EUCLIDEAN ISOSCELES SETS AND TWO-DISTANCE EXTREMAL FUNCTIONS

Abstract. A finite set in Euclidean space is called isosceles if every three of its points span an isosceles triangle. Let $f(d)$ be the largest size of an isosceles subset of $\mathbb{R}^d$, let $g(d)$ be the largest size of a Euclidean two-distance set in $\mathbb{R}^d$, let $s(d)$ be the largest size of a spherical two-distance set in $\mathbb{R}^d$, and let $h(d)$ be the largest size of an isosceles subset of $\mathbb{R}^d$ which is not a two-distance set. We prove that, for every $d \geq 2$,

$$h(d) \leq \max\{s(d)+1,\ s(d-1)+3\},$$

and consequently

$$f(d) = \max\{g(d),\ s(d)+1,\ s(d-1)+3\}.$$

The proof combines Ionin's complete decomposition theorem for Euclidean isosceles sets, the classical absolute bounds for Euclidean and spherical two-distance sets, and a polynomial argument for the only possible two-point block case.

1. Introduction

A finite set $S \subset \mathbb{R}^d$ is called *isosceles* if every three distinct points of S determine an isosceles triangle. Equivalently, every three-point subset of S determines at most two nonzero distances. Let

$$f(d) := \max\{|S| : S \subset \mathbb{R}^d \text{ is isosceles}\}.$$

This is the Euclidean isosceles-set problem appearing as Problem 503 in the Erdős Problems collection [3]. The present note determines $f(d)$ in terms of the standard extremal functions for Euclidean and spherical two-distance sets.

Let

$$g(d) := \max\{|X| : X \subset \mathbb{R}^d \text{ is a Euclidean two-distance set}\},$$

and let

$$s(d) := \max\{|X| : X \subset \mathbb{R}^d \text{ is a spherical two-distance set}\}.$$

Here and throughout, "two-distance" means "at most two nonzero distances". For $d \geq 2$ this convention gives the same values of $g(d)$ and $s(d)$ as the convention "exactly two distances", since the known two-distance examples are larger than a simplex.

We also put

$$h(d) := \max\{|S| : S \subset \mathbb{R}^d \text{ is isosceles and is not a two-distance set}\}.$$

Then

$$f(d) = \max\{g(d), h(d)\}.$$

Our main theorem is the following.

Theorem 1.1. *For every integer $d \geq 2$,*

$$h(d) \leq \max\{s(d)+1,\ s(d-1)+3\}.$$

Consequently,

$$f(d) = \max\{g(d),\ s(d)+1,\ s(d-1)+3\}.$$

Also $f(1) = 3$.

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The lower bound comes from two simple constructions. If $X \subset \mathbb{R}^d$ is a spherical two-distance set and c is the center of a sphere containing X, then $X \cup \{c\}$ is isosceles, giving $f(d) \geq s(d) + 1$. If X is a spherical two-distance set in a hyperplane $H \cong \mathbb{R}^{d-1}$, and if one adds the center of the sphere together with two symmetric points on the perpendicular line through the center, then one obtains an isosceles set of size $s(d-1)+3$. This second construction always has more than two distances.

The upper bound uses Ionin's complete decomposition theorem for Euclidean isosceles sets. The decomposition reduces the problem to two-distance blocks. The only delicate case is when a non-last block consists of two points. In that case all remaining points lie in the perpendicular bisector of the pair. The main new ingredient is a two-shell polynomial lemma, proved in Section 5, which gives exactly the one-point improvement over Blokhuis's general two-distance bound needed for this pair-block case.

2. Preliminaries

For a finite set $X \subset \mathbb{R}^d$, write

$$\operatorname{dimaff} X := \dim(\operatorname{aff} X)$$

for its affine dimension. We use the following notation:

$$U(m) := \frac{m(m+3)}{2}, \qquad L(m) := \binom{m+1}{2}, \qquad G(m) := \binom{m+2}{2} = U(m) + 1.$$

We set $G(0) = 1$ and $s(1) = 2$. The latter convention is only used in the term $s(d-1)+3$ when $d = 2$.

