THE EUCLIDEAN ISOSCELES-SET PROBLEM AND ERDŐS PROBLEM 503

Abstract. Let $f(d)$ be the largest size of a finite set $S \subset \mathbb{R}^d$ such that every three points of S form an isosceles triangle. Let $g(d)$ be the largest size of a Euclidean 2-distance set in $\mathbb{R}^d$, let $s(d)$ be the largest size of a spherical 2-distance set in $\mathbb{R}^d$, and let $h(d)$ be the largest size of an isosceles set in $\mathbb{R}^d$ with more than two distances. Problem #503 on ErdősProblems asks for the determination of $f(d)$ and records the classical exact values in dimensions 2 and 3, Blokhuis's quadratic upper bound, and the simplex-midpoint lower constructions. We prove that for every $d \geq 2$,

$$h(d) \leq \max\{s(d) + 1,\ s(d-1) + 3\},$$

and

$$f(d) = \max\{g(d),\ s(d) + 1,\ s(d-1) + 3\}.$$

Thus the full Euclidean isosceles-set problem is reduced exactly to the Euclidean and spherical 2-distance extremal problems. The proof combines Ionin's complete decomposition theorem for Euclidean isosceles sets with the classical absolute bounds for Euclidean and spherical 2-distance sets. As consequences, $f(22) = 276$, $h(23) = 278$, and

$$f(23) = \max\{g(23), 278\}.$$

We also explain why one should *not* claim the stronger identity $h(d) = \max\{s(d)+1, s(d-1)+3\}$ without an additional argument: adjoining the center to a spherical 2-distance set can leave the number of distances equal to two.

1. Introduction

A finite set $S \subset \mathbb{R}^d$ is called *isosceles* if every three distinct points of S determine an isosceles triangle; equivalently, every 3-point subset of S determines at most two distances. Let

$$f(d) := \max\{|S| : S \subset \mathbb{R}^d \text{ is isosceles}\}.$$

This is the Euclidean isosceles-set problem posed by Erdős in the 1940s. Problem #503 on ErdősProblems formulates the question in modern language; the page records the exact values $f(2) = 6$ and $f(3) = 8$, Blokhuis's upper bound

$$f(d) \leq \binom{d+2}{2},$$

and the lower constructions of sizes $\binom{d+1}{2}$ and $\binom{d+1}{2} + 1$. [6, 10, 4, 3]

To state the main result, let $g(d)$ be the largest size of a Euclidean 2-distance set in $\mathbb{R}^d$, let $s(d)$ be the largest size of a spherical 2-distance set in $\mathbb{R}^d$, and let

$$h(d) := \max\{|S| : S \subset \mathbb{R}^d \text{ is isosceles and has more than two distances}\}.$$

Then

$$f(d) = \max\{g(d), h(d)\}.$$

Our theorem gives an exact formula for $f(d)$ and a universal upper bound for $h(d)$.

Theorem 1.1. *For every integer $d \geq 2$,*

$$h(d) \leq \max\{s(d) + 1,\ s(d-1) + 3\}.$$

Consequently,

$$f(d) = \max\{g(d),\ s(d) + 1,\ s(d-1) + 3\}.$$

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Also $f(1) = 3$.

The theorem settles Erdős Problem #503 in the following precise sense: the maximum size of an isosceles subset of $\mathbb{R}^d$ is now expressed exactly in terms of the already-studied Euclidean and spherical 2-distance extremal functions. The proof is short once one uses Ionin's complete decomposition theorem: every isosceles set with more than two distances splits into 2-distance blocks, every non-last block is spherical, every non-last block has affine dimension at least 2, and the sum of the affine dimensions of the blocks is at most d. Combined with the Delsarte–Goethals–Seidel bound for spherical 2-distance sets and Blokhuis's bound for Euclidean 2-distance sets, this leads to a clean case analysis.

There is one point at which it is easy to overstate the conclusion. If X is a spherical 2-distance set and c is the center of its ambient sphere, then $X \cup \{c\}$ is always isosceles, but it need not have more than two distances. Therefore the center-adjoining construction always contributes to $f(d)$, but it does *not* automatically contribute to $h(d)$. We make this explicit in Remark 3.2; it is exactly why the main theorem is formulated as an exact identity for $f(d)$ together with an upper bound for $h(d)$.

