A fixed-window unconditional reduction for Erdős–Graham's binomial divisor problem

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Abstract

Erdős and Graham asked whether there is an absolute constant $c > 0$ such that every binomial coefficient $\binom{n}{k}$, $1 \le k < n$, has a divisor in $(cn, n]$. Bui, Pratt and Zaharescu proved a stronger negative result conditionally on GRH. Their proof separates into a GRH-dependent covering theorem and an unconditional divisor-counting argument. We give an unconditional replacement for the covering theorem in the fixed-parameter form needed for the original question. For every fixed $B \ge 3$ and arbitrarily large k we construct a residue class $n \equiv \alpha \pmod{N_k}$ and fixed denominators $g_0, \ldots, g_{k-1}$ such that

$$g_i \mid n - i, \qquad g_i \ge B, \qquad \prod_{i=0}^{k-1} g_i = k!,$$

while the quotients $(n-i)/g_i$ have no prime factors below k; after the standard refinement modulo primes in $(k, 2k)$ they have no prime factors below $2k$ and are pairwise coprime. The covering construction uses only the classical Siegel–Walfisz theorem for primes in arithmetic progressions with fixed polylogarithmic moduli. Combining this fixed-B cover with the unconditional divisor propositions of Bui–Pratt–Zaharescu gives, for every sufficiently large fixed B, infinitely many $\binom{n}{k}$ with no divisor in $(n/B, n]$. Choosing $B > 1/c$ gives an unconditional negative answer to the original fixed-c question. The stronger moving endpoint in the Bui–Pratt–Zaharescu theorem is not recovered here.

1 Introduction

We use the convention $P^-(1) = +\infty$ and, for $m > 1$, let $P^-(m)$ denote the least prime divisor of m.

The question is:

> Is there an absolute constant $c > 0$ such that every $\binom{n}{k}$, $1 \le k < n$, has a divisor in $(cn, n]$?

Bui, Pratt and Zaharescu [1] prove, assuming GRH, a negative answer in a stronger moving-window form. Their proof has two parts. Section 5 constructs, using GRH, an arithmetic progression on which the small-prime part of each numerator term can be cancelled in a controlled way. Sections 6–10 then use unconditional sieve and exponential-sum estimates to rule out the remaining, "non-obvious", divisors.

The purpose of this note is to replace only the first part in the fixed-B form sufficient for the Erdős–Graham question. The replacement does not give the BPZ endpoint $n/(2\sqrt{\log\log k})$; it gives the following fixed-window result.

Theorem 1.1. *There is a constant B_* such that, for every fixed $B \ge B_*$, there are infinitely many pairs (n, k) for which $\binom{n}{k}$ has no divisor in $(n/B, n]$.*

Corollary 1.2. *There is no absolute constant $c > 0$ such that every binomial coefficient $\binom{n}{k}$, $1 \le k < n$, has a divisor in $(cn, n]$.*

Indeed, given $c > 0$, choose an integer $B > \max(B_*, 1/c)$. Then $(cn, n] \subseteq (n/B, n]$, so Theorem 1.1 gives infinitely many counterexamples to the proposed constant c.

The proof of Theorem 1.1 has two ingredients. The first is the covering theorem, proved in Section 2. The second is a fixed-B reading of the BPZ divisor propositions, stated and audited in Section 3. The long analytic estimates of BPZ are not reproved; the point is that after the covering data below are supplied, their later arguments use only the component interface listed in Proposition 3.3. This is the same split emphasized in the Erdős problems forum discussion [2] and in the BPZ outline.

2 The fixed-B covering theorem

For $k \ge 1$ and primes $p \le k$, put

$$\alpha_p = \alpha_p(k) := \left\lfloor \frac{\log k}{\log p} \right\rfloor, \qquad N_k := \prod_{p \le k} p^{\alpha_p + 1}.$$

Thus $p^{\alpha_p} \le k < p^{\alpha_p+1}$.