The classical Delsarte–Goethals–Seidel and Blokhuis bounds give [2, 1]

$$s(m) \leq U(m), \qquad g(m) \leq G(m) \qquad (m \geq 1), \tag{1}$$

where the assertions for $m = 1$ are immediate. The regular-simplex midpoint construction gives

$$s(m) \geq L(m) \qquad (m \geq 1). \tag{2}$$

For $m = 1$ this reads $2 \geq 1$. We shall also use without further comment that $g(m)$ and $s(m)$ are nondecreasing in m, which follows by embedding lower-dimensional configurations in a coordinate hyperplane.

We shall use Ionin's complete decomposition theorem in the following form.

Definition 2.1. Let S be a finite metric space. A partition

$$S = S_1 \sqcup \cdots \sqcup S_k$$

is called a *complete decomposition* if

(i) each S_i is a two-distance set;
(ii) $|S_i| \geq 2$ for $1 \leq i < k$, and $|S_k| \geq 1$;
(iii) if $1 \leq i < j \leq k$, then every point of S_j is the center of a sphere containing S_i.

Theorem 2.2 (Ionin)**.** *Every finite isosceles set in Euclidean space admits a complete decomposition. Moreover, if*

$$S = S_1 \sqcup \cdots \sqcup S_k$$

is a complete decomposition of an isosceles set $S \subset \mathbb{R}^d$, *then*

$$\operatorname{dimaff} S \geq \sum_{i=1}^{k} \operatorname{dimaff} S_i.$$

In particular, every non-last block S_i, $i < k$, *is spherical.*

Reference. This is Ionin's complete decomposition theorem for Euclidean isosceles sets; see [4, Definition 4.1 and Proposition 4.2]. The last assertion follows from the defining property that every later block consists of centers of spheres containing S_i. ☐

The following elementary inequality is used to count several non-last blocks at once.

Lemma 2.3. *Let* $a_1, \dots, a_q$ *be positive integers and put* $N = a_1 + \cdots + a_q$*. If* $q \geq 2$*, then*

$$\sum_{i=1}^{q} U(a_i) \leq L(N) + 1.$$

Moreover,

$$\sum_{i=1}^{q} U(a_i) \leq L(N)$$

unless $q = 2$ *and* $\{a_1, a_2\} = \{1, N-1\}$*.*

Proof. A direct calculation gives

$$\begin{aligned} L(N) - \sum_{i=1}^{q} U(a_i) &= \frac{N(N+1) - \sum_i (a_i^2 + 3a_i)}{2} \\ &= \sum_{1 \leq i < j \leq q} a_i a_j - N. \end{aligned}$$

If $q = 2$, this equals

$$a_1 a_2 - a_1 - a_2 = (a_1 - 1)(a_2 - 1) - 1,$$

so it is at least -1, and it is negative exactly when one of a_1, a_2 is 1.

Assume $q \geq 3$. For fixed N, the sum $\sum_{i<j} a_i a_j$ is minimized when one part is as large as possible and all other parts are 1. Hence

$$\sum_{i<j} a_i a_j \geq (q-1)(N-q+1) + \binom{q-1}{2}.$$

The right side is at least N for every $q \geq 3$. Therefore $L(N) - \sum_i U(a_i) \geq 0$ when $q \geq 3$, and the lemma follows. ☐

3. Lower constructions

Proposition 3.1. *For every integer* $d \geq 2$*,*

$$f(d) \geq \max\{g(d),\ s(d) + 1,\ s(d-1) + 3\}.$$

Moreover,

$$h(d) \geq s(d-1) + 3.$$

Proof. The term $g(d)$ is immediate, since every two-distance set is isosceles.

Let $X \subset \mathbb{R}^d$ be a spherical two-distance set of size $s(d)$, and let c be the center of a sphere containing X. Then $X \cup \{c\}$ is isosceles: triples contained in X are already isosceles, and every triple of the form $\{x, y, c\}$ is isosceles because $|x - c| = |y - c|$. Hence $f(d) \geq s(d) + 1$.

