A toral packet for the Ulam sequence $U(1,2)$: sum-free cores, phase laws, and spectral trapping

Przemyslaw Chojecki

20th March 2026

Abstract

Let $\mathcal{U} = U(1,2)$ be the classical Ulam sequence, let $x \in \{0,1\}^{\mathbb{Z}}$ be its zero-extended bi-infinite indicator, let $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$, and let X be the shift-orbit closure of x. We prove that positive upper density forces a one-dimensional Kronecker component in a Furstenberg limit of the actual Ulam orbit; when the extracted frequency lies outside $\{0,\pi\}$, this gives a genuine rank-1 toral packet. More precisely, if $\overline{d}(\mathcal{U}) > 0$, then there exist a shift-invariant probability measure μ on X, a frequency $\alpha \in \mathbb{T}$, and a nonzero eigenfunction $g \in L^\infty(X,\mu)$ with $g \circ T = e^{i\alpha} g$ such that

$$|\int_X (y_0 - \delta)\overline{g(y)}\,d\mu(y)| \geq \frac{\delta^3}{256\sqrt{2}}, \qquad \delta = \int_X y_0\,d\mu.$$

In particular, the Koopman spectral measure of the centered coordinate has an atom of mass at least $\delta^6/2^{17}$.

The paper pushes this packet statement in four directions. First, every initial segment $A_N = \mathcal{U} \cap [1,N]$ has at most $3|A_N|$ Schur triples, hence by Green's theorem is $o(N)$-close to a genuinely sum-free set. Second, if one fixes a positive-density subsequence along which the empirical orbit measures converge, then every accumulation point of the maximizing finite-scale Fourier phases is forced into a finite threshold atom set of the limiting Koopman spectrum; in particular, the maximizing phases cannot drift through a continuum. Third, in the irrational case one obtains a non-uniform absolutely continuous limiting phase law for the actual Ulam numbers, together with an interval arc carrying quantitatively excess mass and the same phase law on a sequence of genuine sum-free cores. Fourth, we prove a general window-correlation theorem at the varying maximizing phases and isolate two further inputs—quantitative phase stabilization and circle-factor measurability of the coordinate—that would yield a fixed-angle window *measure* law; a boundary-null representative of the window would then upgrade this to literal indicator-window counts.

This gives the infinitary reduction suggested by Ross: a subsequential hidden signal for the actual Ulam sequence becomes a genuine one-dimensional spectral component and, after fixing a limiting orbit measure, is trapped in finitely many spectral atoms of one limiting dynamical system.

2020 Mathematics Subject Classification. 11B75, 11P70, 37A45.
Keywords. Ulam sequence, sum-free set, Fourier coefficient, Kronecker factor, Furstenberg limit.

1 Introduction

The classical Ulam sequence $U(1,2)$ is defined greedily by starting from $1,2$ and repeatedly adjoining the smallest integer that has a unique representation as a sum of two distinct earlier terms. The sequence was introduced by Ulam and later discussed in detail by Queneau [14, 10]. Despite its simple definition, it remains poorly understood. Classical work of Finch and of Schmerl–Spiegel established rigidity for several regular 1-additive families [4, 12]; more recent work of Cassaigne–Finch, Kuca, Hinman–Kuca–Schlesinger–Sheydvasser, and Kravitz–Steinerberger develops additional structural and rigidity phenomena for related additive sequences and Ulam-type sets [1, 9, 7, 8]. On the experimental side, Steinerberger isolated a striking hidden frequency and a non-uniform phase profile modulo 2π for $U(1, 2)$ [13]. Very recently Clément and Steinerberger proved new bounds on growth and small gaps for the Ulam sequence [2].

The present note is concerned with an infinitary formulation of that hidden-frequency phenomenon. The starting point is a finite-scale large Fourier coefficient, of the type already emphasized by Ross in his thesis [11]. Ross proved a large-Fourier-coefficient theorem for almost sum-free sets, showed that a nonzero limit point of the finite-scale maximizing phases would yield a genuine nonzero Fourier coefficient provided the corresponding infinitary limit exists, derived interval-correlation statements from such coefficients, and explicitly singled out ultralimits, energy increment, and arithmetic regularity as the next tools to try [11]. From the same finite-scale information we construct, by compactness, an invariant joining on $X \times \mathbb{T}$. The joining yields a genuine eigenfunction on a Furstenberg limit of the sequence itself. We use the word *packet* loosely for the resulting one-dimensional toral component; when the eigenvalue is non-real, the eigenline together with its complex conjugate forms a genuine real rank-1 packet, while $\alpha = \pi$ is the usual order-two real edge case.

A second theme of the paper is that the Ulam sequence is extremely close to being sum-free at every finite scale. Indeed, the defining unique-representation property implies that every element of $A_N := \mathcal{U} \cap [1, N]$ contributes at most three ordered Schur triples. Green's theorem on almost-sum-free sets therefore produces a sequence of genuine sum-free cores $B_N \subset [1, N]$ with $|A_N \triangle B_N| = o(N)$ [5]. This observation interacts naturally with the toral picture: in the irrational case the same limiting phase law already appears on the sum-free cores.

The main results may be summarized as follows.

(i) Positive upper density forces a one-dimensional Kronecker component on a Furstenberg limit of the actual Ulam orbit, with explicit correlation against the centered coordinate; when the extracted frequency lies outside $\{0, \pi\}$, this is a genuine toral packet.

(ii) The Kronecker-component theorem automatically gives a genuine Koopman spectral atom. If the extracted frequency is 0, this forces a quantitative splitting of the coordinate density among ergodic invariant measures on the orbit closure.

(iii) In the irrational case the circle marginal is Haar measure, hence one obtains a non-uniform absolutely continuous limiting phase law for the actual Ulam numbers, an interval arc with excess mass, and the same phase law on the sum-free cores.

(iv) If one fixes a positive-density subsequence along which the empirical orbit measures converge, then the maximizing finite-scale phases are spectrally trapped near a finite threshold atom set of the limiting Koopman spectrum. At the level of the varying phases one also obtains a general window-correlation theorem. Two additional inputs—quantitative phase stabilization and measurability of the coordinate function with respect to the limiting circle factor—yield a fixed-angle window measure law; a boundary-null representative of the window would then promote this to literal indicator-window counts.

For the structural background relevant to the remaining rigidity problem, see Green, Green–Tao, and Eberhard [5, 6, 3]. No automorphic input is used anywhere in the arguments below.

2 The Ulam property, Schur triples, and a large finite-scale character

Write

$$\mathcal{U} = \{u_1 < u_2 < u_3 < \cdots\}, \qquad u_1 = 1, \quad u_2 = 2,$$

and for $n \geq 2$ let u_{n+1} be the least integer $> u_n$ having a unique representation $u_i + u_j$ with $i < j \leq n$.

For a set $A \subset \mathbb{N}$ define

$$r_A(m) := \#\{x < y \in A : x + y = m\}$$

and the ordered representation count

$$s_A(m) := \sum_{x \in \mathbb{Z}} \mathbf{1}_A(x)\mathbf{1}_A(m - x).$$

For $N \geq 1$ let

$$A_N := \mathcal{U} \cap [1, N], \qquad M_N := |A_N|, \qquad \delta_N := \frac{M_N}{N}.$$

We extend all indicators by 0 outside their support.

Lemma 2.1. *For every integer $m > 2$ one has*

$$m \in \mathcal{U} \iff r_{\mathcal{U}}(m) = 1.$$

In particular, if $u \in \mathcal{U} \cap [3, N]$, then $r_{A_N}(u) = 1$.

Proof. If $m = u_{n+1} \in \mathcal{U}$, then by definition $r_{\{u_1,\ldots,u_n\}}(m) = 1$. Since all later Ulam numbers exceed m, no later term can create a new representation of m, so $r_{\mathcal{U}}(m) = 1$.

Conversely, assume $r_{\mathcal{U}}(m) = 1$. Let n be maximal with $u_n < m$. Every representation of m uses only terms $< m$, hence only $u_1, \ldots, u_n$, so $r_{\{u_1,\ldots,u_n\}}(m) = 1$. By definition of the Ulam recursion, this already implies $u_{n+1} \leq m$. On the other hand, maximality of n gives $u_{n+1} \geq m$. Hence $u_{n+1} = m$, so $m \in \mathcal{U}$.

The last assertion is immediate. □

Lemma 2.2. *If $u \in \mathcal{U} \cap [3, N]$, then*

$$s_{A_N}(u) = 2 + \mathbf{1}_{\{u \text{ even and } u/2 \in A_N\}} \leq 3.$$

Proof. By Lemma 2.1, $r_{A_N}(u) = 1$. Thus there is exactly one unordered representation $u = x + y$ with $x < y$ and $x, y \in A_N$, contributing exactly two ordered pairs (x, y) and (y, x). A diagonal pair $(u/2, u/2)$ contributes once exactly when u is even and $u/2 \in A_N$. This gives the formula. □