Theorem 2.1 (Fixed-B denominator allocation)**.** *Fix $B \ge 3$. For every $K \ge 1$ there exist an integer $k \ge K$, a residue class $\alpha \pmod{N_k}$, and positive integers $g_0, \dots, g_{k-1}$ such that, for every integer $n > k$ with $n \equiv \alpha \pmod{N_k}$,*

(i) *$g_i \mid n - i$ for every $0 \le i < k$;*

(ii) *$g_i \ge B$ for every $0 \le i < k$;*

(iii) *$\prod_{i=0}^{k-1} g_i = k!$;*

(iv) *no prime $p \le k$ divides $\binom{n}{k}$;*

(v) *no prime $p \le k$ divides $(n-i)/g_i$ for any $0 \le i < k$.*

Consequently

$$\binom{n}{k} = \prod_{i=0}^{k-1} \frac{n-i}{g_i}, \qquad P^-\left(\frac{n-i}{g_i}\right) > k, \qquad \frac{n-i}{g_i} \le \frac{n}{B}.$$

2.1 The only analytic input

We use the following standard consequence of the Siegel–Walfisz theorem. The intervals have dyadic length; only their endpoints are shifted by a polylogarithmic amount.

Theorem 2.2 (Siegel–Walfisz on shifted dyadic intervals)**.** *For every fixed $C > 0$ there are constants $X_0(C)$ and $c_C > 0$ such that, whenever $X \ge X_0(C)$, $w \le (\log X)^C$, $0 \le h \le (\log X)^C$, and $(a, w) = 1$, one has*

$$\#\{p \text{ prime} : X - h < p \le 2X - h,\ p \equiv a \pmod{w}\} \ge c_C \frac{X}{\varphi(w) \log X}.$$

This follows by applying Siegel–Walfisz to the two endpoints $2X - h$ and $X - h$ and subtracting; see, for example, Montgomery–Vaughan [4, Corollaries 11.19 and 11.21] or Iwaniec–Kowalski [3, Corollary 5.29].

2.2 Anchor, buffers, and the residual set

Fix $0 < \eta < 1/10$. Let $A = A(B, \eta)$ be a sufficiently large integer, to be chosen after Lemma 2.5, let X be large, and put

$$Y = (\log X)^A.$$

Choose an anchor prime

$$Q \in [Y^\eta, 2Y^\eta],$$

which is possible for large X by the prime number theorem. For $t \geq 1$ define

$$z_Q(t) := \frac{t}{Q^{v_Q(t)}}$$

and the provisional contribution

$$C_0(t) := z_Q(t) Q^{\mathbf{1}_{t \equiv 1 \pmod{Q}}}.$$

Let

$$\mathcal{D}_Y := \{1 \leq d \leq Y : C_0(d) < B\}.$$

Since $Q > B$ for large X, every $d \in \mathcal{D}_Y$ satisfies $d \not\equiv 1 \pmod{Q}$ and $z_Q(d) < B$.

Lemma 2.3. *For all sufficiently large X,*

$$|\mathcal{D}_Y| \leq B(1 + \eta^{-1}).$$

Proof. If $d \in \mathcal{D}_Y$, write $d = uQ^a$ with $Q \nmid u$. Then $u = z_Q(d) < B$. Since $d \leq Y$ and $Q \geq Y^\eta$, one has $a \leq \eta^{-1}$. Hence d is determined by $u < B$ and by $0 \leq a \leq \eta^{-1}$. □

For every $d \in \mathcal{D}_Y$ choose a distinct buffer prime

$$b_d \in (Y^2, 2Y^2].$$

This is possible because $|\mathcal{D}_Y| = O_B(1)$. Define

$$\mathcal{S} := \{Q\} \cup \{b_d : d \in \mathcal{D}_Y\}, \qquad W := Q \prod_{d \in \mathcal{D}_Y} b_d.$$

Then $W = (\log X)^{O_B(1)}$.