For the third term, let $H \cong \mathbb{R}^{d-1}$ be a hyperplane, let $X \subset H$ be a spherical two-distance set of size $s(d-1)$, and let $c \in H$ be the center of a sphere containing X. Write $r = |x - c|$ for $x \in X$. Let p, q be the two points on the line through c perpendicular to H such that c is the midpoint of pq and

$$|p - c| = |q - c| = r.$$

Set

$$Y := X \cup \{p, c, q\}.$$

Then Y is isosceles. Indeed, the only nontrivial triples are checked from the identities

$$|x - c| = |p - c| = |q - c| = r, \qquad |p - x| = |q - x| = \sqrt{2}\, r \qquad (x \in X).$$

The set Y has more than two distances, since it realizes r, $\sqrt{2}\,r$, and $2r$. Therefore $h(d) \geq s(d-1)+3$, and the proposition follows. $\square$

Remark 3.2. The construction $X \cup \{c\}$ gives the lower bound $s(d)+1$ for $f(d)$, but it does not necessarily give a lower bound for $h(d)$. It may happen that the radius of the sphere containing X is already one of the two distances determined by X, in which case $X \cup \{c\}$ is still a two-distance set.

4. Reduction to a two-point block

Let

$$M(d) := \max\{s(d)+1,\ s(d-1)+3\}.$$

The purpose of this section is to show that every large non-two-distance isosceles set has a complete decomposition with a two-point first block and one last block of codimension one.

Proposition 4.1. *Let $d \geq 2$, and let $S \subset \mathbb{R}^d$ be an isosceles set which is not a two-distance set. Then either*

$$|S| \leq M(d),$$

or S has a complete decomposition

$$S = S_1 \sqcup S_2$$

such that S_1 consists of two points and $\operatorname{dimaff} S_2 = d-1$.

Proof. Choose a complete decomposition

$$S = S_1 \sqcup \cdots \sqcup S_k.$$

Since S is not a two-distance set, $k \geq 2$. Put $q = k-1$. For $1 \leq i \leq q$, set

$$a_i := \operatorname{dimaff} S_i.$$

The non-last blocks are spherical and have at least two points, so $a_i \geq 1$ and

$$|S_i| \leq U(a_i) \qquad (1 \leq i \leq q).$$

In particular, a one-dimensional non-last block has exactly two points, since a line meets a sphere in at most two points. Let

$$N := a_1 + \cdots + a_q, \qquad e := \operatorname{dimaff} S_k.$$

By Theorem 2.2, $N + e \leq d$.

First suppose $q = 1$, and write $a = a_1$. If $a \geq 2$, then the estimates are standard. If $e = 0$, then

$$|S| \leq s(a)+1 \leq s(d)+1.$$

If $e = 1$, then S_k has at most three points and $a \leq d-1$, so

$$|S| \leq s(a)+3 \leq s(d-1)+3.$$

If $e \geq 2$, then

$$|S| \leq U(a)+G(e),$$

and

$$L(a+e)+1-U(a)-G(e) = ae-a-e \geq 0,$$

because $a, e \geq 2$. Hence

$$|S| \leq L(d)+1 \leq s(d)+1.$$

Thus the case $q = 1$, $a \geq 2$, is bounded by $M(d)$.

It remains in the case $q = 1$ to consider $a = 1$. Then S_1 consists of exactly two points. If $e \leq d-2$, then

$$|S| \leq 2+G(e) \leq 2+G(d-2) \leq L(d)+1 \leq s(d)+1.$$

Therefore the only unbounded alternative with $q = 1$ is precisely $k = 2$, $|S_1| = 2$, and $\operatorname{dimaff} S_2 = d-1$.

Now suppose $q \geq 2$. If the exceptional alternative in Lemma 2.3 does not occur, then

$$\sum_{i=1}^{q} |S_i| \leq L(N).$$

If $e = 0$, then

$$|S| \leq L(N) + 1 \leq L(d) + 1 \leq s(d) + 1.$$

If $e = 1$, then

$$|S| \leq L(N) + 3 \leq L(d-1) + 3 \leq s(d-1) + 3.$$

If $e \geq 2$, then

$$|S| \leq L(N) + G(e),$$

and

$$L(N+e) + 1 - L(N) - G(e) = e(N-1) \geq 0.$$

Thus $|S| \leq s(d) + 1$.

It remains to handle the exceptional alternative in Lemma 2.3. Then $q = 2$ and the two non-last dimensions are 1 and $N - 1$. The one-dimensional block contributes two points. If $e = 0$, then

$$|S| \leq 2 + s(N-1) + 1 \leq s(d-1) + 3.$$

If $e = 1$, then

$$|S| \leq 2 + U(N-1) + 3.$$

Since $N \leq d - 1$, one has

$$2 + U(N-1) + 3 \leq L(d) + 1 \leq s(d) + 1.$$

Finally, if $e \geq 2$, then

$$|S| \leq 2 + U(N-1) + G(e),$$

and

$$L(N+e) + 1 - \big(2 + U(N-1) + G(e)\big) = e(N-1) - 1 \geq 0.$$

Hence $|S| \leq s(d) + 1$ also in this case. The proposition follows. ☐

5. The two-shell polynomial lemma

The following lemma is the main point needed for the two-point block case.