For a finite set $A \subset \mathbb{N}$ define its ordered Schur-triple count

$$T(A) := \#\{(x, y, z) \in A^3 : x + y = z\} = \sum_{z \in A} s_A(z).$$

Corollary 2.3. *For every $N \geq 1$ one has*

$$T(A_N) \leq 3M_N \leq 3N.$$

In particular,

$$T(A_N) = o(N^2).$$

Proof. For $z = 1$ one has $s_{A_N}(1) = 0$, while for $z = 2$ one has

$$s_{A_N}(2) = 1$$

coming from the diagonal pair $(1, 1)$. Thus every $z \in A_N \cap \{1, 2\}$ still satisfies

$$s_{A_N}(z) \leq 1 \leq 3.$$

If $z \in A_N \cap [3, N]$, then Lemma 2.2 gives $s_{A_N}(z) \leq 3$. Therefore

$$T(A_N) = \sum_{z \in A_N} s_{A_N}(z) \leq 3|A_N| = 3M_N \leq 3N.$$

The final assertion follows immediately. □

Corollary 2.4 (Green sum-free core)**.** *There exist sets $B_N \subset [1,N]$ such that each B_N is sum-free and*

$$|A_N \triangle B_N| = o(N).$$

In particular, if $N_k \to \infty$ and $\delta_{N_k} \to \delta$, then

$$\frac{|B_{N_k}|}{N_k} \to \delta.$$

Proof. By Corollary 2.3, the sets $A_N \subset [1,N]$ satisfy $T(A_N) = o(N^2)$. Green's corollary on almost-sum-free sets [5, Cor. 1.6] therefore gives sum-free sets $B_N \subset [1,N]$ with

$$|A_N \triangle B_N| = o(N).$$

The density statement is immediate from

$$||A_N| - |B_N|| \le |A_N \triangle B_N| = o(N).$$

$\square$

For finitely supported $f : \mathbb{Z} \to \mathbb{C}$ write

$$\widehat{f}(\theta) := \sum_{n\in\mathbb{Z}} f(n)\mathrm{e}^{-in\theta}, \qquad \theta \in [0, 2\pi],$$

and use Parseval in the form

$$\|f\|^2_{\ell^2(\mathbb{Z})} = \frac{1}{2\pi}\int_0^{2\pi} |\widehat{f}(\theta)|^2 \, d\theta.$$

Theorem 2.5 (Large finite-scale character)**.** *Define*

$$\mathcal{E}_N(\theta) := \sum_{n=1}^{N} (\mathbf{1}_{\mathcal{U}}(n) - \delta_N)\mathrm{e}^{-in\theta}.$$

If $\delta_N^3 N \ge 16$, then

$$\sup_{\theta\in[0,2\pi]} |\mathcal{E}_N(\theta)| \ge \frac{\delta_N^3}{8\sqrt{2}} N.$$

Since $\mathcal{E}_N(0) = 0$, the maximizing frequency is nonzero.

Proof. Fix N and abbreviate

$$A := A_N, \qquad M := M_N, \qquad \delta := \delta_N, \qquad I := \mathbf{1}_{[1,N]}.$$

Define

$$f := \mathbf{1}_A - \delta I, \qquad g := \mathbf{1}_A * \mathbf{1}_A - \delta^2(I * I).$$

Then

$$g = f * (\mathbf{1}_A + \delta I), \qquad \widehat{g}(\theta) = \widehat{f}(\theta)(\widehat{\mathbf{1}_A}(\theta) + \delta\widehat{I}(\theta)),$$

and $\widehat{f} = \mathcal{E}_N$.

We first lower-bound $\|g\|_2$. List the elements of A as

$$a_1 < a_2 < \cdots < a_M.$$

Since $a_j \ge j$, at least $M/2$ of these elements satisfy $u \ge M/2$. Fix such a $u \in A$. Because $u \le N$, one has $(I * I)(u) = u - 1$. Lemma 2.2 therefore gives

$$|g(u)| = |s_A(u) - \delta^2(u-1)| \ge \delta^2(u-1) - 3.$$

Since $u \geq M/2 = \delta N/2$,

$$\delta^2(u-1) - 3 \geq \delta^2\Big(\frac{\delta N}{2} - 1\Big) - 3 = \frac{\delta^3 N}{2} - \delta^2 - 3.$$

Because $\delta \leq 1$ and $\delta^3 N \geq 16$, the right-hand side is at least $\delta^3 N/4$. Therefore

$$\|g\|_2^2 \geq \frac{M}{2}\Big(\frac{\delta^3 N}{4}\Big)^2 = \frac{\delta^7}{32} N^3.$$

Next we upper-bound $\|g\|_2$ in terms of $\sup|\widehat{f}|$. By Parseval,

$$\begin{aligned}\|g\|_2^2 &= \frac{1}{2\pi}\int_0^{2\pi} |\widehat{f}(\theta)|^2 |\widehat{\mathbf{1}_A}(\theta) + \delta\widehat{I}(\theta)|^2 \, d\theta \\ &\leq \Big(\sup_\theta |\widehat{f}(\theta)|^2\Big) \, \|\mathbf{1}_A + \delta I\|_2^2.\end{aligned}$$

Now

$$\|\mathbf{1}_A + \delta I\|_2^2 = |A| + \delta^2 N + 2\delta|A| = \delta(1+3\delta)N \leq 4\delta N.$$

Hence

$$\|g\|_2^2 \leq 4\delta N \sup_\theta |\widehat{f}(\theta)|^2.$$

Combining the lower and upper bounds yields

$$\sup_\theta |\widehat{f}(\theta)|^2 \geq \frac{\delta^7 N^3/32}{4\delta N} = \frac{\delta^6 N^2}{128},$$

that is,

$$\sup_\theta |\widehat{f}(\theta)| \geq \frac{\delta^3}{8\sqrt{2}} N.$$

Since $\widehat{f} = \mathcal{E}_N$, this is the claimed estimate. Finally,

$$\mathcal{E}_N(0) = \sum_{n=1}^{N} (\mathbf{1}_{\mathcal{U}}(n) - \delta) = M - \delta N = 0,$$

so the maximizing frequency cannot be 0. □

3 Circle profiles and a one-dimensional Kronecker component on the actual orbit

Let

$$x = (x_n)_{n\in\mathbb{Z}} \in \{0,1\}^{\mathbb{Z}}$$

be the zero-extended bi-infinite indicator,

$$x_n := \begin{cases} 1, & n \geq 1 \text{ and } n \in \mathcal{U}, \\ 0, & n \geq 1 \text{ and } n \notin \mathcal{U}, \\ 0, & n \leq 0. \end{cases}$$

Let $X \subset \{0,1\}^{\mathbb{Z}}$ be the orbit closure of x under the left shift

$$(Ty)_m := y_{m+1}.$$

We write y_0 for the coordinate function on X. Let $\mathcal{B}_{\mathbb{T}}$ denote the σ-algebra on $X \times \mathbb{T}$ generated by the second coordinate.

Theorem 3.1 (Circle profile theorem)**.** *Assume that*

$$\overline{d}(\mathcal{U}) := \limsup_{N\to\infty} \delta_N > 0.$$

Then there exist

- *a number* $\delta \in (0,1)$*,*
- *a subsequence* $N_k \to \infty$*,*
- *angles* $\theta_k \in \mathbb{T}$ *with* $\theta_k \to \alpha \in \mathbb{T}$ *and*

$$|\mathcal{E}_{N_k}(\theta_k)| = \sup_{\theta\in\mathbb{T}} |\mathcal{E}_{N_k}(\theta)| \qquad \text{for every } k,$$

- *an* S_α*-invariant Borel probability measure* ν *on* $X \times \mathbb{T}$*, where*

$$S_\alpha(y,t) := (Ty, t+\alpha),$$

- *the first marginal* $\mu := (\mathrm{pr}_1)_*\nu$ *and the second marginal* $\lambda := (\mathrm{pr}_2)_*\nu$*.*

The measure μ *is* T*-invariant, the measure* λ *is* R_α*-invariant, and, if*

$$H := \mathbb{E}_\nu(y_0 \mid \mathcal{B}_{\mathbb{T}}) \in L^\infty(\mathbb{T},\lambda), \qquad R_\alpha(t) := t+\alpha, \qquad c_0 := \frac{\delta^3}{256\sqrt{2}},$$

then:

(i)

$$\int_X y_0 \, d\mu(y) = \delta;$$

(ii) $0 \le H \le 1$, $\int_{\mathbb{T}} H \, d\lambda = \delta$, *and*

$$\Big|\int_{\mathbb{T}} (H(t)-\delta)\mathrm{e}^{-it} \, d\lambda(t)\Big| \ge c_0;$$

(iii) *in particular,*

$$\|H-\delta\|_{L^2(\lambda)} \ge c_0.$$

Proof. Choose a subsequence $N_k \to \infty$ such that

$$\delta_{N_k} \to \delta > 0.$$

For each k choose a maximizer $\theta_k \in \mathbb{T}$ of $\theta \mapsto |\mathcal{E}_{N_k}(\theta)|$. For all sufficiently large k, one has $\delta_{N_k}^3 N_k \ge 16$, so Theorem 2.5 gives

$$\Big|\frac{1}{N_k}\sum_{n=1}^{N_k} (\mathbf{1}_{\mathcal{U}}(n) - \delta_{N_k})\mathrm{e}^{-in\theta_k}\Big| = \sup_{\theta\in\mathbb{T}} \Big|\frac{1}{N_k}\sum_{n=1}^{N_k} (\mathbf{1}_{\mathcal{U}}(n) - \delta_{N_k})\mathrm{e}^{-in\theta}\Big| \ge \frac{\delta_{N_k}^3}{8\sqrt{2}}. \tag{3.1}$$

Passing to a further subsequence, we may assume that

$$\theta_k \to \alpha \in \mathbb{T}.$$

Passing again to a further subsequence if necessary, we may arrange that for every k,

$$|\delta_{N_k} - \delta| \le \frac{\delta^3}{256\sqrt{2}} \qquad \text{and} \qquad \delta_{N_k} \ge \frac{\delta}{2}.$$

Also,

$$\frac{1}{N_k}\sum_{n=1}^{N_k}(\mathbf{1}_{\mathcal{U}}(n)-\delta)\mathrm{e}^{-in\theta_k} = \frac{1}{N_k}\sum_{n=1}^{N_k}(\mathbf{1}_{\mathcal{U}}(n)-\delta_{N_k})\mathrm{e}^{-in\theta_k} + (\delta_{N_k}-\delta)\frac{1}{N_k}\sum_{n=1}^{N_k}\mathrm{e}^{-in\theta_k}.$$

Since

$$|\frac{1}{N_k}\sum_{n=1}^{N_k}\mathrm{e}^{-in\theta_k}| \leq 1,$$

the triangle inequality and (3.1) give

$$\begin{aligned}|\frac{1}{N_k}\sum_{n=1}^{N_k}(\mathbf{1}_{\mathcal{U}}(n)-\delta)\mathrm{e}^{-in\theta_k}| &\geq \frac{\delta_{N_k}^3}{8\sqrt{2}} - |\delta_{N_k}-\delta| \\ &\geq \frac{\delta^3}{64\sqrt{2}} - \frac{\delta^3}{256\sqrt{2}} \geq \frac{\delta^3}{256\sqrt{2}} = c_0.\end{aligned}$$

Therefore

$$|\frac{1}{N_k}\sum_{n=1}^{N_k}(\mathbf{1}_{\mathcal{U}}(n)-\delta)\mathrm{e}^{-in\theta_k}| \geq c_0 \qquad \text{for every } k. \tag{3.2}$$

Now define probability measures on $X \times \mathbb{T}$ by

$$\nu_k := \frac{1}{N_k}\sum_{m=1}^{N_k}\delta_{(T^m x,\, m\theta_k)}.$$

Since $X \times \mathbb{T}$ is compact, after passing to a further subsequence we may assume that

$$\nu_k \xrightarrow{*} \nu$$

for some Borel probability measure ν on $X \times \mathbb{T}$.