For a finite set of primes $\mathcal{R}$, write

$$z_{\mathcal{R}}(t) := \frac{t}{\prod_{r \in \mathcal{R}} r^{v_r(t)}}.$$

The residual set is

$$\begin{aligned} \mathcal{U}_X := \{1 \leq t \leq 2X : z_{\mathcal{S}}(t) < B, \quad t \not\equiv 1 \pmod{Q}, \quad \text{and} \\ t \not\equiv d \pmod{b_d} \text{ for every } d \in \mathcal{D}_Y\}. \end{aligned} \tag{1}$$

Thus $\mathcal{U}_X$ is exactly the set of indices for which the anchor congruence, the buffer congruences, and the zero-residue prime powers outside $\mathcal{S}$ still do not force a denominator of size B.

Lemma 2.4 (No small residuals)**.** *One has $\mathcal{U}_X \cap [1, Y] = \varnothing$.*

Proof. Suppose $t \leq Y$ and $t \in \mathcal{U}_X$. Since every b_d is larger than Y, no buffer prime divides t, and hence $z_{\mathcal{S}}(t) = z_Q(t)$. The defining conditions of $\mathcal{U}_X$ give $z_Q(t) < B$ and $t \not\equiv 1 \pmod{Q}$, so $t \in \mathcal{D}_Y$. Taking $d = t$ in the buffer condition gives $t \equiv d \pmod{b_d}$, a contradiction. □

Lemma 2.5 (The residual set is polylogarithmic)**.** *There is a constant C_B such that, for all sufficiently large X,*

$$|\mathcal{U}_X| \le (\log X)^{C_B}.$$

Proof. If $t \in \mathcal{U}_X$, then $z_{\mathcal{S}}(t) = u < B$. Therefore

$$t = uQ^a \prod_{d \in \mathcal{D}_Y} b_d^{c_d}$$

for some $1 \le u < B$ and nonnegative exponents a, c_d. Since $t \le 2X$, each exponent has $O(\log X)$ possible values. Lemma 2.3 gives $|\mathcal{D}_Y| = O_B(1)$, and the result follows. □

2.3 Scaffold primes near k

For $k \in [X, 2X]$ with $W \mid k$, define

$$A(k) := \#\{p \text{ prime} : k - Y/2 < p \le k,\ p \equiv 1 \pmod{Q}\}.$$

Let

$$\mathcal{H} := \{h \in \mathbb{N} : 0 \le h < Y/2,\ h \equiv Q - 1 \pmod{Q}\}.$$

For large X, $|\mathcal{H}| \ge Y/(4Q)$.

Lemma 2.6 (Average scaffold supply)**.** *For all sufficiently large X,*

$$\sum_{\substack{X \le k \le 2X \\ W \mid k}} A(k) \gg_B \frac{X}{W} \frac{Y}{Q \log X}.$$

Proof. Write $p = k - h$. The conditions $W \mid k$ and $p \equiv 1 \pmod{Q}$ are implied by

$$p \equiv W - h \pmod{W}, \qquad h \in \mathcal{H}.$$

For $h \in \mathcal{H}$, one has $(h, W) = 1$: indeed $h \equiv Q - 1 \pmod{Q}$, and $h < Y/2 < Y^2 < b_d$ for every buffer prime. Hence $(W - h, W) = 1$.

Since $W = (\log X)^{O_B(1)}$ and $h \le Y = (\log X)^A$, Theorem 2.2 gives

$$\#\{p : X - h < p \le 2X - h,\ p \equiv W - h \pmod{W}\} \gg_B \frac{X}{W \log X}$$

for every $h \in \mathcal{H}$; here we used $\varphi(W) \asymp_B W$, since W has $O_B(1)$ prime factors, all tending to infinity with X. Each pair (h, p) gives a unique $k = p + h \in [X, 2X]$ with $W \mid k$, and then $p \in (k - Y/2, k]$ and $p \equiv 1 \pmod{Q}$. Summing over $h \in \mathcal{H}$ proves the claim. □