Lemma 5.1. *Let $m \geq 1$, let $X \subset \mathbb{R}^m$ be a two-distance set with squared distances $0 < \alpha < \beta$, and let $h > 0$. Suppose that*

$$|x|^2 + h^2 \in \{\alpha, \beta\} \qquad (x \in X).$$

Put

$$X_1 := \{x \in X : |x|^2 + h^2 = \alpha\}, \qquad X_2 := \{x \in X : |x|^2 + h^2 = \beta\}.$$

If $|X_1| \geq 2$ and $|X_2| \geq 2$, then

$$|X| \leq \binom{m+2}{2} - 1 = U(m).$$

Proof. Let $\eta := h^2$, $r_1 := \alpha - \eta$, $r_2 := \beta - \eta$, and $\Delta := \beta - \alpha$. Since $|X_1| \geq 2$, the first shell has positive radius; hence

$$0 < \eta < \alpha.$$

If $\operatorname{dimaff} X < m$, then Blokhuis's bound in the affine span of X gives

$$|X| \leq G(m-1) \leq G(m) - 1 = U(m),$$

so we may assume that X has affine dimension m.

By Blokhuis's bound, $|X| \le G(m)$. We shall exclude equality. Suppose, for contradiction, that

$$|X| = G(m) = \binom{m+2}{2}.$$

Let $u = (u_1, \dots, u_m)$, let z be an additional variable, and set

$$\sigma := u_1^2 + \cdots + u_m^2 + z^2, \qquad \rho := u_1^2 + \cdots + u_m^2.$$

For $x \in X$, write $r_x := |x|^2$, and define

$$F_x(u, z) := \frac{(\sigma - 2x \cdot u + r_x - \alpha)(\sigma - 2x \cdot u + r_x - \beta)}{\alpha\beta}.$$

Then

$$F_x(y, 0) = \delta_{xy} \qquad (x, y \in X).$$

Also set

$$Q(u, z) := (\sigma - r_1)(\sigma - r_2).$$

Since $|y|^2 \in \{r_1, r_2\}$ for every $y \in X$, one has

$$Q(y, 0) = 0 \qquad (y \in X).$$

Let $R \in \mathbb{R}$ be a parameter, and fix a nonzero real number ε. Define

$$\Phi_x := F_x + \left(-\frac{1}{\alpha\beta} + \varepsilon(2r_x - R)\right) Q \qquad (x \in X).$$

Then

$$\Phi_x(y, 0) = \delta_{xy} \qquad (x, y \in X).$$

Each Φ_x lies in the vector space

$$W := \operatorname{span}\{\sigma^2, \sigma u_1, \dots, \sigma u_m, u_i u_j\ (1 \le i \le j \le m), \sigma, u_1, \dots, u_m, 1\}.$$

The displayed spanning polynomials are linearly independent. Indeed, terms of different total degree cannot cancel; in degree two, the polynomial σ is independent of the polynomials $u_i u_j$ because it contains the term z^2, while no polynomial $u_i u_j$ does. Hence

$$\dim W = 1 + m + \binom{m+1}{2} + 1 + m + 1 = \binom{m+3}{2}.$$

Since

$$|X| + m + 3 = G(m) + m + 3 = \binom{m+3}{2} + 1,$$

the family

$$\{\Phi_x : x \in X\} \cup \{u_1, \dots, u_m, \sigma, \rho, 1\}$$

is linearly dependent. Thus there are real coefficients c_x, a_i, b, c, and d, not all zero, such that

$$\sum_{x \in X} c_x \Phi_x + \sum_{i=1}^{m} a_i u_i + b\sigma + c\rho + d = 0 \tag{3}$$

as a polynomial identity.

