A STRICT-ORDER RESOLUTION OF ERDŐS PROBLEM #415 FOR CONSECUTIVE TOTIENTS

CIRCULATION DRAFT

Abstract. The current discussion thread for Erdős problem #415 asks three questions about the ordering patterns of

$$\varphi(m+1), \ldots, \varphi(m+k):$$

whether the universal pattern threshold has the form $(c+o(1))\log_3 x$, whether the first missing pattern is always the decreasing one, and whether the "natural" ordering coming from $\varphi(1), \ldots, \varphi(k)$ is the most likely.

For the strict-permutation interpretation forced by the phrase "the $k!$ possible ordering patterns", we give a complete asymptotic solution. Let $F_{\mathrm{str}}(x)$ be the largest integer k such that every permutation in S_k occurs as the strict order pattern of some block

$$(\varphi(n+1), \ldots, \varphi(n+k)), \qquad n+k \le x.$$

We prove

$$F_{\mathrm{str}}(x) = \frac{\log_3 x}{\log_6 x} + (\alpha - \gamma + o(1))\frac{\log_3 x}{(\log_6 x)^2},$$

where $e^{\alpha} = \prod_p (1-1/p)^{-1/p}$. We also show that the longest strictly increasing and strictly decreasing consecutive blocks of totients have the same asymptotic expansion, and that every fixed permutation occurs infinitely often.

Thus the first thread question has an exact strict-order answer, and the second has an asymptotic threshold answer: the monotone patterns are asymptotically extremal. The third question is not a strict-permutation question at all, because $\varphi(1), \ldots, \varphi(k)$ has ties; it belongs to the weak-order setting, which already contains Erdős problem #1003 on the infinitude of solutions to $\varphi(n) = \varphi(n+1)$.

The only genuinely new ingredient is a permutation-equivariant reindexing of the lower-bound construction in Section 8 of Pollack–Pomerance–Treviño.

1. The thread questions and the strict-order problem

Write $\log_1 x := \log x$ and inductively $\log_{j+1} x := \log(\log_j x)$ whenever this is defined. All logarithms are natural.

The current thread for problem #415 asks the following three questions; see [2].

> For any n let $F(n)$ be the largest k such that any of the $k!$ possible ordering patterns appears in some sequence of $\varphi(m+1), \ldots, \varphi(m+k)$ with $m+k \le n$. Is it true that
>
> $$F(n) = (c+o(1))\log\log\log n$$
>
> for some constant c? Is the first pattern which fails to appear always
>
> $$\varphi(m+1) > \varphi(m+2) > \cdots > \varphi(m+k)?$$
>
> Is it true that "natural" ordering which mimics what happens to $\varphi(1), \ldots, \varphi(k)$ is the most likely to appear?

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The phrase "the k! possible ordering patterns" points to a strict-permutation interpretation. That is the version resolved in this paper.

Definition 1.1. *For a k-tuple of distinct real numbers $(a_1, \dots, a_k)$, define*

$$\mathrm{pat}(a_1, \dots, a_k) = \pi \in S_k$$

by the rule

$$a_i < a_j \iff \pi(i) < \pi(j) \qquad (1 \le i, j \le k).$$

For $x \ge 1$, let $F_{\mathrm{str}}(x)$ be the largest integer k such that for every $\pi \in S_k$ there exists an integer n with $n + k \le x$ and

$$\mathrm{pat}\big(\varphi(n+1), \dots, \varphi(n+k)\big) = \pi.$$

To address the second question from the thread, we also define

$$D_{\mathrm{str}}(x) := \max\big\{k : \exists n \le x - k \text{ with } \varphi(n+1) > \cdots > \varphi(n+k)\big\},$$
$$I_{\mathrm{str}}(x) := \max\big\{k : \exists n \le x - k \text{ with } \varphi(n+1) < \cdots < \varphi(n+k)\big\}.$$

Theorem 1.2. *Let*

$$\exp(\alpha) := \prod_p \left(1 - \frac{1}{p}\right)^{-1/p}.$$

Then, as $x \to \infty$,

$$F_{\mathrm{str}}(x) = \frac{\log_3 x}{\log_6 x} + (\alpha - \gamma + o(1)) \frac{\log_3 x}{(\log_6 x)^2}.$$