Lemma 2.7 (Endpoint exclusions)**.** *The number of multiples $k \in [X, 2X]$, $W \mid k$, for which*

$$\mathcal{U}_X \cap (k - Y, k] \ne \varnothing$$

is at most

$$|\mathcal{U}_X| \left(\frac{Y}{W} + 2 \right).$$

The total contribution of these excluded k to $\sum_{W \mid k} A(k)$ is

$$o\left(\frac{X}{W} \frac{Y}{Q \log X} \right).$$

Proof. For a fixed $t \in \mathcal{U}_X$, the conditions $t \in (k-Y, k]$ and $W \mid k$ put k among the multiples of W in an interval of length Y, giving at most $Y/W + 2$ possibilities. This proves the first assertion. Since trivially $A(k) \le Y$, the excluded contribution is

$$O\left(|\mathcal{U}_X| \left(\frac{Y}{W} + 2\right) Y\right).$$

By Lemma 2.5, and since Y, Q, W are all powers of $\log X$ with exponents depending only on B, η, A, the ratio of this bound to $(X/W)Y/(Q \log X)$ tends to 0. □

Proposition 2.8 (Good k and matching)**.** *If $A = A(B, \eta)$ is sufficiently large, then for all sufficiently large X there exists $k \in [X, 2X]$ such that $W \mid k$,*

$$A(k) \ge 3|\mathcal{U}_X|, \qquad \mathcal{U}_X \cap (k-Y, k] = \varnothing.$$

Consequently the elements of $\mathcal{U}_X \cap [1, k]$ can be matched injectively to distinct primes

$$p_t \in (k - Y/2, k], \qquad p_t \equiv 1 \pmod{Q}.$$

Proof. The average value of $A(k)$ over multiples $k \in [X, 2X]$ of W is, by Lemma 2.6,

$$\gg_B \frac{Y}{Q \log X} \gg_B (\log X)^{A(1-\eta)-1}.$$

Choose A so large that this average is at least $10|\mathcal{U}_X|$ for large X, using Lemma 2.5. Lemma 2.7 shows that removing the endpoint-excluded k loses $o(1)$ of the total scaffold mass. Therefore some non-excluded multiple of W has $A(k) \ge 3|\mathcal{U}_X|$. Since $A(k)$ counts available primes and $A(k)$ is larger than $|\mathcal{U}_X \cap [1, k]|$, the matching follows. □

2.4 Residues, lifts, and denominator exponents

Fix k from Proposition 2.8. Define residues for primes $p \le k$ by

$$a_Q = 1, \qquad a_{b_d} = d \quad (d \in \mathcal{D}_Y), \qquad a_{p_t} = t \quad (t \in \mathcal{U}_X \cap [1, k]),$$

and put $a_p = 0$ for all remaining primes $p \le k$.

Lemma 2.9 (Level-one non-excess)**.** *If $a_p \ne 0$, then a_p is represented by an integer in $\{1, \dots, p-1\}$ and*

$$a_p > k \bmod p.$$

Proof. For $p = Q$, $Q \mid W \mid k$ and $a_Q = 1$. For $p = b_d$, $b_d \mid W \mid k$ and $1 \le d \le Y < b_d$. Thus the claim is immediate in these two cases.

For a scaffold prime $p = p_t$, we have $p_t > k - Y/2$, so $k \bmod p_t = k - p_t < Y/2$. Lemma 2.4 gives $t > Y$, so $t > k \bmod p_t$. The endpoint exclusion gives $t \le k - Y$, while $p_t > k - Y/2 > k - Y$, so $t < p_t$. □

For $a_p \ne 0$, define compatible lifts

$$a_p^{(1)} := a_p, \qquad a_p^{(u)} := p^u - p + a_p \quad (u \ge 2).$$

These satisfy $a_p^{(u+1)} \equiv a_p^{(u)} \pmod{p^u}$.