Put

$$C := \sum_{x \in X} c_x, \qquad V := \sum_{x \in X} c_x r_x x, \qquad M := \sum_{x \in X} c_x x x^T.$$

We compare coefficients in (3). The coefficient of σ^2 gives, after division by ε,

$$\sum_{x \in X} c_x (2r_x - R) = 0, \tag{4}$$

and hence

$$\sum_{x\in X} c_x r_x = \frac{R}{2}C. \tag{5}$$

The coefficients of the terms σu_i give

$$\sum_{x\in X} c_x x = 0. \tag{6}$$

Comparing quadratic terms gives

$$M = \mu I_m, \qquad \mu = \frac{RC}{2m}, \tag{7}$$

where the value of μ follows by taking traces and using (5). The linear coefficients give

$$a := (a_1, \dots, a_m) = \frac{4}{\alpha\beta}V. \tag{8}$$

If $P := b + c$, then the coefficients of z^2 and u_i^2 give

$$P = \frac{C}{\alpha\beta}\left(2\eta - \frac{m+2}{m}R\right). \tag{9}$$

Finally, the constant coefficient gives

$$d = \frac{C}{\alpha\beta}\left(\eta R + \alpha\beta - 2(\alpha+\beta)\eta + 2\eta^2\right). \tag{10}$$

For completeness, we note that the terms involving ε vanish in these comparisons because of (4).

Evaluating (3) at $(y, 0)$, with $y \in X$, gives

$$c_y + a\cdot y + P|y|^2 + d = 0. \tag{11}$$

Let

$$C_2 := \sum_{y\in X_2} c_y, \qquad V_2 := \sum_{y\in X_2} c_y y.$$

From (5) and (6),

$$C_2 = \frac{(R/2 - r_1)C}{\Delta}, \qquad V_2 = \frac{V}{\Delta}. \tag{12}$$

Now multiply (11) by c_y and sum over $y \in X_2$. Using (8) and (12), we obtain

$$0 = \sum_{y\in X_2} c_y^2 + \frac{4}{\alpha\beta\Delta}\|V\|^2 + \frac{C^2}{\alpha\beta\Delta}(R/2 - r_1)\ell(R), \tag{13}$$

where

$$\ell(R) := \alpha(\beta - 2\eta) + \frac{2(m+1)\eta - (m+2)\beta}{m}R. \tag{14}$$

Indeed, (9) and (10) give

$$Pr_2 + d = \frac{C}{\alpha\beta}\ell(R).$$

Suppose first that R can be chosen so that

$$(R/2 - r_1)\ell(R) > 0. \tag{15}$$

For this choice of R, all three terms in (13) are nonnegative, and the last term is positive unless $C = 0$. Therefore

$$C = 0, \qquad V = 0, \qquad c_y = 0 \quad (y \in X_2).$$

Then (8), (9), and (10) give $a = 0$, $P = 0$, and $d = 0$. Equation (11) gives $c_y = 0$ for every $y \in X$. The relation (3) then reduces to

$$b\sigma + c\rho = 0.$$

Since $P = b + c = 0$, this forces $b = c = 0$. The dependence relation is trivial, a contradiction.

It remains to consider the case where no real R satisfies (15). The polynomial $(R/2 - r_1)\ell(R)$ is a product of two linear polynomials in R. It is not identically zero: the first factor is not zero, and ℓ cannot vanish identically, since that would imply both $\beta = 2\eta$ and $2(m+1)\eta = (m+2)\beta$, hence $\eta = 0$. Therefore, if it is never positive, the two factors must have the same zero. Hence

$$\ell(2r_1) = 0. \tag{16}$$

Now set $R = 2r_1$. Then the last term in (13) is zero, so

$$\sum_{y \in X_2} c_y^2 + \frac{4}{\alpha\beta\Delta}\|V\|^2 = 0.$$

Thus

$$c_y = 0 \quad (y \in X_2), \qquad V = 0, \qquad a = 0. \tag{17}$$

If $C = 0$, then (9) and (10) give $P = d = 0$, and (11) again gives $c_y = 0$ for every $y \in X$, leading to a trivial relation as above. Hence $C \neq 0$.