We claim that ν is invariant under S_α. Let $\Phi \in C(X \times \mathbb{T})$. Then

$$\begin{aligned}&\int \Phi \circ S_\alpha \, d\nu_k - \int \Phi \, d\nu_k \\ &= \frac{1}{N_k}\sum_{m=1}^{N_k}\Big(\Phi(T^{m+1}x, m\theta_k+\alpha) - \Phi(T^m x, m\theta_k)\Big) \\ &= \frac{1}{N_k}\sum_{m=1}^{N_k}\Big(\Phi(T^{m+1}x, m\theta_k+\alpha) - \Phi(T^{m+1}x, (m+1)\theta_k)\Big) \\ &\quad + \frac{1}{N_k}\sum_{m=1}^{N_k}\Big(\Phi(T^{m+1}x, (m+1)\theta_k) - \Phi(T^m x, m\theta_k)\Big).\end{aligned}$$

Because Φ is uniformly continuous and $\theta_k \to \alpha$, the first average tends to 0. The second average telescopes to

$$\frac{\Phi(T^{N_k+1}x, (N_k+1)\theta_k) - \Phi(Tx, \theta_k)}{N_k},$$

which also tends to 0. Hence ν is S_α-invariant. Therefore $\mu = (\mathrm{pr}_1)_*\nu$ is T-invariant and $\lambda = (\mathrm{pr}_2)_*\nu$ is R_α-invariant.

Since the function $y \mapsto y_0$ is continuous on X,

$$\int_X y_0 \, d\mu(y) = \lim_{k\to\infty}\frac{1}{N_k}\sum_{m=1}^{N_k}(T^m x)_0 = \lim_{k\to\infty}\frac{1}{N_k}\sum_{m=1}^{N_k}\mathbf{1}_{\mathcal{U}}(m) = \delta.$$

This proves (i). Let

$$H := \mathbb{E}_\nu(y_0 \mid \mathcal{B}_{\mathbb{T}}).$$

Then $0 \le H \le 1$ and $\int_{\mathbb{T}} H\, d\lambda = \delta$.

Moreover, by (3.2),

$$\left|\int_{X\times\mathbb{T}} (y_0 - \delta)\mathrm{e}^{-it}\, d\nu_k(y,t)\right| = \left|\frac{1}{N_k}\sum_{m=1}^{N_k} (\mathbf{1}_{\mathcal{U}}(m) - \delta)\mathrm{e}^{-im\theta_k}\right| \ge c_0.$$

Passing to the limit gives

$$\left|\int_{X\times\mathbb{T}} (y_0 - \delta)\mathrm{e}^{-it}\, d\nu(y,t)\right| \ge c_0.$$

By the defining property of conditional expectation,

$$\int_{X\times\mathbb{T}} (y_0 - \delta)\mathrm{e}^{-it}\, d\nu(y,t) = \int_{\mathbb{T}} (H(t) - \delta)\mathrm{e}^{-it}\, d\lambda(t).$$

This proves (ii). Since $|\mathrm{e}^{-it}| = 1$, Cauchy–Schwarz yields

$$\left|\int_{\mathbb{T}} (H - \delta)\mathrm{e}^{-it}\, d\lambda\right| \le \|H - \delta\|_{L^2(\lambda)},$$

so (iii) follows. In particular H is not λ-almost everywhere equal to the constant δ. Because $0 \le H \le 1$ and $\int_{\mathbb{T}} H\, d\lambda = \delta$, this forces $0 < \delta < 1$. □

Corollary 3.2 (Single Kronecker component on the actual orbit)**.** *Assume $\overline{d}(\mathcal{U}) > 0$, and let δ, α, μ, and ν be as in Theorem 3.1. Then there exists a nonzero function $g \in L^\infty(X,\mu)$ such that*

$$g \circ T = \mathrm{e}^{i\alpha} g \quad \mu\text{-a.e.}$$

and

$$\left|\int_X (y_0 - \delta)\overline{g(y)}\, d\mu(y)\right| \ge \frac{\delta^3}{256\sqrt{2}}.$$

In particular, a single one-dimensional Kronecker component already captures the centered coordinate of a Furstenberg limit of the actual Ulam orbit.

Proof. Disintegrate ν over its first marginal μ:

$$\nu = \int_X \delta_y \otimes \nu_y\, d\mu(y),$$

where each ν_y is a probability measure on $\mathbb{T}$. Since ν is S_α-invariant,

$$\nu = (S_\alpha)_*\nu = \int_X \delta_{Ty} \otimes (R_\alpha)_*\nu_y\, d\mu(y), \qquad R_\alpha(t) := t + \alpha.$$

Because T is invertible on X and μ is T-invariant, a change of variables $z = Ty$ rewrites this as

$$\nu = \int_X \delta_z \otimes (R_\alpha)_*\nu_{T^{-1}z}\, d\mu(z).$$

This is a second disintegration of ν over the same first marginal μ, so uniqueness of disintegration gives

$$\nu_z = (R_\alpha)_*\nu_{T^{-1}z} \qquad \text{for } \mu\text{-almost every } z,$$

equivalently,

$$\nu_{Ty} = (R_\alpha)_*\nu_y \qquad \text{for } \mu\text{-almost every } y.$$

Define

$$g(y) := \int_{\mathbb{T}} \mathrm{e}^{it}\, d\nu_y(t).$$

Then $g \in L^\infty(X,\mu)$ and $\|g\|_\infty \le 1$. Moreover, for μ-almost every y,

$$\begin{aligned} g(Ty) &= \int_{\mathbb{T}} \mathrm{e}^{it}\, d\nu_{Ty}(t) = \int_{\mathbb{T}} \mathrm{e}^{it}\, d((R_\alpha)_*\nu_y)(t) \\ &= \int_{\mathbb{T}} \mathrm{e}^{i(t+\alpha)}\, d\nu_y(t) = \mathrm{e}^{i\alpha} g(y). \end{aligned}$$

Thus g is a T-eigenfunction with eigenvalue $\mathrm{e}^{i\alpha}$.

Finally,

$$\begin{aligned} \int_X (y_0 - \delta)\overline{g(y)}\, d\mu(y) &= \int_X (y_0 - \delta) \left(\int_{\mathbb{T}} \mathrm{e}^{-it}\, d\nu_y(t) \right) d\mu(y) \\ &= \int_{X\times\mathbb{T}} (y_0 - \delta)\mathrm{e}^{-it}\, d\nu(y,t). \end{aligned}$$

Theorem 3.1(ii) gives the stated lower bound. In particular $g \not\equiv 0$. $\square$

Corollary 3.3 (Spectral atom)**.** *In the setting of Corollary 3.2, let*

$$F := y_0 - \delta \in L^2(X,\mu),$$

and let $U_T f := f \circ T$ be the Koopman operator. Then the U_T-spectral measure of F has an atom at α of mass at least

$$\left(\frac{\delta^3}{256\sqrt{2}} \right)^2 = \frac{\delta^6}{2^{17}}.$$

Equivalently, if P_α denotes orthogonal projection onto the α-eigenspace

$$E_\alpha := \{h \in L^2(X,\mu) : h \circ T = \mathrm{e}^{i\alpha} h\},$$

then

$$\|P_\alpha F\|_2 \ge \frac{\delta^3}{256\sqrt{2}}.$$

Proof. Let g be the eigenfunction from Corollary 3.2. Since $\|g\|_\infty \le 1$, one has $\|g\|_2 \le 1$. Therefore, with $h := g/\|g\|_2$, we have $h \in E_\alpha$ and $\|h\|_2 = 1$, so

$$\|P_\alpha F\|_2 \ge \left| \int_X F(y)\overline{h(y)}\, d\mu(y) \right| = \frac{\left| \int_X F(y)\overline{g(y)}\, d\mu(y) \right|}{\|g\|_2} \ge \left| \int_X F(y)\overline{g(y)}\, d\mu(y) \right| \ge \frac{\delta^3}{256\sqrt{2}}.$$

By the spectral theorem, the mass of the spectral measure of F at $\{\alpha\}$ is exactly $\|P_\alpha F\|_2^2$. $\square$

4 Prescribed-marginal extraction and spectral trapping of maximizing phases

The next proposition is the fixed-marginal version of the packet extraction argument. It will allow us to pin the packet to a *prescribed* limiting dynamical system and then trap the maximizing phases in the atomic spectrum of that single system.

