Signed Zero-Sums of Reciprocal Squares on the Squares: A Computational Attack Plan

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1 Problem statement (squares case)

Erdős posed the following problem [1]. Let

$$A = \{n^2 : n \in \mathbb{N},\ n \geq 2\}$$

(the squares excluding 1). Given a non-constant function

$$f : A \to \{-1, +1\},$$

must there exist a finite non-empty $S \subset A$ such that

$$\sum_{x \in S} \frac{f(x)}{x} = 0\ ?$$

2 Reformulation in square roots

Define $g : \{2, 3, 4, \dots\} \to \{-1, +1\}$ by

$$g(m) := f(m^2).$$

Then the question becomes:

> Given non-constant $g(m) \in \{\pm 1\}$ for $m \geq 2$, must there exist a finite non-empty $T \subset \{2, 3, \dots\}$ such that
>
> $$\sum_{m \in T} \frac{g(m)}{m^2} = 0\ ?$$

Equivalently, does every non-constant $\{\pm 1\}$-coloring of $\{2, 3, \dots\}$ contain a finite configuration

$$\sum_{u \in U} \frac{1}{u^2} = \sum_{v \in V} \frac{1}{v^2}$$

such that all $u \in U$ have one color and all $v \in V$ have the opposite color?

Mapping back to squares: if such U, V are found (possibly after scaling, see below), then

$$S = \{(mu)^2 : u \in U\}\ \cup\ \{(mv)^2 : v \in V\} \subset A$$

satisfies $\sum_{x \in S} f(x)/x = 0$ with the corresponding sign assignment.

3 Scaling lemma (critical for computation)

Lemma 1 (Scale invariance)**.** *If U, V are finite sets of positive integers with*

$$\sum_{u\in U}\frac{1}{u^2}=\sum_{v\in V}\frac{1}{v^2},$$

then for every $m \in \mathbb{N}$,

$$\sum_{u\in U}\frac{1}{(mu)^2}=\sum_{v\in V}\frac{1}{(mv)^2}.$$

Thus each "base identity" generates an infinite family of identities by scaling.

4 Constraint view: forbidden monochromatic splits

Fix a base identity (U, V) with

$$\sum_{u\in U}\frac{1}{u^2}=\sum_{v\in V}\frac{1}{v^2}.$$

For each scale m, define the scaled sets $mU = \{mu : u \in U\}$ and $mV = \{mv : v \in V\}$. Then a zero-sum is obtained if either

$$g|_{mU} \equiv +1 \ \text{ and } \ g|_{mV} \equiv -1, \quad \text{or} \quad g|_{mU} \equiv -1 \ \text{ and } \ g|_{mV} \equiv +1.$$

Therefore, any *counterexample coloring* g must avoid both sign patterns for every m and every identity.

SAT/CNF encoding (sketch). Represent $g(n)$ by a Boolean variable X_n (e.g. $X_n = \text{true}$ means $g(n) = +1$). For each (U, V) and each admissible scale m (within a finite search universe), add two forbidden assignments:

$$(\forall u \in U,\ X_{mu} = \text{true}) \wedge (\forall v \in V,\ X_{mv} = \text{false}) \ \text{ is forbidden},$$

and the opposite assignment is forbidden. Each can be converted into CNF by a single clause that forces at least one variable to flip:

$$\Big(\bigvee_{u\in U} \neg X_{mu}\Big) \ \vee \ \Big(\bigvee_{v\in V} X_{mv}\Big),$$

and similarly with $(X, \neg X)$ swapped.

5 Identity library (all exact; suitable as a base set)

Below, each identity is written as a pair (U, V) meaning

$$\sum_{u\in U}\frac{1}{u^2}=\sum_{v\in V}\frac{1}{v^2}.$$

All identities listed here are exact and can be verified by clearing denominators or by exact rational arithmetic.

5.1 (I) A classical "singleton = sum" identity

$$\mathbf{I_1}: \quad \Big\{2\Big\} = \Big\{3, 4, 5, 7, 12, 15, 20, 28, 35\Big\}.$$
$$\text{i.e. } \frac{1}{2^2} = \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{12^2} + \frac{1}{15^2} + \frac{1}{20^2} + \frac{1}{28^2} + \frac{1}{35^2}.$$

5.2 (II) Small singleton decompositions (local "propagation rules")