Lemma 2.10 (All-level non-excess)**.** *If $a_p \ne 0$ and $1 \le u \le \alpha_p$, then*

$$a_p^{(u)} > k \bmod p^u.$$

Moreover $a_p^{(\alpha_p+1)} > k$.

Proof. For $u = 1$ this is Lemma 2.9. If $p = Q$ or $p = b_d$, then $p \mid k$, and hence for $u \geq 2$,

$$k \bmod p^u \leq p^u - p < a_p + p^u - p = a_p^{(u)}.$$

Also $k \leq p^{\alpha_p+1} - p$ because k is a multiple of p below p^{α_p+1}, so $a_p^{(\alpha_p+1)} > k$.

If $p = p_t$ is a scaffold prime, then $p_t > k - Y/2 > k/2$ for large X, so $\alpha_p = 1$. There are no levels $u \geq 2$ with $u \leq \alpha_p$, and the top lift $p_t^2 - p_t + t$ is $> k$ because $p_t^2 > k$. □

For $1 \leq j \leq k$ and primes $p \leq k$, define

$$e_{j,p} := \begin{cases} v_p(j), & a_p = 0, \\ \max\bigl(\{0\} \cup \{1 \leq u \leq \alpha_p : j \equiv a_p^{(u)} \pmod{p^u}\}\bigr), & a_p \neq 0, \end{cases}$$

Put

$$h_j := \prod_{p \leq k} p^{e_{j,p}} \qquad (1 \leq j \leq k).$$

Lemma 2.11 (Valuation sum)**.** *For every prime $p \leq k$,*

$$\sum_{j=1}^{k} e_{j,p} = v_p(k!).$$

Consequently $\prod_{j=1}^{k} h_j = k!$.

Proof. If $a_p = 0$, this is Legendre's formula. Suppose $a_p \neq 0$. For each $1 \leq u \leq \alpha_p$, the condition $e_{j,p} \geq u$ is equivalent to

$$j \equiv a_p^{(u)} \pmod{p^u}.$$

By Lemma 2.10, this nonzero residue lies above $k \bmod p^u$, so it occurs exactly $\lfloor k/p^u \rfloor$ times in $1 \leq j \leq k$. Summing over u gives

$$\sum_{j=1}^{k} e_{j,p} = \sum_{u \geq 1} \left\lfloor \frac{k}{p^u} \right\rfloor = v_p(k!).$$

Multiplying over primes gives the product identity. □

Lemma 2.12 (Every denominator is large)**.** *For every $1 \leq j \leq k$ one has $h_j \geq B$.*

Proof. If $j \equiv 1 \pmod{Q}$, then $Q \mid h_j$, so $h_j \geq B$. If $j \equiv d \pmod{b_d}$ for some $d \in \mathcal{D}_Y$, then $b_d \mid h_j$, so $h_j \geq B$. If $j \in \mathcal{U}_X \cap [1, k]$, then j is matched to a scaffold prime p_j with $a_{p_j} = j$, and hence $p_j \mid h_j$; again $h_j \geq B$.

It remains to treat j in none of these cases. Then $j \leq k \leq 2X$, $j \not\equiv 1 \pmod{Q}$, $j \not\equiv d \pmod{b_d}$ for every $d \in \mathcal{D}_Y$, and $j \notin \mathcal{U}_X$. By the definition of $\mathcal{U}_X$, this forces $z_{\mathcal{S}}(j) \geq B$.