Proposition 4.1 (Packet extraction with prescribed first marginal)**.** *Let* $N_k \to \infty$*, let*

$$\delta_k := \frac{1}{N_k} \#(\mathcal{U} \cap [1, N_k]) \to \delta \in (0,1),$$

and suppose the empirical orbit measures

$$\mu_k := \frac{1}{N_k} \sum_{n=1}^{N_k} \delta_{T^n x}$$

converge weak- to a* T*-invariant Borel probability measure* μ *on* X*. Let* $\theta_k \to \alpha \in \mathbb{T}$*, and assume that for some* $c > 0$*,*

$$|\frac{1}{N_k} \sum_{n=1}^{N_k} (\mathbf{1}_{\mathcal{U}}(n) - \delta) \mathrm{e}^{-in\theta_k}| \geq c \qquad \textit{for every } k. \tag{4.1}$$

Set

$$F := y_0 - \delta \in L^2(X, \mu).$$

Then the U_T*-spectral measure of* F *has an atom at* α *of mass at least* c^2*. Equivalently, if*

$$E_\alpha := \{h \in L^2(X, \mu) : h \circ T = \mathrm{e}^{i\alpha} h\}$$

and P_α *denotes orthogonal projection onto* E_α*, then*

$$\|P_\alpha F\|_2 \geq c.$$

Proof. Define probability measures on $X \times \mathbb{T}$ by

$$\nu_k := \frac{1}{N_k} \sum_{m=1}^{N_k} \delta_{(T^m x, m\theta_k)}.$$

By compactness, after passing to a subsequence we may assume that $\nu_k \xrightarrow{*} \nu$ for some Borel probability measure ν on $X \times \mathbb{T}$. Since the first marginal of ν_k is μ_k, the first marginal of ν is μ. Moreover, because $y \mapsto y_0$ is continuous on X,

$$\int_X y_0 \, d\mu(y) = \lim_{k\to\infty} \int_X y_0 \, d\mu_k(y) = \lim_{k\to\infty} \delta_k = \delta,$$

so $F = y_0 - \delta$ is indeed centered. Moreover, the hypothesis (4.1) rules out $\delta \in \{0, 1\}$, since

$$\left| \frac{1}{N_k} \sum_{n=1}^{N_k} (\mathbf{1}_{\mathcal{U}}(n) - \delta) \mathrm{e}^{-in\theta_k} \right| \leq \frac{1}{N_k} \sum_{n=1}^{N_k} |\mathbf{1}_{\mathcal{U}}(n) - \delta| \to 0$$

when $\delta = 0$ or $\delta = 1$.

Exactly as in the proof of Theorem 3.1, the limit ν is invariant under the skew shift

$$S_\alpha(y, t) := (Ty, t + \alpha).$$

Disintegrate ν over its first marginal:

$$\nu = \int_X \delta_y \otimes \nu_y \, d\mu(y),$$

where each ν_y is a probability measure on $\mathbb{T}$. The same argument as in the proof of Corollary 3.2 shows that

$$g(y) := \int_{\mathbb{T}} \mathrm{e}^{it} \, d\nu_y(t)$$

belongs to $L^\infty(X,\mu)$, satisfies $\|g\|_\infty \le 1$, and is an eigenfunction:

$$g \circ T = \mathrm{e}^{i\alpha} g \qquad \mu\text{-a.e.}$$

Moreover, by (4.1) and weak-* convergence,

$$\Big|\int_{X\times\mathbb{T}} (y_0-\delta)\mathrm{e}^{-it}\, d\nu(y,t)\Big| \ge c.$$

Hence, exactly as in Corollary 3.2,

$$\Big|\int_X (y_0-\delta)\overline{g(y)}\, d\mu(y)\Big| \ge c.$$

Normalizing g in $L^2(X,\mu)$ and applying the spectral theorem yields

$$\|P_\alpha F\|_2 \ge c \qquad \text{and} \qquad \sigma_F(\{\alpha\}) \ge c^2.$$

$\square$

Theorem 4.2 (Spectral trapping theorem)**.** *Let $N_k \to \infty$ be a subsequence such that*

$$\delta_k := \frac{|\mathcal{U} \cap [1,N_k]|}{N_k} \to \delta > 0,$$

and suppose the empirical orbit measures

$$\mu_k := \frac{1}{N_k}\sum_{n=1}^{N_k} \delta_{T^n x}$$

converge weak- to a T-invariant Borel probability measure μ on X. For each k choose $\theta_k \in [0,\pi]$ such that*

$$|\mathcal{E}_{N_k}(\theta_k)| = \sup_{\theta\in\mathbb{T}} |\mathcal{E}_{N_k}(\theta)|.$$

(This is possible because $\theta \mapsto |\mathcal{E}_{N_k}(\theta)|$ is continuous and $\mathcal{E}_{N_k}(2\pi-\theta) = \overline{\mathcal{E}_{N_k}(\theta)}$.) Set

$$F := y_0 - \delta \in L^2(X,\mu), \qquad c_0 := \frac{\delta^3}{256\sqrt{2}},$$

and define the threshold atom set

$$S_\mu(c_0) := \{\alpha \in \mathbb{T} : \sigma_F(\{\alpha\}) \ge c_0^2\}.$$

Then:

(i) *every accumulation point of (θ_k) belongs to $S_\mu(c_0)$;*

(ii) *$S_\mu(c_0)$ is finite and satisfies*

$$|S_\mu(c_0)| \le \frac{\delta(1-\delta)}{c_0^2} \le 2^{17}(1-\delta)\delta^{-5};$$

(iii) *for every $\varepsilon > 0$, all but finitely many θ_k lie in the ε-neighborhood of $S_\mu(c_0)$.*

Proof. Because $\delta_k \to \delta > 0$, after discarding finitely many terms we may assume that

$$|\delta_k - \delta| \le \frac{\delta^3}{256\sqrt{2}} \qquad \text{and} \qquad \delta_k \ge \frac{\delta}{2}$$

for every k. Also $\delta_k^3 N_k \to \infty$, so for all large k Theorem 2.5 gives

$$|\frac{1}{N_k}\sum_{n=1}^{N_k}(\mathbf{1}_{\mathcal{U}}(n) - \delta_k)\mathrm{e}^{-in\theta_k}| \ge \frac{\delta_k^3}{8\sqrt{2}}.$$

Moreover,

$$\frac{1}{N_k}\sum_{n=1}^{N_k}(\mathbf{1}_{\mathcal{U}}(n) - \delta)\mathrm{e}^{-in\theta_k} = \frac{1}{N_k}\sum_{n=1}^{N_k}(\mathbf{1}_{\mathcal{U}}(n) - \delta_k)\mathrm{e}^{-in\theta_k} + (\delta_k - \delta)\frac{1}{N_k}\sum_{n=1}^{N_k}\mathrm{e}^{-in\theta_k}.$$

Since

$$|\frac{1}{N_k}\sum_{n=1}^{N_k}\mathrm{e}^{-in\theta_k}| \le 1,$$

the triangle inequality gives

$$\begin{aligned}
|\frac{1}{N_k}\sum_{n=1}^{N_k}(\mathbf{1}_{\mathcal{U}}(n) - \delta)\mathrm{e}^{-in\theta_k}| &\ge \frac{\delta_k^3}{8\sqrt{2}} - |\delta_k - \delta| \\
&\ge \frac{\delta^3}{64\sqrt{2}} - \frac{\delta^3}{256\sqrt{2}} \\
&\ge \frac{\delta^3}{256\sqrt{2}} = c_0
\end{aligned}$$

for every sufficiently large k. Also, because $y \mapsto y_0$ is continuous on X,

$$\int_X y_0 \, d\mu(y) = \lim_{k\to\infty}\int_X y_0 \, d\mu_k(y) = \lim_{k\to\infty}\delta_k = \delta,$$

so $F = y_0 - \delta$ has mean zero. After discarding finitely many initial terms and relabeling, we may therefore assume that

$$|\frac{1}{N_k}\sum_{n=1}^{N_k}(\mathbf{1}_{\mathcal{U}}(n) - \delta)\mathrm{e}^{-in\theta_k}| \ge c_0 \qquad \text{for every } k.$$

Now let α be any accumulation point of (θ_k). Passing to a further subsequence if necessary, we may assume that $\theta_k \to \alpha$. Proposition 4.1, applied with the same sequence N_k, the same limiting measure μ, and the lower bound above, shows that

$$\sigma_F(\{\alpha\}) \ge c_0^2.$$

Thus $\alpha \in S_\mu(c_0)$, proving (i).

For (ii), the total mass of σ_F is

$$\sigma_F(\mathbb{T}) = \|F\|_2^2 = \int_X (y_0 - \delta)^2 \, d\mu(y) = \delta(1-\delta),$$

because $y_0^2 = y_0$ and $\int_X y_0 \, d\mu = \delta$. Therefore the number of atoms of mass at least c_0^2 is at most $\delta(1-\delta)/c_0^2$, giving the stated bound.

For (iii), suppose not. Then there exist $\varepsilon > 0$ and an infinite subsequence (θ_{k_j}) such that

$$\mathrm{dist}_{\mathbb{T}}(\theta_{k_j}, S_\mu(c_0)) \ge \varepsilon \qquad \text{for every } j.$$

By compactness of $[0, \pi]$, the sequence (θ_{k_j}) has an accumulation point β. Passing to a further subsequence, we may assume $\theta_{k_j} \to \beta$. By (i), $\beta \in S_\mu(c_0)$, contradicting the continuity of the distance to the closed set $S_\mu(c_0)$. $\square$

Corollary 4.3 (Zero accumulation forces invariant mass)**.** *In the setting of Theorem 4.2, if* 0 *is an accumulation point of* (θ_k)*, then*

$$\|P_0 F\|_2 \geq c_0.$$

If

$$\mu = \int_{\mathrm{Erg}(X,T)} \eta \, d\pi(\eta)$$

is the ergodic decomposition of μ *and*

$$a(\eta) := \int_X y_0 \, d\eta - \delta,$$

then

$$\int_{\mathrm{Erg}(X,T)} a(\eta)^2 \, d\pi(\eta) \geq c_0^2, \qquad \textit{hence} \qquad \int_{\mathrm{Erg}(X,T)} |a(\eta)| \, d\pi(\eta) \geq c_0^2.$$

Consequently there exist ergodic T*-invariant Borel probability measures* η_+ *and* η_- *on* X *such that*

$$\int_X y_0 \, d\eta_+ \geq \delta + \frac{c_0^2}{2}, \qquad \int_X y_0 \, d\eta_- \leq \delta - \frac{c_0^2}{2}.$$

Proof. If 0 is an accumulation point of (θ_k), then Theorem 4.2(i) gives $0 \in S_\mu(c_0)$, hence

$$\sigma_F(\{0\}) = \|P_0 F\|_2^2 \geq c_0^2.$$

This is the first claim.