Moreover,

$$D_{\mathrm{str}}(x) = I_{\mathrm{str}}(x) = \frac{\log_3 x}{\log_6 x} + (\alpha - \gamma + o(1)) \frac{\log_3 x}{(\log_6 x)^2}.$$

Finally, for every fixed $k \ge 1$ and every permutation $\pi \in S_k$, the pattern π occurs infinitely often among the blocks

$$\big(\varphi(n+1), \dots, \varphi(n+k)\big).$$

Corollary 1.3 (Answers to the three thread questions in the strict setting)**.** *Under Definition 1.1, the three questions from problem #415 have the following answers.*

(1) *The first question has an exact answer: the correct strict-order asymptotic is*

$$\frac{\log_3 x}{\log_6 x} + (\alpha - \gamma + o(1)) \frac{\log_3 x}{(\log_6 x)^2},$$

not $(c + o(1)) \log_3 x$.

(2) *The second question has an asymptotic threshold answer: both monotone patterns are asymptotically extremal, since $D_{\mathrm{str}}(x)$, $I_{\mathrm{str}}(x)$, and $F_{\mathrm{str}}(x)$ all have the same asymptotic expansion. The theorem does* not *prove the stronger finite-x claim that the first missing pattern is always the decreasing one.*

(3) *The third question is not a strict-permutation question, because $\varphi(1), \varphi(2), \varphi(3), \varphi(4) = 1, 1, 2, 2$ already has ties. It belongs to the weak-order setting discussed in Section 5.*

Proof. This is just a restatement of Theorem 1.2 and Proposition 5.2 below. □

Remark 1.4 (On the discrepancy with older summaries)**.** *The current problem page for #415 states that Erdős proved $F(n) \asymp \log_3 n$; see* [2]*. Under Definition 1.1, this cannot be the relevant function, because the increasing and decreasing permutations are among the required patterns, so*

$$F_{\mathrm{str}}(x) \le \min\{D_{\mathrm{str}}(x), I_{\mathrm{str}}(x)\},$$

and Theorem 1.2 shows that the right-hand side has order $\log_3 x/\log_6 x$*. We therefore treat the strict-permutation problem on its own terms and do not attempt a bibliographical reconciliation here. Compare also the citation of Erdős's 1958 paper in Pollack–Pomerance–Treviño; see* [7, p. 3].

2. External input from monotone consecutive blocks

Let $M^{\downarrow}_{\mathrm{con}}(x)$ be the maximum size of a block of consecutive integers contained in $[1,x]$ on which φ is nonincreasing, and let $M^{\uparrow}_{\mathrm{con}}(x)$ be the analogous quantity with "nondecreasing" in place of "nonincreasing".

Theorem 2.1 (Pollack–Pomerance–Treviño)**.** *As* $x\to\infty$*,*

$$M^{\downarrow}_{\mathrm{con}}(x)=\frac{\log_3 x}{\log_6 x}+(\alpha-\gamma+o(1))\frac{\log_3 x}{(\log_6 x)^2},$$

and the same asymptotic formula holds for $M^{\uparrow}_{\mathrm{con}}(x)$*.*

Proof. This is exactly [7, Theorem 1.5]. □

Remark 2.2 (Strict monotonicity in the lower bound)**.** *Remark 8.1 of* [7] *states that the lower-bound construction for Theorem 2.1 already gives strictly monotone consecutive blocks. Consequently,*

$$D_{\mathrm{str}}(x)\le M^{\downarrow}_{\mathrm{con}}(x),\qquad I_{\mathrm{str}}(x)\le M^{\uparrow}_{\mathrm{con}}(x),$$

with matching lower bounds at the asymptotic level.

Proposition 2.3. *As* $x\to\infty$*,*

$$D_{\mathrm{str}}(x)=I_{\mathrm{str}}(x)=\frac{\log_3 x}{\log_6 x}+(\alpha-\gamma+o(1))\frac{\log_3 x}{(\log_6 x)^2}.$$

Proof. The upper bounds follow from the definitions and Theorem 2.1. The matching lower bounds are supplied by Remark 2.2. □

Proposition 2.4 (Upper bound for $F_{\mathrm{str}}(x)$)**.** *As* $x\to\infty$*,*

$$F_{\mathrm{str}}(x)\le\frac{\log_3 x}{\log_6 x}+(\alpha-\gamma+o(1))\frac{\log_3 x}{(\log_6 x)^2}.$$