We claim that $z_{\mathcal{S}}(j) \mid h_j$. Every prime power in $z_{\mathcal{S}}(j)$ comes from a prime outside $\mathcal{S}$. Such a prime cannot be a scaffold prime: if $p_t \mid j$, then $p_t > k/2$ and $j \leq k$ force $j = p_t$, but then $j \equiv 1 \pmod{Q}$, contrary to the present case. Therefore every prime factor of $z_{\mathcal{S}}(j)$ has residue zero, and for residue-zero primes the exponent in h_j is exactly $v_p(j)$. Thus $z_{\mathcal{S}}(j) \mid h_j$, and $h_j \geq B$. □

2.5 The CRT progression

For $a_p = 0$, impose
$$n \equiv k \pmod{p^{\alpha_p+1}}.$$
For $a_p \neq 0$, impose
$$n \equiv k - a_p^{(\alpha_p+1)} \pmod{p^{\alpha_p+1}}.$$
The Chinese remainder theorem gives a residue class $\alpha \pmod{N_k}$ satisfying all these congruences.

Lemma 2.13 (Exact local valuations)**.** *If $n \equiv \alpha \pmod{N_k}$, then for every prime $p \leq k$ and every $1 \leq j \leq k$,*
$$v_p(n-k+j) = e_{j,p}.$$

Proof. If $a_p = 0$, then $n-k+j \equiv j \pmod{p^{\alpha_p+1}}$. Since $1 \leq j \leq k < p^{\alpha_p+1}$, this gives $v_p(n-k+j) = v_p(j) = e_{j,p}$.

If $a_p \neq 0$, then
$$n-k+j \equiv j - a_p^{(\alpha_p+1)} \pmod{p^{\alpha_p+1}}.$$
The top lift is compatible with all lower lifts, so for $1 \leq u \leq \alpha_p$, divisibility by p^u is equivalent to $j \equiv a_p^{(u)} \pmod{p^u}$. Divisibility by p^{α_p+1} is impossible for $1 \leq j \leq k$, since $a_p^{(\alpha_p+1)}$ is represented by an integer between k and p^{α_p+1}. Hence the exact valuation is $e_{j,p}$. □

Proof of Theorem 2.1. Choose X large enough in terms of B and K that the construction above gives a $k \in [X, 2X]$ with $k \geq K$. Define
$$g_i := h_{k-i} \qquad (0 \leq i < k).$$
Lemma 2.12 gives $g_i \geq B$, and Lemma 2.11 gives $\prod_i g_i = k!$. Lemma 2.13, with $j = k-i$, gives $g_i \mid n-i$ for every $n \equiv \alpha \pmod{N_k}$.

For every prime $p \leq k$, summing Lemma 2.13 over $j = 1, \ldots, k$ gives exactly $v_p(k!)$, so $p \nmid \binom{n}{k}$. The same exact valuation statement shows that, for each fixed i, the factor g_i removes the full p-adic valuation of $n-i$; hence $p \nmid (n-i)/g_i$. The displayed factorization follows from $n-i = g_i((n-i)/g_i)$ and $\prod_i g_i = k!$, and $(n-i)/g_i \leq n/B$ follows from $g_i \geq B$. □

3 The BPZ divisor machine in fixed-B form

Fix $B \geq 3$ and data k, α, g_i supplied by Theorem 2.1. Refine the progression by imposing
$$n \equiv k \pmod{q} \qquad (k < q < 2k,\ q \text{ prime}).$$
Let
$$M := N_k \prod_{k<q<2k} q,$$
and let $\gamma \pmod{M}$ be the residue class obtained by CRT from this refinement and $\alpha \pmod{N_k}$.

For $n \equiv \gamma \pmod{M}$, write
$$m_i(n) := \frac{n-i}{g_i} \qquad (0 \leq i < k).$$

Lemma 3.1 (Component interface)**.** *For every $n > k$ with $n \equiv \gamma \pmod{M}$,*
$$\binom{n}{k} = \prod_{i=0}^{k-1} m_i(n), \qquad m_i(n) \leq \frac{n}{B}, \qquad P^-(m_i(n)) > 2k,$$
and the $m_i(n)$ are pairwise coprime. Moreover $g_i \mid M$ and g_i depends only on i.