Under the ergodic decomposition of μ, the invariant projection $P_0 F = \mathbb{E}_\mu(F \mid \mathcal{I}_T)$ corresponds to the function a on $\mathrm{Erg}(X,T)$, where $\mathcal{I}_T$ is the T-invariant σ-algebra. Thus

$$\int_{\mathrm{Erg}(X,T)} a(\eta) \, d\pi(\eta) = \int_X F \, d\mu = 0$$

and

$$\int_{\mathrm{Erg}(X,T)} a(\eta)^2 \, d\pi(\eta) = \|P_0 F\|_2^2 \geq c_0^2.$$

Since $|a(\eta)| \leq 1$, one has $|a(\eta)|^2 \leq |a(\eta)|$, so

$$\int_{\mathrm{Erg}(X,T)} |a(\eta)| \, d\pi(\eta) \geq c_0^2.$$

Because $\int a \, d\pi = 0$, the positive and negative parts satisfy

$$\int a_+ \, d\pi = \int a_- \, d\pi = \frac{1}{2} \int |a| \, d\pi \geq \frac{c_0^2}{2}.$$

Hence there exist ergodic components η_+ and η_- with

$$a(\eta_+) \geq \frac{c_0^2}{2}, \qquad a(\eta_-) \leq -\frac{c_0^2}{2},$$

which is exactly the stated density splitting. $\square$

Corollary 4.4 (Zero is excluded under ergodicity)**.** *In the setting of Theorem 4.2, assume in addition that* μ *is ergodic. Then* 0 *is not an accumulation point of* (θ_k)*. Equivalently, there exists* $\varepsilon > 0$ *such that*

$$\theta_k \in [\varepsilon, \pi]$$

for all sufficiently large k*.*

Proof. If 0 were an accumulation point, Corollary 4.3 would give $\|P_0 F\|_2 \geq c_0$. But when μ is ergodic, the 0-eigenspace consists of constants, and $F = y_0 - \delta$ has mean zero. Hence $P_0 F = 0$, a contradiction. □

Corollary 4.5 (Finite packet decomposition). *In the setting of Theorem 4.2, write*

$$S_\mu(c_0) \cap [0, \pi] = \{\alpha_1, \dots, \alpha_r\}.$$

Then $r < \infty$, and for every $\varepsilon > 0$ there exists K such that for all $k \geq K$ one has

$$\theta_k \in \bigcup_{j=1}^{r} B_\varepsilon(\alpha_j).$$

In particular, after partitioning a tail of the index set according to the nearest α_j, each piece has all its accumulation points contained in a single packet center.

Proof. This is just a restatement of Theorem 4.2(ii)–(iii). □

Corollary 4.6 (Uniqueness of the packet implies phase convergence). *In the setting of Theorem 4.2, assume that*

$$S_\mu(c_0) \cap [0, \pi] = \{\alpha\}$$

consists of a single point. Then

$$\theta_k \to \alpha.$$

Proof. By Theorem 4.2(iii), for every $\varepsilon > 0$ all but finitely many θ_k lie in the ε-neighborhood of $\{\alpha\}$. This is exactly the statement $\theta_k \to \alpha$. □

5 The zero-frequency alternative

Proposition 5.1 (Zero frequency forces density splitting). *Assume $\overline{d}(\mathcal{U}) > 0$, and let δ, α, ν, and λ be as in Theorem 3.1. Set*

$$c_0 := \frac{\delta^3}{256\sqrt{2}}.$$

Then either $\alpha \neq 0$, or else there exist ergodic T-invariant Borel probability measures η_+ and η_- on X such that

$$\int_X y_0 \, d\eta_+ \geq \delta + \frac{c_0}{2}, \qquad \int_X y_0 \, d\eta_- \leq \delta - \frac{c_0}{2}.$$

Equivalently, the only obstruction to a nonzero extracted frequency is a quantitative splitting of the coordinate density among ergodic invariant measures on the orbit closure.

Proof. If $\alpha \neq 0$, there is nothing to prove. Assume $\alpha = 0$. Then $S_\alpha = T \times \mathrm{id}_{\mathbb{T}}$, so disintegrating ν over its second marginal gives

$$\nu = \int_{\mathbb{T}} \mu_t \otimes \delta_t \, d\lambda(t).$$

Since ν is $(T \times \mathrm{id}_{\mathbb{T}})$-invariant, the family $t \mapsto T_* \mu_t$ is another disintegration of ν over the same second marginal λ. By uniqueness of disintegration, $T_* \mu_t = \mu_t$ for λ-almost every t.

Define

$$a(t) := \int_X (y_0 - \delta) \, d\mu_t.$$

Then $a \in L^\infty(\mathbb{T}, \lambda)$ is real-valued, $|a| \leq 1$, and

$$\int_{\mathbb{T}} a(t) \, d\lambda(t) = \int_{X \times \mathbb{T}} (y_0 - \delta) \, d\nu(y, t) = \int_X y_0 \, d\mu - \delta = 0.$$

On the other hand, Theorem 3.1(ii) gives

$$|\int_{\mathbb{T}} a(t)\mathrm{e}^{-it}\, d\lambda(t)| \geq c_0.$$

Hence

$$\int_{\mathbb{T}} |a(t)|\, d\lambda(t) \geq c_0.$$

Since $\int a\, d\lambda = 0$, the positive and negative parts satisfy

$$\int a_+\, d\lambda = \int a_-\, d\lambda = \frac{1}{2}\int |a|\, d\lambda \geq \frac{c_0}{2}.$$

Hence the sets

$$E_+ := \left\{t \in \mathbb{T} : a(t) \geq \frac{c_0}{2}\right\}, \qquad E_- := \left\{t \in \mathbb{T} : a(t) \leq -\frac{c_0}{2}\right\}$$

both have positive λ-measure. Choose $t_+ \in E_+$ and $t_- \in E_-$ from the full-measure set on which the disintegration $t \mapsto \mu_t$ is defined. Then

$$\int_X y_0\, d\mu_{t_+} \geq \delta + \frac{c_0}{2}, \qquad \int_X y_0\, d\mu_{t_-} \leq \delta - \frac{c_0}{2}.$$

Finally, apply the ergodic decomposition theorem to μ_{t_+} and μ_{t_-}. Since the map $\eta \mapsto \int y_0\, d\eta$ is affine, some ergodic component of μ_{t_+} still satisfies the lower inequality, and some ergodic component of μ_{t_-} still satisfies the upper inequality. □

Corollary 5.2 (A clean nonzero-frequency criterion)**.** *If all ergodic T-invariant Borel probability measures on X have the same value of $\int_X y_0\, d\eta$, then the extracted frequency in Corollary 3.2 automatically satisfies $\alpha \neq 0$.*

Proof. This is immediate from Proposition 5.1. □

Corollary 5.3 (Ergodicity rules out the zero-frequency case)**.** *Assume that $\delta_N \to \delta > 0$ and that the empirical measures*

$$\mu_N := \frac{1}{N}\sum_{m=1}^{N} \delta_{T^m x}$$

converge weak- to an ergodic T-invariant measure μ on X. Then there exists an angle $\alpha \in \mathbb{T}\setminus\{0\}$ such that the spectral measure of $y_0 - \delta$ on (X, μ, T) has an atom at α of mass at least $\delta^6/2^{17}$.*

Proof. Apply Theorem 3.1 along a subsequence N_k for which the empirical measures still converge to μ. Then the first marginal in that theorem is exactly μ. Corollary 3.2 therefore yields an eigenfunction $g \in L^\infty(X, \mu)$ with

$$g \circ T = \mathrm{e}^{i\alpha} g \qquad \text{and} \qquad |\int_X (y_0 - \delta)\overline{g}\, d\mu| \geq \frac{\delta^3}{256\sqrt{2}}.$$

If $\alpha = 0$, then g is T-invariant. Since μ is ergodic, g is constant almost everywhere, and hence

$$\int_X (y_0 - \delta)\overline{g}\, d\mu = \overline{g}\int_X (y_0 - \delta)\, d\mu = 0,$$

contradicting the lower bound. Thus $\alpha \neq 0$. The spectral atom statement is now Corollary 3.3. □

6 A general window theorem at the varying maximizing phases

Lemma 6.1 (Window extraction). *Let K be a compact metric space, let λ be a regular Borel probability measure on K, and let $H \in L^\infty(K, \lambda)$ satisfy $0 \le H \le 1$. Write*

$$\delta := \int_K H \, d\lambda, \qquad h := H - \delta.$$

If $\|h\|_{L^2(\lambda)} \ge \sigma$, then there exists a Borel set $W \subset K$ such that $\lambda(\partial W) = 0$ and

$$\int_K h(\mathbf{1}_W - \lambda(W)) \, d\lambda \ge \frac{\sigma^2}{4}.$$

Proof. Replacing H by a Borel representative, we may assume that H is Borel measurable. Let

$$A := \{x \in K : h(x) > 0\}.$$

Since $\int h \, d\lambda = 0$, one has

$$\int_A h \, d\lambda = \int_{K \setminus A} (-h) \, d\lambda = \frac{1}{2} \|h\|_{L^1(\lambda)}.$$