Proof. If $F_{\mathrm{str}}(x)\ge k$, then in particular the increasing permutation $1\,2\,\ldots\,k$ and the decreasing permutation $k\,(k-1)\,\ldots\,1$ both occur among the blocks

$$\big(\varphi(n+1),\ldots,\varphi(n+k)\big),\qquad n+k\le x.$$

Hence $F_{\mathrm{str}}(x)\le I_{\mathrm{str}}(x)$ and $F_{\mathrm{str}}(x)\le D_{\mathrm{str}}(x)$. Now apply Proposition 2.3. □

3. A permutation-equivariant lower bound

This section contains the only ingredient that does not seem to be explicitly written down in the literature: a rank-symmetric version of the Section 8 lower-bound construction in [7]. Once this is established, the remainder of the proof is a direct relabelling of Pollack–Pomerance–Treviño.

Fix $\delta>0$, let x be large, and put

$$k:=\left\lfloor\frac{\log_3 x}{\log_6 x}+(\alpha-\gamma-\delta)\frac{\log_3 x}{(\log_6 x)^2}\right\rfloor. \tag{1}$$

Let $L:=\log_2 x$, and let A be the largest primorial not exceeding k. Then

$$\frac{\varphi(i)}{i}\ge\frac{\varphi(A)}{A}\qquad(1\le i\le k), \tag{2}$$

since A uses the smallest possible prime factors.

We partition the primes into three classes:

$$\begin{aligned}&\text{small primes: } p \le k,\\ &\text{medium primes: } k < p \le \tfrac{1}{2}\log x,\\ &\text{large primes: } p > \tfrac{1}{2}\log x.\end{aligned}$$

As in [7, (8.2)], the contribution of the large prime factors of any $m \le x$ to $\varphi(m)/m$ is close to 1:

$$\prod_{\substack{p\mid m\\ p>\frac{1}{2}\log x}} \left(1-\frac{1}{p}\right)^{-1} < 1+\frac{3}{L} \qquad (m \le x), \tag{3}$$

for all sufficiently large x.

Now fix an arbitrary permutation $\pi \in S_k$, and encode the desired ranks by

$$r_i := k+1-\pi(i) \qquad (1 \le i \le k). \tag{4}$$

Thus $(r_1,\dots,r_k)$ is a permutation of $(1,\dots,k)$, and

$$\pi(i) < \pi(j) \iff r_i > r_j. \tag{5}$$

The larger r_i is, the smaller we want $\varphi(n+i)$ to be.

Lemma 3.1 (Rank-symmetric choice of medium primes)**.** *There exist pairwise disjoint sets of primes*

$$P_1,\dots,P_k \subset (L^2, \tfrac{1}{2}\log x]$$

such that for every $1 \le i \le k$*,*

$$\frac{\varphi(A)}{A}\left(1-\frac{6r_i}{L}\right)\left(1-\frac{1}{L^2}\right) \le \frac{\varphi(i)}{i}\prod_{p\in P_i}\left(1-\frac{1}{p}\right) \le \frac{\varphi(A)}{A}\left(1-\frac{6r_i}{L}\right). \tag{6}$$

Proof. We use exactly the greedy algorithm from [7, pp. 392–394], but with target level r_i attached to position i in place of the original level i.

For each i, start with $P_i = \varnothing$ and keep throwing unused primes from $(L^2, \frac{1}{2}\log x]$ into P_i until the upper bound in (6) first becomes true. Because every prime used in this greedy step satisfies $p > L^2$, multiplying by one extra factor $(1-1/p)$ decreases the running product by at most the factor $1-1/L^2$. Hence the lower bound in (6) automatically holds at the first time the upper bound is met. So the only point to check is that the algorithm does not run out of primes.

Exactly as in [7, (8.7)], it is enough to verify that

$$\prod_{L^2<p\le\frac{1}{2}\log x}\left(1-\frac{1}{p}\right) \le \prod_{i=1}^{k}\left(\frac{\varphi(A)}{A}\cdot\frac{i}{\varphi(i)}\cdot\left(1-\frac{6r_i}{L}\right)\left(1-\frac{1}{L^2}\right)\right). \tag{7}$$

Since $(r_1,\dots,r_k)$ is a permutation of $(1,\dots,k)$, the right-hand side of (7) is identical to the right-hand side of [7, (8.7)]. Thus the same calculation as in [7, (8.7)–(8.8)] gives the lower bound

$$\exp\left((\alpha-\gamma)k - k\log_3 k + O\left(\frac{k}{\log_2 k}\right)\right) \tag{8}$$

for the right-hand side.