Proof. Theorem 2.1 gives the factorization, the bound $m_i(n) \le n/B$, and the absence of prime factors $\le k$ in $m_i(n)$. If $k < q < 2k$ is prime, then $n \equiv k \pmod q$, so

$$n - i \equiv k - i \pmod q, \qquad 1 \le k - i \le k < q.$$

Thus no such q divides any $n-i$, and hence none divides any $m_i(n)$. This proves $P^-(m_i(n)) > 2k$.

If a prime ℓ divided both $m_i(n)$ and $m_j(n)$, with $i \ne j$, then $\ell > 2k$ would divide $i - j$, impossible because $0 < |i - j| < k$. Finally, $g_i = h_{k-i}$ has p-adic exponent at most α_p for every $p \le k$, so $g_i \mid N_k \mid M$. ☐

For primes $p \ge 2k$, the local density is unchanged from BPZ.

Lemma 3.2 (Large-prime local density)**.** *Let $p \ge 2k$ be prime. Among residue classes modulo p, exactly the k classes*

$$n \equiv 0, 1, \dots, k-1 \pmod p$$

make $p \mid \binom{n}{k}$. Equivalently, these are exactly the classes for which p divides one of the components $m_i(n)$.

Proof. Since $p > k$, $p \nmid k!$ and $p \nmid \prod_i g_i$. Therefore $p \mid \binom{n}{k}$ if and only if $p \mid n - i$ for some $0 \le i < k$, equivalently if and only if $p \mid m_i(n)$ for some i. The k residue classes are distinct because $k < p$. ☐

Let x be large compared with k, assume

$$k \le \vartheta(\log\log x)^{1/2} \tag{2}$$

with $\vartheta > 0$ sufficiently small, and put

$$\varepsilon_k := 3^{-k}, \qquad z := x^{\varepsilon_k}.$$

We always take x so large that $z > 2k$. Write $n \sim x$ for $x/2 < n \le x$ and define

$$S := \sum_{\substack{n \sim x \\ n \equiv \gamma\ (M) \\ P^-(\binom{n}{k}) \ge z}} 1.$$

Let E_B be the number of n counted by S for which $\binom{n}{k}$ has a divisor in $(n/B, n]$.

Proposition 3.3 (BPZ divisor estimates, fixed-B form)**.** *There is an absolute constant B_* such that the following holds. Fix $B \ge B_*$. For all sufficiently large k in terms of B, and then all sufficiently large x satisfying (2),*

$$S \ge (1 + o_{k\to\infty}(1))\mathfrak{S}_k \frac{x}{2M}(\log z)^{-k} > 0,$$

where $\mathfrak{S}_k$ is the BPZ singular series, and

$$E_B \le \big(O_B(k^{-1}) + o_{B,k;x\to\infty}(1)\big)S.$$

In particular, after taking k sufficiently large in terms of B and then x sufficiently large in terms of B, k, one has $E_B < S$.

Audit of the import from BPZ. This is the fixed-B reading of [1, Propositions 6.1 and 6.3–6.7]. BPZ state the argument after their Theorem 5.1, but the proofs of the divisor propositions use the covering theorem only through the following data.

First, the lower-bound sieve for S uses that the refined progression eliminates primes below $2k$, and that for every prime $p \geq 2k$ exactly k residue classes modulo p make $p \mid \binom{n}{k}$. These are Lemmas 3.1 and 3.2. This gives precisely the local factor used in [1, Proposition 6.1].

Second, if n is counted by S, then each $m_i(n)$ has no prime factor below z, because

$$\binom{n}{k} = \prod_i m_i(n)$$

and $P^-(\binom{n}{k}) \geq z$. Hence every divisor $d \mid \binom{n}{k}$ with $d \in (n/B, n]$ decomposes uniquely as

$$d = d_0 d_1 \cdots d_{k-1}, \qquad d_i \mid m_i(n),$$

with the d_i pairwise coprime and with congruences $n \equiv i \pmod{d_i}$ that combine by CRT. If all but one d_i were 1, then the remaining component would be at most $m_i(n) \leq n/B$, contradicting $d > n/B$. Thus every bad divisor uses at least two nontrivial components. This is the replacement for the one-component exclusion in BPZ Section 6.