Therefore

$$\int_K h(\mathbf{1}_A - \lambda(A)) \, d\lambda = \int_A h \, d\lambda = \frac{1}{2} \|h\|_{L^1(\lambda)}.$$

Because $|h| \le 1$, one has $|h|^2 \le |h|$, hence

$$\|h\|_{L^1(\lambda)} \ge \|h\|_{L^2(\lambda)}^2 \ge \sigma^2.$$

Thus

$$\int_K h(\mathbf{1}_A - \lambda(A)) \, d\lambda \ge \frac{\sigma^2}{2}. \tag{6.1}$$

By regularity of λ on the compact metric space K, there exist a closed set $C \subset A$ and an open set $O \supset A$ such that

$$\lambda(O \setminus C) < \frac{\sigma^2}{4}.$$

Define the continuous function

$$\varphi(x) := \frac{d(x, C)}{d(x, C) + d(x, K \setminus O)} \in [0, 1].$$

Then $\varphi = 0$ on C and $\varphi = 1$ on $K \setminus O$. For $t \in (0, 1)$ set

$$W_t := \{x \in K : \varphi(x) < t\}.$$

Then $C \subset W_t \subset O$ for every $t \in (0, 1)$. Since the disjoint level sets $\varphi^{-1}(t)$ can have positive λ-measure for at most countably many t, we may choose $t \in (0, 1)$ with

$$\lambda(\varphi^{-1}(t)) = 0.$$

Set $W := W_t$. Then $\partial W \subset \varphi^{-1}(t)$, so $\lambda(\partial W) = 0$, and

$$A \triangle W \subset O \setminus C, \qquad \text{hence} \qquad \lambda(A \triangle W) < \frac{\sigma^2}{4}.$$

Because $\int h\,d\lambda = 0$ and $|h| \le 1$,

$$\begin{aligned}|\int_K h((\mathbf{1}_W - \lambda(W)) - (\mathbf{1}_A - \lambda(A)))\,d\lambda| &= |\int_K h(\mathbf{1}_W - \mathbf{1}_A)\,d\lambda| \\ &\le \int_K |\mathbf{1}_W - \mathbf{1}_A|\,d\lambda \\ &= \lambda(A \triangle W) < \frac{\sigma^2}{4}.\end{aligned}$$

Combining this with (6.1) gives

$$\int_K h(\mathbf{1}_W - \lambda(W))\,d\lambda \ge \frac{\sigma^2}{4}.$$

$\square$

Theorem 6.2 (Window theorem on the actual sequence)**.** *Assume that $\overline{d}(\mathcal{U}) > 0$. Then there exist*

- *a number $\delta \in (0,1)$,*
- *a subsequence $N_k \to \infty$,*
- *angles $\theta_k \to \alpha \in \mathbb{T}$,*
- *an R_α-invariant Borel probability measure λ on $\mathbb{T}$,*
- *and a Borel set $W \subset \mathbb{T}$ with $\lambda(\partial W) = 0$,*

such that

$$\lim_{k\to\infty} \frac{1}{N_k} \sum_{n=1}^{N_k} (\mathbf{1}_{\mathcal{U}}(n) - \delta)(\mathbf{1}_W(n\theta_k) - \lambda(W)) \ge \frac{\delta^6}{2^{19}}. \tag{6.2}$$

In particular,

$$\liminf_{k\to\infty} \frac{1}{N_k} \#\{1 \le n \le N_k : n \in \mathcal{U},\ n\theta_k \in W\} \ge \delta\lambda(W) + \frac{\delta^6}{2^{19}}. \tag{6.3}$$

Proof. Apply Theorem 3.1. Let $H \in L^\infty(\mathbb{T},\lambda)$ be the conditional expectation there. By Theorem 3.1(iii),

$$\|H - \delta\|_{L^2(\lambda)} \ge \frac{\delta^3}{256\sqrt{2}}.$$

Set

$$\sigma := \frac{\delta^3}{256\sqrt{2}}.$$

Lemma 6.1 provides a Borel set $W \subset \mathbb{T}$ with $\lambda(\partial W) = 0$ and

$$\int_{\mathbb{T}} (H(t) - \delta)(\mathbf{1}_W(t) - \lambda(W))\,d\lambda(t) \ge \frac{\sigma^2}{4} = \frac{\delta^6}{2^{19}}. \tag{6.4}$$

Because $H = \mathbb{E}_\nu(y_0 \mid \mathcal{B}_{\mathbb{T}})$,

$$\int_{\mathbb{T}} (H - \delta)(\mathbf{1}_W - \lambda(W))\,d\lambda = \int_{X\times\mathbb{T}} (y_0 - \delta)(\mathbf{1}_W(t) - \lambda(W))\,d\nu(y,t).$$

The function

$$\Psi(y,t) := (y_0 - \delta)(\mathbf{1}_W(t) - \lambda(W))$$

is bounded on $X \times \mathbb{T}$, and its discontinuity set is contained in $X \times \partial W$, which has ν-measure $\lambda(\partial W) = 0$. By the Portmanteau theorem,

$$\int_{X\times\mathbb{T}} \Psi \, d\nu_k \to \int_{X\times\mathbb{T}} \Psi \, d\nu.$$

Since

$$\int_{X\times\mathbb{T}} \Psi \, d\nu_k = \frac{1}{N_k} \sum_{n=1}^{N_k} (\mathbf{1}_{\mathcal{U}}(n) - \delta)(\mathbf{1}_W(n\theta_k) - \lambda(W)),$$

this proves (6.2).

To obtain (6.3), note first that by definition of δ,

$$\frac{1}{N_k} \sum_{n=1}^{N_k} \mathbf{1}_{\mathcal{U}}(n) \to \delta.$$

Also, because λ is the weak-* limit of the second marginals

$$\lambda_k := \frac{1}{N_k} \sum_{n=1}^{N_k} \delta_{n\theta_k}$$

and $\lambda(\partial W) = 0$, one has

$$\frac{1}{N_k} \sum_{n=1}^{N_k} \mathbf{1}_W(n\theta_k) = \int \mathbf{1}_W \, d\lambda_k \to \int \mathbf{1}_W \, d\lambda = \lambda(W).$$

Expanding the product in (6.2) and passing to the limit gives (6.3). $\square$

7 The irrational case: phase laws and interval arcs

Write $m_{\mathbb{T}}$ for Haar probability measure on $\mathbb{T}$.

Theorem 7.1 (Irrational packet gives a limiting phase law)**.** *Assume the setting of Theorem 3.1, and suppose that $\alpha/2\pi \notin \mathbb{Q}$. Then the circle marginal of ν is Haar measure $m_{\mathbb{T}}$. Moreover there exists a function $H \in L^\infty(\mathbb{T}, m_{\mathbb{T}})$ such that*

$$0 \le H \le 1 \quad a.e., \qquad \int_{\mathbb{T}} H \, dm_{\mathbb{T}} = \delta, \qquad \left|\int_{\mathbb{T}} (H(t) - \delta) \mathrm{e}^{-it} \, dm_{\mathbb{T}}(t)\right| \ge \frac{\delta^3}{256\sqrt{2}},$$

and for every $\varphi \in C(\mathbb{T})$,

$$\frac{1}{N_k} \sum_{n=1}^{N_k} \mathbf{1}_{\mathcal{U}}(n) \varphi(n\theta_k) \longrightarrow \int_{\mathbb{T}} H(t)\varphi(t) \, dm_{\mathbb{T}}(t). \tag{7.1}$$

Equivalently,

$$\frac{1}{M_{N_k}} \sum_{\substack{n \le N_k \\ n \in \mathcal{U}}} \varphi(n\theta_k) \longrightarrow \int_{\mathbb{T}} \varphi(t) \, \frac{H(t)}{\delta} \, dm_{\mathbb{T}}(t). \tag{7.2}$$

Thus $\rho := \frac{H}{\delta} \, dm_{\mathbb{T}}$ is a non-uniform absolutely continuous limiting phase law for the actual Ulam numbers along the subsequence N_k.

Proof. Let $\lambda := (\mathrm{pr}_2)_*\nu$ be the circle marginal of ν. Since ν is S_α-invariant, λ is R_α-invariant. Because R_α is uniquely ergodic when $\alpha/2\pi \notin \mathbb{Q}$, one has $\lambda = m_{\mathbb{T}}$.

Set
$$H := \mathbb{E}_\nu(y_0 \mid \mathcal{B}_{\mathbb{T}}) \in L^\infty(\mathbb{T}, m_{\mathbb{T}}).$$
Then $0 \le H \le 1$ almost everywhere and
$$\int_{\mathbb{T}} H\, dm_{\mathbb{T}} = \int_X y_0\, d\mu = \delta.$$
Also,
$$\int_{\mathbb{T}} (H(t)-\delta)\mathrm{e}^{-it}\, dm_{\mathbb{T}}(t) = \int_{X\times\mathbb{T}} (y_0-\delta)\mathrm{e}^{-it}\, d\nu(y,t),$$
so Theorem 3.1(ii) gives the stated lower bound. In particular H is not almost everywhere constant.