Let $n_1 := |X_1|$. Since the coefficients on X_2 vanish and $a = 0$, equation (11) shows that all coefficients c_y with $y \in X_1$ are equal. As their sum is C,

$$c_y = \frac{C}{n_1} \qquad (y \in X_1). \tag{18}$$

Equation (6) gives

$$\sum_{y \in X_1} y = 0. \tag{19}$$

Moreover, substituting (18) into (11) gives

$$\frac{C}{n_1} + Pr_1 + d = 0.$$

With $R = 2r_1$, equations (9) and (10) give

$$Pr_1 + d = \frac{C}{\alpha\beta}B,$$

where

$$B = r_1\left(2\eta - \frac{2(m+2)}{m}r_1\right) + 2\eta r_1 + \alpha\beta - 2(\alpha+\beta)\eta + 2\eta^2.$$

Using (16) to eliminate β, this simplifies to

$$B = -\frac{\beta}{\eta m}D,$$

where

$$D := (m+4)\alpha^2 - 4(m+2)\alpha\eta + 4(m+1)\eta^2.$$

Therefore

$$n_1 = \frac{\alpha\eta m}{D}. \tag{20}$$

Fix $y \in X_1$, and let k_y be the number of points $y' \in X_1 \setminus \{y\}$ such that

$$|y - y'|^2 = \beta.$$

Since the squared distances inside X are α and β, equation (19) gives

$$(n_1 - 1)\alpha + k_y(\beta - \alpha) = \sum_{y' \in X_1,\ y' \neq y} |y - y'|^2 = 2n_1 r_1.$$

Solving for k_y gives

$$k_y = \frac{2n_1 r_1 - (n_1 - 1)\alpha}{\beta - \alpha}.$$

The denominator in the next display is nonzero. Indeed, if

$$2(m+2)\eta = (m+4)\alpha,$$

then substituting this into $\ell(2r_1)$ gives

$$-\frac{(m+4)\alpha^2}{(m+2)^2},$$

contrary to (16). Hence (16) is equivalent to

$$\beta = \frac{2\eta\big((m+2)\alpha - 2(m+1)\eta\big)}{(m+4)\alpha - 2(m+2)\eta}.$$

Using this identity together with (20), the preceding expression simplifies to

$$k_y = \frac{\alpha(\eta - \alpha)\big(2(m+2)\eta - (m+4)\alpha\big)^2}{D^2}. \tag{21}$$

The denominator is strictly positive. Indeed, as a quadratic polynomial in η, the discriminant of D is

$$16\alpha^2\big((m+2)^2 - (m+1)(m+4)\big) = -16m\alpha^2 < 0,$$

and its leading coefficient is positive. Also $0 < \eta < \alpha$, and the square factor in (21) is nonzero by the preceding paragraph. Therefore (21) gives $k_y < 0$, impossible because k_y is a cardinality. This contradiction excludes $|X| = G(m)$, and the lemma follows. □

6. THE TWO-POINT BLOCK CASE

We now complete the analysis of the alternative left by Proposition 4.1. Suppose

$$S = \{p, q\} \cup X \subset \mathbb{R}^d,$$

where X is a two-distance set contained in the perpendicular bisector H of the segment pq, and $\operatorname{dimaff} X = d - 1$. Let o be the midpoint of pq, identify H with $\mathbb{R}^{d-1}$ using o as the origin, and write

$$p = (0, \dots, 0, h), \qquad q = (0, \dots, 0, -h), \qquad h > 0.$$

Put $n := d - 1$.

If X determines at most one nonzero distance, then $|X| \le n + 1 = d$, and therefore

$$|S| \le d + 2 \le L(d) + 1 \le s(d) + 1.$$

Thus we may assume that X determines exactly two squared distances

$$0 < \alpha < \beta.$$

For $x \in X$, put

$$t(x) := |p - x|^2 = |q - x|^2 = |x|^2 + h^2.$$

Lemma 6.1. *Among the values $t(x)$, there is at most one value outside $\{\alpha, \beta\}$.*

Proof. Suppose $x, y \in X$ and $t(x) \neq t(y)$. The triangle pxy is isosceles, and its squared side lengths are

$$t(x), \qquad t(y), \qquad |x-y|^2.$$

Since $t(x) \neq t(y)$, the third squared side length must be one of $t(x)$ and $t(y)$. But $|x-y|^2 \in \{\alpha, \beta\}$. Hence two distinct values of t cannot both lie outside $\{\alpha, \beta\}$. $\square$

Lemma 6.2. *If some value $t(x)$ lies outside $\{\alpha, \beta\}$, then*

$$|S| \leq M(d).$$

Proof. By Lemma 6.1, there is a unique value $\gamma \notin \{\alpha, \beta\}$ attained by t outside $\{\alpha, \beta\}$. Put

$$C := \{x \in X : t(x) = \gamma\}.$$

If $C = X$, then X is spherical in H with center o, so

$$|S| = |X| + 2 \leq s(n) + 2 \leq s(n) + 3 = s(d-1) + 3.$$

Assume now that $C \neq X$.