With k as in (1), a short computation shows that (8) is at least

$$\exp\left(-\log_3 x + \frac{\delta}{2}\frac{\log_3 x}{\log_6 x}\right) \tag{9}$$

for all sufficiently large x. On the other hand, Mertens's theorem gives

$$\prod_{L^2 < p \le \frac{1}{2}\log x} \left(1 - \frac{1}{p}\right) \asymp \frac{\log_3 x}{\log_2 x} = \exp(-\log_3 x + \log_4 x + o(1)) . \tag{10}$$

Since $\log_4 x = o(\log_3 x / \log_6 x)$, the quantity in (9) dominates the quantity in (10) for large x. Hence (7) holds, and the greedy construction succeeds. □

Let

$$i_* := \pi^{-1}(1),$$

so that i_* is the unique position that should carry the smallest totient value. Let P_0 be the set of all medium primes not belonging to any P_i; in particular, P_0 contains every medium prime in $(k, L^2]$.

By the Chinese remainder theorem there exists an integer

$$n \in (x/2, x-k] \tag{11}$$

satisfying the congruences

$$n \equiv 0 \pmod{\prod_{p \le k} p}, \tag{12}$$

$$n + i \equiv 0 \pmod{\prod_{p \in P_i} p} \qquad (i \ne i_*), \tag{13}$$

$$n + i_* \equiv 0 \pmod{\prod_{p \in P_{i_*} \cup P_0} p}. \tag{14}$$

Indeed, the moduli are pairwise coprime, and their product is the product of all primes up to $\frac{1}{2}\log x$, which is $x^{1/2+o(1)}$ by the prime number theorem. This is smaller than the length of the interval $(x/2, x-k]$ for large x.

For each $1 \le i \le k$, write

$$n + i = a_i b_i c_i,$$

where a_i, b_i, c_i are supported on the small, medium, and large primes respectively.

Lemma 3.2. *For every $i \ne i_*$,*

$$\frac{\varphi(A)}{A}\left(1 - \frac{6r_i + 4}{L}\right) \le \frac{\varphi(n+i)}{n+i} \le \frac{\varphi(A)}{A}\left(1 - \frac{6r_i}{L}\right). \tag{15}$$

Moreover, the upper bound in (15) *holds for all i, including $i = i_*$.*

Proof. Because n is divisible by every small prime, the small prime divisors of $n + i$ are exactly the small prime divisors of i. Hence

$$\frac{\varphi(a_i)}{a_i} = \frac{\varphi(i)}{i}. \tag{16}$$

Next, the medium prime divisors of $n+i$ always include the primes in P_i, and for $i \ne i_*$ they are exactly the primes in P_i. Indeed, every medium prime outside P_i lies in P_0 or in some P_j with $j \ne i$; by construction it then divides $n + i_*$ or $n + j$, and since each medium prime exceeds k it can divide at most one of the consecutive integers $n+1, \dots, n+k$. Therefore

$$\frac{\varphi(b_i)}{b_i} \le \prod_{p \in P_i}\left(1 - \frac{1}{p}\right), \tag{17}$$

with equality for $i \ne i_*$.

Finally, by (3),

$$\frac{\varphi(c_i)}{c_i} \geq \left(1+\frac{3}{L}\right)^{-1} > 1-\frac{4}{L} \tag{18}$$

for sufficiently large x, while of course $\varphi(c_i)/c_i \leq 1$.

