 4 yields $P_3(B \cup \{2\}) = P_3(\mathcal{P})$. ☐

Remark 6. Corollaries 3 and 5 show why a universal strong $K = 1$ or $K = 2$ theorem cannot simply be imported as a routine known fact. Such a theorem would have immediate consequences far beyond the currently established literature.

3 What Erdős–Graham actually support

The relevant discussion on p. 29 of Erdős–Graham [2, p. 29] is about the density-one formulation, not the strong one. In their notation, if

$$\sum_i \frac{1}{n_i} = \infty,$$

then one can choose residues so that *almost all* integers are of the form

$$a_i + tn_i \qquad (t > 2),$$

and they immediately add that this is certainly not true for *all large integers*. They then formulate the stronger finite-exception question and say that even the case corresponding to $P_3(\mathcal{P})$ seems difficult [2, p. 29].

So the primary source does *not* provide a proof of the normalized strong properties P_1 or P_2. What it points to is the density-one property D_3, and in fact the same probabilistic idea gives D_K for every fixed K.

4 A self-contained proof of the density-one theorem

Theorem 7. *Let $A \subseteq \mathbb{N}$ satisfy*

$$\sum_{n\in A} \frac{1}{n} = \infty.$$

Fix an integer $K \geq 1$. Then there exist least residues $a_n \in \{0, 1, \ldots, n-1\}$ $(n \in A)$ such that the exceptional set

$$E_K(A) = \{m \geq 1 : m \neq a_n + tn \text{ for every } n \in A \text{ and every integer } t \geq K\}$$

has natural density 0.

Proof. Choose the residues a_n independently and uniformly from $\{0, 1, \ldots, n-1\}$.

Fix $m \geq 1$ and $n \in A$. Let $B_{m,n}$ be the event that

$$m = a_n + tn$$

for some integer $t \geq K$. Because a_n is required to be the least residue, this happens exactly when a_n is the residue class of m modulo n and $\lfloor m/n \rfloor \geq K$. Equivalently,

$$\mathbb{P}(B_{m,n}) = \begin{cases} 1/n, & n \leq m/K, \\ 0, & n > m/K. \end{cases}$$

Since the choices of the residues are independent across n, the probability that m is exceptional equals

$$q_m := \mathbb{P}(m \in E_K(A)) = \prod_{\substack{n\in A \\ n\leq m/K}} \left(1 - \frac{1}{n}\right).$$

The sequence (q_m) is nonincreasing, and

$$\log q_m = \sum_{\substack{n\in A\\ n\le m/K}} \log\left(1-\frac{1}{n}\right) \le -\sum_{\substack{n\in A\\ n\le m/K}} \frac{1}{n}.$$

Because the reciprocal sum over A diverges, the right-hand side tends to $-\infty$, so

$$q_m \to 0 \qquad (m\to\infty).$$

Choose an increasing sequence

$$1 = M_1 < M_2 < M_3 < \cdots$$

so that

$$q_m \le 4^{-j} \qquad \text{for all } m \ge M_j.$$

Let

$$I_j := [M_j, M_{j+1}) \cap \mathbb{N}, \qquad X_j := |E_K(A) \cap I_j|.$$

Then

$$\mathbb{E}X_j = \sum_{m\in I_j} q_m \le 4^{-j}|I_j|.$$

By Markov's inequality,

$$\mathbb{P}(X_j > 2^{-j}|I_j|) \le 2^j \frac{\mathbb{E}X_j}{|I_j|} \le 2^{-j}.$$

Since $\sum_j 2^{-j} < \infty$, the first Borel–Cantelli lemma shows that with probability 1 all but finitely many j satisfy

$$X_j \le 2^{-j}|I_j|.$$

Fix a realization of the residues with this property.

Now let x be large, and choose J so that $x \in I_J$. Then

$$|E_K(A) \cap [1,x]| \le C + \sum_{j_0\le j<J} 2^{-j}|I_j| + 2^{-J}|I_J|$$

for some constants C and j_0 accounting for the finitely many bad initial blocks. Given $\varepsilon > 0$, choose $J_\varepsilon \ge j_0$ so large that $2^{-j} < \varepsilon$ for all $j \ge J_\varepsilon$. Then, for large enough x,

$$|E_K(A) \cap [1,x]| \le C_\varepsilon + \varepsilon \sum_{J_\varepsilon\le j\le J} |I_j| \le C_\varepsilon + \varepsilon x.$$

Dividing by x and letting $x \to \infty$, we obtain

$$\limsup_{x\to\infty} \frac{|E_K(A)\cap[1,x]|}{x} \le \varepsilon.$$

Because $\varepsilon > 0$ is arbitrary, the natural density of $E_K(A)$ is 0. □

Corollary 8. *If $\sum_{n\in A} 1/n = \infty$, then $D_K(A)$ holds for every fixed integer $K \ge 1$. In particular, $D_1(A)$, $D_2(A)$, and $D_3(A)$ all hold.*

Remark 9. Thus one really can prove "$k = 1$" and "$k = 2$" statements in the density-one sense, and in fact for every fixed threshold. What is *not* justified by the source—and would in fact have unexpectedly strong consequences by Section 2—is the corresponding strong finite-exception statement P_1 or P_2.

Conclusion

The corrected picture is this.

- The strong problem relevant to Erdős Problem #279 is $P_3(\mathcal{P})$ and remains open on the current webpage [1].
- A universal strong theorem P_1 would imply P_K for every K by the multiplication lemma.
- A universal strong theorem P_2 would already imply $P_3(\mathcal{P})$ by the parity lift.
- The statement that one can actually prove cleanly and self-containedly is the density-one theorem $D_K(A)$ of Theorem 7, valid for every fixed K whenever $\sum_{n\in A} 1/n = \infty$.

So the honest completion is not a proof of the strong $K = 1$ or $K = 2$ assertions, but a correction of them to the density-one statement together with a full proof of that corrected theorem.

References

[1] T. F. Bloom, *Erdős Problem #279*, https://www.erdosproblems.com/279, accessed 16 April 2026.

[2] P. Erdős and R. L. Graham, *Old and New Problems and Results in Combinatorial Number Theory*, Monographies de L'Enseignement Mathématique, vol. 28, L'Enseignement Mathématique, Geneva, 1980.