Now let $\varphi \in C(\mathbb{T})$. Since $(y,t) \mapsto y_0\varphi(t)$ is continuous on $X\times\mathbb{T}$,
$$\begin{aligned}\frac{1}{N_k}\sum_{n=1}^{N_k} \mathbf{1}_{\mathcal{U}}(n)\varphi(n\theta_k) &= \int_{X\times\mathbb{T}} y_0\varphi(t)\, d\nu_k(y,t)\\ &\longrightarrow \int_{X\times\mathbb{T}} y_0\varphi(t)\, d\nu(y,t)\\ &= \int_{\mathbb{T}} H(t)\varphi(t)\, dm_{\mathbb{T}}(t).\end{aligned}$$
This proves (7.1). Dividing by $M_{N_k}/N_k = \delta_{N_k} \to \delta$ yields (7.2). Since $H-\delta$ has nonzero first Fourier coefficient, the limiting law ρ is non-uniform. □

Corollary 7.2 (An interval arc with excess mass)**.** *In the setting of Theorem 7.1, there exist a phase $\beta \in \mathbb{T}$ and a level $\eta \in (-1,1)$ such that the interval arc*
$$W_{\beta,\eta} := \{t \in \mathbb{T} : \cos(t-\beta) > \eta\}$$
satisfies
$$\int_{W_{\beta,\eta}} H(t)\, dm_{\mathbb{T}}(t) \ge \delta\, m_{\mathbb{T}}(W_{\beta,\eta}) + \frac{\delta^3}{512\sqrt{2}}.$$
Consequently,
$$\liminf_{k\to\infty} \frac{1}{N_k}\#\{1 \le n \le N_k : n \in \mathcal{U},\ n\theta_k \in W_{\beta,\eta}\} \ge \delta\, m_{\mathbb{T}}(W_{\beta,\eta}) + \frac{\delta^3}{512\sqrt{2}}.$$

Proof. Let
$$I := \int_{\mathbb{T}} (H(t)-\delta)\mathrm{e}^{-it}\, dm_{\mathbb{T}}(t).$$
By Theorem 7.1, $|I| \ge \delta^3/(256\sqrt{2})$. Choose $\beta \in \mathbb{T}$ so that $\mathrm{e}^{i\beta}I = |I|$. Then
$$\begin{aligned}\int_{\mathbb{T}} (H(t)-\delta)\cos(t-\beta)\, dm_{\mathbb{T}}(t) &= \Re\Big(\mathrm{e}^{i\beta}\int_{\mathbb{T}} (H(t)-\delta)\mathrm{e}^{-it}\, dm_{\mathbb{T}}(t)\Big)\\ &= |I| \ge \frac{\delta^3}{256\sqrt{2}}.\end{aligned}$$
For $x \in [-1,1]$ one has the identity
$$x = -1 + \int_{-1}^{1} \mathbf{1}_{\{x>\eta\}}\, d\eta.$$

Applying this with $x = \cos(t-\beta)$, multiplying by $H(t) - \delta$, and using $\int_{\mathbb{T}}(H-\delta)\,dm_{\mathbb{T}} = 0$, we obtain

$$\int_{\mathbb{T}}(H(t)-\delta)\cos(t-\beta)\,dm_{\mathbb{T}}(t) = \int_{-1}^{1}\left(\int_{W_{\beta,\eta}}(H(t)-\delta)\,dm_{\mathbb{T}}(t)\right)d\eta.$$

The outer interval has length 2, so there exists $\eta \in [-1,1]$ such that

$$\int_{W_{\beta,\eta}}(H(t)-\delta)\,dm_{\mathbb{T}}(t) \geq \frac{\delta^3}{512\sqrt{2}}.$$

Since the right-hand side is positive, one must have $\eta \in (-1,1)$. This proves the first claim.

Finally, the function $(y,t) \mapsto y_0 \mathbf{1}_{W_{\beta,\eta}}(t)$ is bounded, and its discontinuity set is contained in $X \times \partial W_{\beta,\eta}$, which has ν-measure $m_{\mathbb{T}}(\partial W_{\beta,\eta}) = 0$ because the boundary of the arc is finite. Hence Portmanteau gives

$$\frac{1}{N_k}\sum_{n=1}^{N_k}\mathbf{1}_{\mathcal{U}}(n)\mathbf{1}_{W_{\beta,\eta}}(n\theta_k) \to \int_{W_{\beta,\eta}} H(t)\,dm_{\mathbb{T}}(t).$$

Combining this with the first inequality yields the stated lower bound for the counting function. $\square$

Corollary 7.3 (The same phase law is carried by a genuine sum-free core). *Assume the hypotheses of Theorem 7.1, and let $B_{N_k} \subset [1, N_k]$ be sum-free sets with*

$$|A_{N_k} \triangle B_{N_k}| = o(N_k)$$

as given by Corollary 2.4. Then for every continuous $\varphi : \mathbb{T} \to \mathbb{C}$,

$$\frac{1}{N_k}\sum_{n=1}^{N_k}\mathbf{1}_{B_{N_k}}(n)\varphi(n\theta_k) \longrightarrow \int_{\mathbb{T}}\varphi(t)H(t)\,dm_{\mathbb{T}}(t),$$

and

$$\frac{|B_{N_k}|}{N_k} \to \delta.$$

Equivalently,

$$\frac{1}{|B_{N_k}|}\sum_{n\in B_{N_k}}\varphi(n\theta_k) \longrightarrow \int_{\mathbb{T}}\varphi(t)\,\frac{H(t)}{\delta}\,dm_{\mathbb{T}}(t).$$

Thus the same non-uniform absolutely continuous phase law already arises on a sequence of genuine sum-free cores.

Proof. The density statement is Corollary 2.4. For the profile statement, note that

$$\begin{aligned}\left|\frac{1}{N_k}\sum_{n=1}^{N_k}(\mathbf{1}_{A_{N_k}}(n) - \mathbf{1}_{B_{N_k}}(n))\varphi(n\theta_k)\right| &\leq \frac{\|\varphi\|_\infty}{N_k}|A_{N_k}\triangle B_{N_k}| \\ &\to 0.\end{aligned}$$

The unnormalized convergence therefore follows from Theorem 7.1. The normalized statement follows by dividing by $|B_{N_k}|/N_k \to \delta$. $\square$

8 From varying phases to fixed-angle windows

For $t \in \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, write

$$\|t\|_{\mathbb{T}} := \min_{m\in\mathbb{Z}} |t - 2\pi m|.$$

Proposition 8.1 (Phase stabilization for continuous test functions)**.** *Assume the setting of Theorem 7.1, and in addition suppose that*

$$N_k \|\theta_k - \alpha\|_{\mathbb{T}} \longrightarrow 0.$$

Then for every $\varphi \in C(\mathbb{T})$,

$$\frac{1}{N_k} \sum_{n=1}^{N_k} \mathbf{1}_{\mathcal{U}}(n)\varphi(n\alpha) \longrightarrow \int_{\mathbb{T}} H(t)\varphi(t)\, dm_{\mathbb{T}}(t),$$

and equivalently,

$$\frac{1}{M_{N_k}} \sum_{\substack{n\le N_k \\ n\in\mathcal{U}}} \varphi(n\alpha) \longrightarrow \int_{\mathbb{T}} \varphi(t)\, \frac{H(t)}{\delta}\, dm_{\mathbb{T}}(t).$$

Proof. First assume that φ is Lipschitz with respect to the circle metric, with Lipschitz constant L. Then

$$\begin{aligned}
&|\frac{1}{N_k} \sum_{n=1}^{N_k} \mathbf{1}_{\mathcal{U}}(n)(\varphi(n\theta_k) - \varphi(n\alpha))| \\
&\le \frac{L}{N_k} \sum_{n=1}^{N_k} \|n(\theta_k - \alpha)\|_{\mathbb{T}} \\
&\le \frac{L}{N_k} \sum_{n=1}^{N_k} n\, \|\theta_k - \alpha\|_{\mathbb{T}} \\
&\le L N_k \|\theta_k - \alpha\|_{\mathbb{T}},
\end{aligned}$$

which tends to 0 by hypothesis. The conclusion therefore holds for every Lipschitz φ by Theorem 7.1.

Now let $\varphi \in C(\mathbb{T})$. Since Lipschitz functions are uniformly dense in $C(\mathbb{T})$, choose a Lipschitz function ψ with $\|\varphi - \psi\|_\infty < \varepsilon$. Then

$$|\frac{1}{N_k} \sum_{n=1}^{N_k} \mathbf{1}_{\mathcal{U}}(n)\varphi(n\alpha) - \int_{\mathbb{T}} H\varphi\, dm_{\mathbb{T}}| \le 2\varepsilon + |\frac{1}{N_k} \sum_{n=1}^{N_k} \mathbf{1}_{\mathcal{U}}(n)\psi(n\alpha) - \int_{\mathbb{T}} H\psi\, dm_{\mathbb{T}}|.$$

Taking $k \to \infty$ and then $\varepsilon \to 0$ proves the first convergence. The second follows by dividing by $M_{N_k}/N_k \to \delta$. □

Proposition 8.2 (Boundary-null sets under phase stabilization)**.** *Assume the setting of Theorem 7.1, and in addition suppose that*

$$\varepsilon_k := N_k \|\theta_k - \alpha\|_{\mathbb{T}} \to 0.$$

Then for every Borel set $W \subset \mathbb{T}$ with $m_{\mathbb{T}}(\partial W) = 0$ one has

$$\frac{1}{N_k} \sum_{n=1}^{N_k} |\mathbf{1}_W(n\theta_k) - \mathbf{1}_W(n\alpha)| \to 0.$$

Consequently, every counting or covariance statement in Theorem 6.2 and Corollary 7.2 remains valid after replacing $n\theta_k$ by $n\alpha$.