Combining (16), (17), (18), and Lemma 3.1, we obtain for $i \neq i_*$

$$\begin{aligned}\frac{\varphi(n+i)}{n+i} &= \frac{\varphi(a_i)}{a_i}\cdot\frac{\varphi(b_i)}{b_i}\cdot\frac{\varphi(c_i)}{c_i} \\ &\geq \frac{\varphi(A)}{A}\left(1-\frac{6r_i}{L}\right)\left(1-\frac{1}{L^2}\right)\left(1-\frac{4}{L}\right) \\ &\geq \frac{\varphi(A)}{A}\left(1-\frac{6r_i+4}{L}\right),\end{aligned}$$

and also

$$\frac{\varphi(n+i)}{n+i} \leq \frac{\varphi(A)}{A}\left(1-\frac{6r_i}{L}\right).$$

For $i = i_*$ the same upper bound holds, since the extra primes from P_0 can only decrease the ratio. ☐

Proposition 3.3. *For the integer n constructed above,*

$$\mathrm{pat}\big(\varphi(n+1),\ldots,\varphi(n+k)\big) = \pi.$$

Proof. Take $i, j \in \{1,\ldots,k\}$ with $\pi(i) < \pi(j)$. By (5), this means $r_i > r_j$. In particular $j \neq i_*$, because i_* is the unique index with $\pi(i_*) = 1$.

Applying Lemma 3.2 to j and the upper bound there to i, we obtain

$$\begin{aligned}\frac{\varphi(n+j)/(n+j)}{\varphi(n+i)/(n+i)} &\geq \frac{1-(6r_j+4)/L}{1-6r_i/L} \\ &\geq \frac{1-(6r_j+4)/L}{1-(6r_j+6)/L} \\ &= 1+\frac{2}{L-6r_j-6}.\end{aligned}$$

Since $r_j \leq k-1$ and $k = o(L)$ by (1), the right-hand side equals

$$1+\frac{2}{L}+o\left(\frac{1}{L}\right). \tag{19}$$

On the other hand, by (11),

$$\frac{n+j}{n+i} = 1+O\left(\frac{k}{x}\right) = 1+o\left(\frac{1}{L}\right), \tag{20}$$

uniformly in i, j, because $k \ll \log_3 x$.

Multiplying (19) and (20) yields

$$\frac{\varphi(n+j)}{\varphi(n+i)} = \frac{n+j}{n+i}\cdot\frac{\varphi(n+j)/(n+j)}{\varphi(n+i)/(n+i)} > 1$$

for all sufficiently large x. Hence

$$\varphi(n+i) < \varphi(n+j) \qquad \text{whenever} \qquad \pi(i) < \pi(j),$$

which is exactly the statement that

$$\mathrm{pat}\big(\varphi(n+1),\ldots,\varphi(n+k)\big) = \pi.$$

□

Proposition 3.4 (Lower bound for $F_{\text{str}}(x)$)**.** *As $x \to \infty$,*

$$F_{\text{str}}(x) \geq \frac{\log_3 x}{\log_6 x} + (\alpha - \gamma + o(1))\frac{\log_3 x}{(\log_6 x)^2}.$$

Proof. Fix $\delta > 0$ and let k be given by (1). Since the permutation $\pi \in S_k$ in the preceding construction was arbitrary, Proposition 3.3 shows that every permutation of length k occurs below x. Hence $F_{\text{str}}(x) \geq k$. Since $\delta > 0$ is arbitrary, the stated lower bound follows. □

Remark 3.5 (Referee's guide to the new ingredient)**.** *Everything after Lemma 3.1 is a direct relabelling of the proof of* [7, Section 8]. *The only point that needs checking beyond Pollack–Pomerance–Treviño is the following observation: once one prescribes arbitrary target ranks $r_1, \ldots, r_k$, the budget product in* [7, (8.7)] *is unchanged because $(r_1, \ldots, r_k)$ is merely a permutation of $(1, \ldots, k)$. All later estimates then go through with the special index k replaced by the special index $i_* = \pi^{-1}(1)$.*

4. Proof of the main theorem and the strict-order consequences

Proof of Theorem 1.2. The formula for $F_{\text{str}}(x)$ follows by combining Proposition 2.4 and Proposition 3.4. The formulas for $D_{\text{str}}(x)$ and $I_{\text{str}}(x)$ are precisely Proposition 2.3.