Let $c \in C$ and $y \in X \setminus C$. Since $t(c) = \gamma \notin \{\alpha, \beta\}$ and $t(y) \in \{\alpha, \beta\}$, the isosceles triangle pcy forces

$$|c-y|^2 = t(y). \tag{22}$$

Thus every point of $X \setminus C$ is the center of a sphere containing C.

If $|C| = 1$, say $C = \{c\}$, then (22) gives

$$|c-y|^2 = |y|^2 + h^2 \qquad (y \in X \setminus C).$$

Equivalently,

$$c \cdot y = \frac{|c|^2 - h^2}{2} \qquad (y \in X \setminus C).$$

Hence $X \setminus C$ lies in a hyperplane of H. Therefore

$$|X| \leq 1 + G(n-1),$$

and so

$$|S| \leq 3 + G(n-1) \leq L(d) + 1 \leq s(d) + 1.$$

It remains to consider $|C| \geq 2$. Put $b := \operatorname{dimaff} C$. Since C lies on the sphere about o of squared radius $\gamma - h^2$, it is spherical, and

$$|C| \leq s(b) \leq U(b).$$

Moreover, because every point of $X \setminus C$ is the center of a sphere containing C, the set $X \setminus C$ lies in an affine subspace of H of dimension at most $n - b$.

If $b = n$, this affine subspace has dimension zero, so $|X \setminus C| \leq 1$. Hence

$$|S| \leq s(n) + 3 = s(d-1) + 3.$$

If $1 \leq b < n$, then

$$|X \setminus C| \leq G(n-b),$$

and therefore

$$|X| \leq U(b) + G(n-b) = G(n) - b(n-b) \leq G(n) - 1.$$

It follows that

$$|S| = |X| + 2 \leq G(n) + 1 = L(d) + 1 \leq s(d) + 1.$$

This proves the lemma. $\square$

Proposition 6.3. *In the two-point block case,*

$$|S| \leq M(d).$$

Proof. By Lemma 6.2, we may assume

$$t(x) \in \{\alpha, \beta\} \qquad (x \in X). \tag{23}$$

Define

$$X_1 := \{x \in X : t(x) = \alpha\}, \qquad X_2 := \{x \in X : t(x) = \beta\}.$$

If one of X_1, X_2 is empty, then X is spherical in H with center o, and hence

$$|S| \le s(n) + 2 \le s(d-1) + 3.$$

If one of X_1, X_2 has exactly one point, then the other is a spherical two-distance subset of H. Hence

$$|S| \le s(n) + 3 = s(d-1) + 3.$$

Finally, suppose that both X_1 and X_2 have at least two points. Applying Lemma 5.1 in dimension $m = n = d-1$ gives

$$|X| \le G(n) - 1.$$

Therefore

$$|S| = |X| + 2 \le G(n) + 1 = L(d) + 1 \le s(d) + 1.$$

In every case, $|S| \le M(d)$. ☐

7. Proof of the main theorem

Proof of Theorem 1.1. Let $d \ge 2$, and let $S \subset \mathbb{R}^d$ be an isosceles set which is not a two-distance set. By Proposition 4.1, either $|S| \le M(d)$ or S lies in the two-point block case. In the latter case, Proposition 6.3 again gives $|S| \le M(d)$. Thus

$$h(d) \le M(d) = \max\{s(d)+1,\ s(d-1)+3\}.$$

Now let $S \subset \mathbb{R}^d$ be any isosceles set. If S is a two-distance set, then $|S| \le g(d)$. If S is not a two-distance set, then the preceding paragraph gives $|S| \le M(d)$. Hence

$$f(d) \le \max\{g(d),\ s(d)+1,\ s(d-1)+3\}.$$

The reverse inequality is Proposition 3.1. Therefore

$$f(d) = \max\{g(d),\ s(d)+1,\ s(d-1)+3\}.$$

Finally, $f(1) = 3$: three equally spaced points on a line are isosceles, while any four distinct points on a line contain three points which are not equally spaced and hence form a non-isosceles triple. This completes the proof. ☐

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