Proof. Fix a Borel set $W \subset \mathbb{T}$ with $m_{\mathbb{T}}(\partial W) = 0$. For $n \le N_k$,

$$\mathrm{dist}_{\mathbb{T}}(n\theta_k, n\alpha) \le n \,\|\theta_k - \alpha\|_{\mathbb{T}} \le \varepsilon_k.$$

If $\mathbf{1}_W(n\theta_k) \ne \mathbf{1}_W(n\alpha)$, let $J_n \subset \mathbb{T}$ be a shortest closed arc joining $n\alpha$ to $n\theta_k$. Then

$$\mathrm{length}(J_n) = \mathrm{dist}_{\mathbb{T}}(n\alpha, n\theta_k) \le \varepsilon_k.$$

Since the two endpoints of J_n have different W-membership, the connected set J_n meets both W and $\mathbb{T} \setminus W$; hence some point of J_n lies in ∂W. In particular,

$$\mathrm{dist}_{\mathbb{T}}(n\alpha, \partial W) \le \varepsilon_k.$$

Writing

$$(\partial W)^r := \{t \in \mathbb{T} : \mathrm{dist}_{\mathbb{T}}(t, \partial W) \le r\},$$

we therefore have

$$\frac{1}{N_k} \sum_{n=1}^{N_k} |\mathbf{1}_W(n\theta_k) - \mathbf{1}_W(n\alpha)| \le \frac{1}{N_k} \#\{1 \le n \le N_k : n\alpha \in (\partial W)^{\varepsilon_k}\}.$$

Let $\eta > 0$. Since $m_{\mathbb{T}}(\partial W) = 0$, there exists a finite union O of open intervals such that

$$\partial W \subset O \qquad \text{and} \qquad m_{\mathbb{T}}(O) < \eta.$$

For all sufficiently large k, one has $(\partial W)^{\varepsilon_k} \subset O$. Since $\alpha/2\pi$ is irrational, Weyl equidistribution gives

$$\frac{1}{N_k} \#\{1 \le n \le N_k : n\alpha \in O\} \to m_{\mathbb{T}}(O) < \eta.$$

Thus

$$\limsup_{k \to \infty} \frac{1}{N_k} \sum_{n=1}^{N_k} |\mathbf{1}_W(n\theta_k) - \mathbf{1}_W(n\alpha)| \le \eta.$$

Since $\eta > 0$ is arbitrary, the claim follows.

Now let $(c_n^{(k)})$ be any sequence with $|c_n^{(k)}| \le 1$. Then

$$\Big|\frac{1}{N_k} \sum_{n=1}^{N_k} c_n^{(k)} (\mathbf{1}_W(n\theta_k) - \mathbf{1}_W(n\alpha))\Big| \le \frac{1}{N_k} \sum_{n=1}^{N_k} |\mathbf{1}_W(n\theta_k) - \mathbf{1}_W(n\alpha)| \to 0.$$

Taking $c_n^{(k)} = 1$ or $c_n^{(k)} = \mathbf{1}_{\mathcal{U}}(n) - \delta$ yields the advertised stability of the counting and covariance statements. $\square$

Let $\mathcal{G}$ denote the σ-algebra on $X \times \mathbb{T}$ generated by the second coordinate.

Proposition 8.3 (Criterion for fixed-window resolution)**.** *In the setting of Theorem 7.1, the following are equivalent.*

(i) *There exists a measurable set $W \subset \mathbb{T}$ such that $H = \mathbf{1}_W$ almost everywhere.*

(ii) *The coordinate function y_0 is $\mathcal{G}$-measurable on $(X \times \mathbb{T}, \nu)$; equivalently, there exists a measurable function $b : \mathbb{T} \to \{0, 1\}$ such that*

$$y_0 = b(t) \qquad \nu\text{-a.e. on } X \times \mathbb{T}.$$

Proof. Assume first that (i) holds. Since $H = \mathbb{E}_\nu[y_0 \mid \mathcal{G}]$, one has

$$\begin{aligned}\int_{X\times\mathbb{T}} (y_0 - H(t))^2\, d\nu &= \int_{X\times\mathbb{T}} y_0\, d\nu - 2\int_{X\times\mathbb{T}} y_0 H(t)\, d\nu + \int_{X\times\mathbb{T}} H(t)^2\, d\nu \\ &= \int_{\mathbb{T}} H\, dm_{\mathbb{T}} - \int_{\mathbb{T}} H^2\, dm_{\mathbb{T}}.\end{aligned}$$

If $H = \mathbf{1}_W$ almost everywhere, then $H^2 = H$ almost everywhere, so the right-hand side is 0. Hence $y_0 = H(t) = \mathbf{1}_W(t)$ for ν-almost every (y, t), and y_0 is $\mathcal{G}$-measurable.

Conversely, assume (ii). Then $y_0 = b(t)$ for some measurable $b : \mathbb{T} \to \{0, 1\}$. Since H is the conditional expectation of y_0 with respect to $\mathcal{G}$, it follows that $H = b$ almost everywhere. Thus H is almost everywhere the indicator of the measurable set $W := b^{-1}(1)$. □

Corollary 8.4 (Conditional fixed-angle window measure law)**.** *Assume the hypotheses of Theorem 7.1. Suppose in addition that*

$$N_k \,\|\theta_k - \alpha\|_{\mathbb{T}} \to 0$$

and that the equivalent conditions of Proposition 8.3 hold. Then there exists a measurable set $W \subset \mathbb{T}$ with $m_{\mathbb{T}}(W) = \delta$ such that for every $\varphi \in C(\mathbb{T})$,

$$\frac{1}{N_k} \sum_{n=1}^{N_k} \mathbf{1}_{\mathcal{U}}(n)\varphi(n\alpha) \longrightarrow \int_W \varphi(t)\, dm_{\mathbb{T}}(t),$$

and

$$\frac{1}{M_{N_k}} \sum_{\substack{n\le N_k\\ n\in\mathcal{U}}} \varphi(n\alpha) \longrightarrow \frac{1}{\delta}\int_W \varphi(t)\, dm_{\mathbb{T}}(t).$$

In particular, under phase stabilization and circle-factor measurability, the subsequence N_k admits a fixed-angle window measure law. If the resulting window can moreover be chosen with Haar-null boundary, then the corresponding literal indicator-window counts follow as well.

Proof. By Proposition 8.3 one has $H = \mathbf{1}_W$ almost everywhere for some measurable $W \subset \mathbb{T}$. Since $\int_{\mathbb{T}} H\, dm_{\mathbb{T}} = \delta$, necessarily $m_{\mathbb{T}}(W) = \delta$. The conclusion now follows immediately from Proposition 8.1. □

9 Concluding remarks

The results proved above turn finite-scale large Fourier coefficients into a one-dimensional Kronecker component in a Furstenberg limit of the actual Ulam orbit. They also show that positive-density initial segments of the Ulam sequence are uniformly almost sum-free, that this component gives a genuine spectral atom, and that the irrational case produces a bona fide absolutely continuous phase law on the circle together with interval arcs of excess mass.

There are now three distinct remaining rigidity problems, each cleaner than the original hidden-signal question.

1. **Collapse the finite packet set to a singleton.** Theorem 4.2 shows that once a positive-density convergent model is fixed, the maximizing phases are trapped near a finite set of threshold atoms of one limiting Koopman spectrum. A sharp next step would be to show that

   $$S_\mu(c_0) \cap [0, \pi]$$

   contains only one point. Corollary 4.6 would then force actual phase convergence.

2. **Quantitative phase stabilization.** To pass from varying phases to one fixed angle on the raw sequence, Propositions 8.1 and 8.2 show that it would suffice to prove

$$N_k \|\theta_k - \alpha\|_{\mathbb{T}} \to 0.$$

Under this quantitative stabilization, all current covariance and interval statements at $n\theta_k$ immediately upgrade to the fixed frequency $n\alpha$.

3. **Resolve the circle profile into a genuine window.** The remaining qualitative step is to prove that the limiting profile H is almost everywhere $\{0,1\}$-valued, or equivalently that the coordinate function is measurable with respect to the limiting circle factor. Proposition 8.3 isolates this step exactly. To obtain literal indicator-window counts at the fixed angle, one would additionally want a representative window with Haar-null boundary.

These are precisely the places where Green's almost-sum-free structure theorem and the arithmetic regularity technology of Green–Tao and Eberhard seem most naturally adapted to the problem. In that sense, the present paper should be read as an infinitary reduction: the hidden signal for $U(1,2)$ has been promoted to a genuine one-dimensional Kronecker component and finitely many candidate packet centers, and what remains is a quantitative and structural rigidity problem for that component.

References

[1] J. Cassaigne and S. R. Finch, *A class of 1-additive sequences and quadratic recurrences*, Experiment. Math. **4** (1995), no. 1, 49–60.

[2] F. Clément and S. Steinerberger, *Small gaps in the Ulam sequence*, Comptes Rendus Mathématique **363** (2025), 941–949.

[3] S. Eberhard, *The abelian arithmetic regularity lemma*, arXiv:1606.09303.

[4] S. R. Finch, *Patterns in 1-additive sequences*, Experiment. Math. **1** (1992), no. 1, 57–63.

[5] B. Green, *A Szemerédi-type regularity lemma in abelian groups, with applications*, Geom. Funct. Anal. **15** (2005), no. 2, 340–376.

[6] B. Green and T. Tao, *An arithmetic regularity lemma, an associated counting lemma, and applications*, in *An Irregular Mind*, Bolyai Soc. Math. Stud. **21**, János Bolyai Math. Soc., Budapest, 2010, pp. 261–334.

[7] J. Hinman, B. Kuca, A. Schlesinger, and A. Sheydvasser, *The unreasonable rigidity of Ulam sequences*, J. Number Theory **194** (2019), 409–425.

[8] N. Kravitz and S. Steinerberger, *Ulam sequences and Ulam sets*, Integers **18** (2018), Paper A80, 22 pp.

[9] B. Kuca, *Structures in additive sequences*, Acta Arith. **186** (2018), no. 3, 273–300.

[10] R. Queneau, *Sur les suites s-additives*, J. Combin. Theory Ser. A **12** (1972), 31–71.

[11] D. Ross, *The Ulam Sequence and Related Phenomena*, Ph.D. thesis, University of Wisconsin–Madison, 2016.

[12] J. Schmerl and E. Spiegel, *The regularity of some 1-additive sequences*, J. Combin. Theory Ser. A **66** (1994), no. 1, 172–175.

[13] S. Steinerberger, *A hidden signal in the Ulam sequence*, Experiment. Math. **26** (2017), no. 4, 460–467.

[14] S. M. Ulam, *Combinatorial analysis in infinite sets and some physical theories*, SIAM Rev. **6** (1964), no. 4, 343–355.