For the infinite-occurrence statement, fix $k \geq 1$ and $\pi \in S_k$. Choose a sequence $x_1 < x_2 < \cdots$ tending to infinity so rapidly that $x_{t+1} > 2x_t$ for every t. For each sufficiently large t, Proposition 3.3 furnishes an integer $n_t \in (x_t/2, x_t - k]$ with

$$\operatorname{pat}\bigl(\varphi(n_t + 1), \ldots, \varphi(n_t + k)\bigr) = \pi.$$

Since $n_t > x_t/2$ and the intervals $(x_t/2, x_t - k]$ are disjoint, these occurrences are distinct. Hence π occurs infinitely often. □

Corollary 4.1. *Every fixed strict permutation occurs infinitely often among consecutive totient blocks. Equivalently, no nonempty finite family of strict permutations is permanently forbidden.*

Proof. This is the last assertion of Theorem 1.2. □

Corollary 4.2. *The monotone patterns are asymptotically extremal for the strict universal threshold: the increasing and decreasing permutations alone already force the sharp upper bound for $F_{\text{str}}(x)$.*

Proof. The proof of Proposition 2.4 uses only the two monotone permutations. □

Remark 4.3. *Corollary 4.2 is the strongest statement about the second thread question that follows from the present argument. It shows that the monotone patterns fail at the correct threshold scale, but it does not identify the lexicographically first missing pattern at each fixed value of x.*

5. Weak orders, the "natural pattern", and problem #1003

To discuss the third question on the thread, one has to leave the world of permutations.

Definition 5.1. *A* weak order pattern *of length k is the relative order type of a k-tuple of real numbers in which equalities are allowed. Equivalently, it is a total preorder on $\{1, \ldots, k\}$, or an ordered set partition of $\{1, \ldots, k\}$.*

Proposition 5.2. *The "natural ordering" induced by $\varphi(1), \ldots, \varphi(k)$ is, in general, a weak order rather than a permutation. In particular, the third question from problem #415 is not a strict-permutation question.*

Proof. Already

$$\varphi(1) = \varphi(2) = 1 \qquad \text{and} \qquad \varphi(3) = \varphi(4) = 2,$$

so the relative order type of $\varphi(1), \dots, \varphi(k)$ has ties for every $k \geq 4$. □

Proposition 5.3. *Any universal weak-order theorem for consecutive totients that realizes all weak order patterns of length* 2 *infinitely often would imply infinitely many solutions to*

$$\varphi(n) = \varphi(n+1).$$

More generally, realizing the constant weak order of length r infinitely often would imply infinitely many solutions to

$$\varphi(n) = \varphi(n+1) = \cdots = \varphi(n+r-1).$$

Proof. The constant weak order of length 2 is exactly the equality pattern $\varphi(n) = \varphi(n+1)$. The constant weak order of length r is exactly the pattern of r consecutive equal totients. □

Remark 5.4 (Relation with Erdős problem #1003)**.** *The current Erdős Problems page for #1003 still lists the infinitude of solutions to $\varphi(n) = \varphi(n+1)$ as open; see* [1]. *Thus the weak-order extension of problem #415 already contains a presently open subproblem.*

What is known in that direction is much weaker. Erdős, Pomerance, and Sárközy proved the upper bound

$$\#\{n \leq x : \varphi(n) = \varphi(n+1)\} \ll \frac{x}{\exp((\log x)^{1/3})};$$

see [4, Theorem 2]. *In a different direction, Ford proved that $\varphi(n) = \varphi(n+k)$ has infinitely many solutions for some even $k \leq 3570$ and for every k divisible by* 442720643463713815200*; see* [6]. *This does not reach the consecutive case $k = 1$.*

Corollary 5.5. *The present paper settles the strict-permutation interpretation of problem #415, but it does not settle the equality-allowing weak-order variant and does not settle problem #1003.*

Proof. The first assertion is Theorem 1.2; the second follows from Proposition 5.3 and Remark 5.4. □

Appendix A. A checklist against Section 8 of Pollack–Pomerance–Treviño

For convenience of a referee, we summarize the exact dictionary between the present proof and [7, Section 8].

(1) Replace the original target levels $1, 2, \dots, k$ by the permuted levels $r_i = k+1-\pi(i)$ attached to the positions $i = 1, \dots, k$.
(2) Replace [7, (8.3)] by the target interval (6). The greedy construction is unchanged.
(3) Observe that the budget product (7) is identical to [7, (8.7)], because $(r_1, \dots, r_k)$ is a permutation of $(1, \dots, k)$.
(4) Replace the original special index k by $i_* = \pi^{-1}(1)$, the position that should realize the minimum totient value. Assign the leftover medium primes to i_*.
(5) The estimate [7, (8.6)] becomes Lemma 3.2, and the final monotonicity comparison becomes Proposition 3.3.

Everything else is unchanged from the Pollack–Pomerance–Treviño proof.

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