A counterexample to the literal formulation of Erdős Problem #326

On `erdosproblems.com`, an additive basis of order 2 means a set $A \subseteq \mathbb{N}$ such that all sufficiently large integers are sums of at most 2 elements of A. Under that definition, the statement

> *"For every additive basis $A \subseteq \mathbb{N}$ of order 2, there is a basis $B = \{b_1 < b_2 < \cdots\} \subseteq A$ such that $\lim b_k/k^2$ does not exist"*

is false.

Define

$$C := \left\{ \sum_{j \geq 0} \varepsilon_j 3^j : \ \varepsilon_j \in \{0,1\}, \ \varepsilon_j = 0 \text{ for all but finitely many } j \right\}$$

and

$$A := C \setminus \{0\}.$$

Thus A is the set of positive integers whose ternary digits are all 0 or 1.

Lemma 1. *Every $n \in \mathbb{N}$ can be written as $n = x + y$ with $x, y \in C$. Consequently A is an additive basis of order 2.*

Proof. Write

$$n = \sum_{j \geq 0} \delta_j 3^j, \qquad \delta_j \in \{0,1,2\}.$$

Set

$$x := \sum_{j \geq 0} \min(\delta_j, 1) 3^j, \qquad y := \sum_{j \geq 0} \mathbf{1}_{\{\delta_j = 2\}} 3^j.$$

Then $x, y \in C$ and $n = x + y$. If one of x, y is 0, then $n \in A$; otherwise both lie in A. Hence every positive integer is the sum of at most two elements of A. □

Lemma 2. *For each $c \in C$, the equation $x + y = 2c$ with $x, y \in C$ implies $x = y = c$.*

Proof. We induct on c. The case $c = 0$ is trivial. Write

$$x = x_0 + 3x_1, \qquad y = y_0 + 3y_1, \qquad c = c_0 + 3c_1,$$

with $x_0, y_0, c_0 \in \{0, 1\}$ and $x_1, y_1, c_1 \in C$. From $x + y = 2c$ we obtain

$$x_0 + y_0 \equiv 2c_0 \pmod 3.$$

Both sides lie in $\{0, 1, 2\}$, so in fact $x_0 + y_0 = 2c_0$. Hence $x_0 = y_0 = c_0$. Subtracting and dividing by 3 gives

$$x_1 + y_1 = 2c_1.$$

By the induction hypothesis, $x_1 = y_1 = c_1$, and therefore $x = y = c$. $\square$

Lemma 3. *If $B \subseteq A$ is an additive basis of order 2, then $A \setminus B$ is finite.*

Proof. Let $c \in A \setminus B$. Since $c > 0$, the ternary expansion of $2c$ has at least one digit 2, so $2c \notin A$. Also, if $u, v \in A$ and $u + v = 2c$, then $u = v = c$ by the previous lemma. Thus $2c$ is not representable as a sum of at most two elements of B.

If $A \setminus B$ were infinite, then the distinct integers $2c$ with $c \in A \setminus B$ would give infinitely many unbounded exceptions to the basis property, contradicting that B is an additive basis of order 2. $\square$

Lemma 4. *If $A = \{a_1 < a_2 < \cdots\}$, then $a_k/k^2 \to 0$.*

Proof. For each $m \geq 1$, the elements of $A \cap [1, 3^m)$ are exactly the positive integers whose first m ternary digits lie in $\{0, 1\}$. Therefore

$$|A \cap [1, 3^m)| = 2^m - 1.$$

Hence, whenever $2^m - 1 \leq k < 2^{m+1} - 1$, we have $a_k < 3^{m+1}$. Therefore

$$0 \leq \frac{a_k}{k^2} \leq \frac{3^{m+1}}{(2^m - 1)^2} \to 0,$$

since $3/4 < 1$. $\square$

Theorem 1. *There exists an additive basis $A \subseteq \mathbb{N}$ of order 2 such that for every additive basis $B = \{b_1 < b_2 < \cdots\} \subseteq A$ of order 2, the limit $\lim_{k \to \infty} b_k/k^2$ exists; in fact it equals 0.*

Proof. Take the set A above, and let $B \subseteq A$ be an additive basis of order 2. By the previous lemma, $A \setminus B$ is finite; say $|A \setminus B| = r$. Then

$$a_k \leq b_k \leq a_{k+r} \qquad (k \geq 1).$$

Since $a_k/k^2 \to 0$, we also have

$$0 \leq \frac{b_k}{k^2} \leq \frac{a_{k+r}}{k^2} = \frac{a_{k+r}}{(k+r)^2} \cdot \frac{(k+r)^2}{k^2} \to 0.$$

Thus $b_k/k^2 \to 0$. In particular, no such B can make b_k/k^2 fail to converge. ☐

Remark. If one uses the alternative convention that a basis of order 2 means a set from which all sufficiently large integers are representable as the sum of exactly two positive elements, then the same idea works after shifting by 1: one may use $1 + C$ in place of A.