$$\mathbf{I_2}: \quad \{12\} = \{15, 20\}, \qquad \text{i.e. } \frac{1}{12^2} = \frac{1}{15^2} + \frac{1}{20^2}.$$
$$\mathbf{I_3}: \quad \{6\} = \{7, 14, 21\}, \qquad \text{i.e. } \frac{1}{6^2} = \frac{1}{7^2} + \frac{1}{14^2} + \frac{1}{21^2}.$$
$$\mathbf{I_4}: \quad \{14\} = \{15, 42, 105\}, \qquad \text{i.e. } \frac{1}{14^2} = \frac{1}{15^2} + \frac{1}{42^2} + \frac{1}{105^2}.$$
$$\mathbf{I_5}: \quad \{20\} = \{21, 84, 105\}, \qquad \text{i.e. } \frac{1}{20^2} = \frac{1}{21^2} + \frac{1}{84^2} + \frac{1}{105^2}.$$
$$\mathbf{I_6}: \quad \{5\} = \{6, 10, 30, 35, 70, 105\}, \qquad \text{i.e. } \frac{1}{5^2} = \frac{1}{6^2} + \frac{1}{10^2} + \frac{1}{30^2} + \frac{1}{35^2} + \frac{1}{70^2} + \frac{1}{105^2}.$$
$$\mathbf{I_7}: \quad \{7\} = \{10, 12, 20, 35, 84, 420\}, \qquad \text{i.e. } \frac{1}{7^2} = \frac{1}{10^2} + \frac{1}{12^2} + \frac{1}{20^2} + \frac{1}{35^2} + \frac{1}{84^2} + \frac{1}{420^2}.$$
$$\mathbf{I_8}: \quad \{10\} = \{12, 20, 60, 70, 140, 210\}, \qquad \text{i.e. } \frac{1}{10^2} = \frac{1}{12^2} + \frac{1}{20^2} + \frac{1}{60^2} + \frac{1}{70^2} + \frac{1}{140^2} + \frac{1}{210^2}.$$
$$\mathbf{I_9}: \quad \{70\} = \{75, 210, 525\}, \qquad \text{i.e. } \frac{1}{70^2} = \frac{1}{75^2} + \frac{1}{210^2} + \frac{1}{525^2}.$$

5.3 (III) Two different decompositions of $1/3^2$ (useful for cancellations)

$$\mathbf{I_{10}}: \quad \{3\} = \{4, 6, 10, 12, 20, 30, 60\},$$
$$\mathbf{I_{11}}: \quad \{3\} = \{5, 6, 7, 10, 14, 15, 21, 30\}.$$

5.4 (IV) A decomposition of $1/5^2$ (alternative to $\mathbf{I_6}$)

$$\mathbf{I_{12}}: \quad \{5\} = \{6, 10, 25, 50, 75, 150\}.$$

5.5 (V) Balanced "bridge" identities (small and highly structured)

These are particularly valuable as direct targets: if g is constant on each side with opposite signs (after some scaling), one immediately obtains a zero-sum.

$$\mathbf{I_{13}}: \quad \{5, 21, 35\} = \{6, 10, 14, 70\}.$$
$$\mathbf{I_{14}}: \quad \{5, 35, 63\} = \{6, 10, 18, 70\}.$$
$$\mathbf{I_{15}}: \quad \{5, 7, 15, 21\} = \{4, 20, 28, 42, 60\}.$$

5.6 (VI) A key cancellation identity from $\mathbf{I_{10}}$ and $\mathbf{I_{11}}$

Cancelling common terms between $\mathbf{I_{10}}$ and $\mathbf{I_{11}}$ yields:

$$\mathbf{I_{16}}: \quad \{4, 12, 20, 60\} = \{5, 7, 14, 15, 21\}.$$

5.7 (VII) Additional balanced identities derived by scaling and cancellation

$$\mathbf{I_{17}}: \quad \{15, 25, 35, 60, 75, 140, 175\} = \{12, 50, 150, 300\}.$$
$$\mathbf{I_{18}}: \quad \{20, 50, 60, 100, 300\} = \{18, 75, 225, 450\}.$$

5.8 (VIII) Small 2-vs-2 "gadgets" (many exist; these are examples)

$$\mathbf{I_{19}}: \quad \{6, 9\} = \{5, 90\},$$
$$\mathbf{I_{20}}: \quad \{18, 21\} = \{14, 63\}.$$

6 Suggested computational workflow

1. Choose a finite search universe $K \subset \{2, 3, \dots\}$ (e.g. all $n \leq N$, or all divisors of a large smooth number, or all 7-smooth numbers up to a bound).
2. Fix a base identity list $\mathcal{B} = \{\mathbf{I_1}, \dots, \mathbf{I_{20}}\}$.
3. For each $(U, V) \in \mathcal{B}$ and for each scale m such that $mU \cup mV \subseteq K$, add the two forbidden monochromatic-split constraints (both orientations).
4. Add "non-constant" constraints (e.g. $\exists a, b \in K$ with $g(a) \neq g(b)$).
5. Run SAT to find an avoiding coloring. If SAT says UNSAT, you have a *finite certificate* that K forces a zero-sum using only identities in $\mathcal{B}$.
6. If SAT says SAT (an avoiding coloring exists), enlarge K and/or enrich $\mathcal{B}$ by automatically searching for additional identities supported on K:
$$\sum_{u \in U} \frac{1}{u^2} = \sum_{v \in V} \frac{1}{v^2}, \quad U, V \subseteq K, \ |U|, |V| \leq L,$$
then add them to $\mathcal{B}$ and repeat.

Remark (identity search). A practical identity miner is to clear denominators using $M = \mathrm{lcm}(K)$ and search for equal-sum-of-squares relations

$$\sum_{u \in U} \left(\frac{M}{u}\right)^2 = \sum_{v \in V} \left(\frac{M}{v}\right)^2,$$

then translate back to reciprocal-square identities.

References

[1] T. Bloom, Erdős Problem #318 `https://www.erdosproblems.com/318`