Third, in every later divisor estimate BPZ write, for a selected component,

$$n - i = g_i d_i b_i,$$

where g_i is fixed, divides the small-prime part, and d_i, b_i have prime factors at least z. Lemma 3.1 supplies exactly this relation with the present g_i, since $m_i = d_i b_i$ and $g_i \mid M$. The factors g_i are fixed once the progression is fixed, have prime factors at most k, and satisfy $g_i \leq N_k \leq \exp(O(k))$; BPZ's estimates use no further information from their GRH covering section.

The five bad-divisor ranges in [1, Propositions 6.3–6.7] are then unchanged: one very large component, one intermediate component, a component with a convenient factorization, the close almost-prime case, and the separated almost-prime case. Replacing BPZ's original components by the divisors of the quotients $m_i(n)$ only restricts the set being counted while preserving the same congruences and coprimality. The parameter B enters the divisor problem only through the interval $d \in (n/B, n]$ and the inequality $m_i(n) \leq n/B$; for fixed B, all implicit constants in the upper bounds may depend on B. Consequently the conclusions of BPZ Propositions 6.1 and 6.3–6.7 give the displayed estimates. □

4 Completion of the proof

Proof of Theorem 1.1. Fix $B \geq B_*$. Choose $K = K(B)$ so large that Proposition 3.3 applies for every $k \geq K$ and the $O_B(k^{-1})$ term is less than 1/3. Apply Theorem 2.1 with this K, obtaining k, a class $\alpha \pmod{N_k}$, and denominators g_i. Refine to the class $\gamma \pmod{M}$ as in Section 3.

Now take x arbitrarily large subject to (2) and large enough that the $o_{B,k;x\to\infty}(1)$ term in Proposition 3.3 is less than 1/3. Then $E_B < S$, while $S > 0$. Hence some $n \sim x$, $n \equiv \gamma \pmod{M}$, is counted by S but not by E_B. For this n, the binomial coefficient $\binom{n}{k}$ has no divisor in $(n/B, n]$.

Since x may be taken arbitrarily large after k is fixed, this gives infinitely many such n. □

Proof of Corollary 1.2. Let $c > 0$ be given. Choose an integer $B > \max(B_*, 1/c)$. Theorem 1.1 gives infinitely many $\binom{n}{k}$ with no divisor in $(n/B, n]$. Since $1/B < c$, we have $(cn, n] \subseteq (n/B, n]$, so these same binomial coefficients have no divisor in $(cn, n]$. □

5 Audit checklist and scope

The covering theorem above is unconditional. Its only analytic input is Theorem 2.2. A Lean formalization of this fixed-B covering construction is described in [5]; the formalization does not include the BPZ analytic divisor estimates.

The precise plug-in facts supplied to BPZ Sections 6–10 are:

BPZ divisor input	Supplied here
Progression modulo N_k and no primes $\le k$ in $\binom{n}{k}$	Theorem 2.1 and Lemma 2.13
Fixed denominators $g_i \mid n-i$, $\prod_i g_i = k!$, $g_i \ge B$	Theorem 2.1
The quotient $m_i = (n-i)/g_i$ has no primes $< 2k$	Refinement modulo primes $q \in (k, 2k)$
The components m_i are pairwise coprime	Lemma 3.1
A one-component divisor cannot lie in $(n/B, n]$	$m_i \le n/B$
For $p \ge 2k$ the local density is k/p	Lemma 3.2
The g_i are fixed and divide M	Lemma 3.1

This proves the qualitative fixed-window Erdős–Graham statement. It does not prove BPZ's moving-window theorem with lower endpoint $n/(2\sqrt{\log\log k})$, because the covering construction here fixes B before choosing k.

References

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