The isosceles problem is also related to the theory of *locally two-distance sets*, studied systematically by Nozaki and Shinohara. Among other results, they proved an upper bound for locally two-distance sets in Euclidean space and an equivalence between extremal spherical 2-distance sets in dimension $d-1$ and locally two-distance but non-2-distance sets in dimension d above the threshold $d(d+1)/2$. [15] Our theorem identifies the precise contribution of the analogous constructions to the isosceles problem.

Finally, Blokhuis's 2-distance bound is the case $s = 2$ of the general Bannai–Bannai–Stanton upper bound for Euclidean s-distance sets, and Petrov and Pohoata later gave a short proof of the general theorem. [1, 16]

2. Preliminaries

Throughout, a finite set will be called a 2-*distance set* if it determines at most two distances. This is the convention used by Ionin in the complete decomposition theorem. For the extremal quantities $g(d)$ and $s(d)$ with $d \geq 2$, this agrees with the more common "exactly two distances" convention: in dimension 2 one already has exact 2-distance examples of size 5, while for $d \geq 3$ the simplex-midpoint construction has size $\binom{d+1}{2} > d+1$, larger than any one-distance set in $\mathbb{R}^d$ or on a sphere in $\mathbb{R}^d$.

For convenience we set

$$s(1) := 2.$$

This is the only nonstandard low-dimensional value needed below, and it is used only in the term $s(d-1)+3$ when $d = 2$.

We shall use the monotonicity of $g(d)$ and $s(d)$ without further comment: if $X \subset \mathbb{R}^d$, then the same set can be viewed inside $\mathbb{R}^{d+1}$. We also write

$$\dim X := \dim(\operatorname{aff} X)$$

for the affine dimension of a finite set X.

Write

$$U(m) := \frac{m(m+3)}{2}, \qquad L(m) := \binom{m+1}{2}, \qquad G(m) := \binom{m+2}{2} = U(m) + 1.$$

The quantity $U(m)$ is the classical Delsarte–Goethals–Seidel absolute upper bound for spherical 2-distance sets, while $G(m)$ is the Blokhuis upper bound for Euclidean 2-distance sets. [5, 3]

We need Ionin's decomposition theorem.

Definition 2.1 (Ionin)**.** Let S be a nonempty subset of a metric space. A partition $(S_1, S_2, \dots, S_k)$ of S is called a *complete decomposition* of S if it satisfies the following conditions:

(i) for $1 \le i \le k$, the set S_i is a 2-distance set;
(ii) for $1 \le i \le k-1$, one has $|S_i| \ge 3$, and also $|S_k| \ge 1$;
(iii) for $1 \le i < j \le k$, each point of S_j is the center of a sphere containing S_i.

Theorem 2.2 (Ionin)**.** *Every finite isosceles set $S \subset \mathbb{R}^d$ admits a complete decomposition. Moreover, if $(S_1, S_2, \dots, S_k)$ is a complete decomposition of S, then*

$$\dim S \ge \dim S_1 + \dim S_2 + \cdots + \dim S_k.$$

In particular, every block S_i with $i < k$ is spherical.

Reference. This is [9, Definition 4.1 and Proposition 4.2]. The final assertion is immediate from Definition 2.1: every point of a later block is the center of a sphere containing S_i. □

We also need the classical absolute bounds.

Theorem 2.3 (Classical absolute bounds)**.** *For every integer $m \ge 2$,*

$$s(m) \le U(m), \qquad g(m) \le G(m), \qquad s(m) \ge L(m).$$

References. The upper bound for spherical 2-distance sets is the Delsarte–Goethals–Seidel harmonic bound. [5] The upper bound for Euclidean 2-distance sets is due to Blokhuis; more generally, Bannai, Bannai, and Stanton proved that every Euclidean s-distance set in $\mathbb{R}^d$ has size at most $\binom{d+s}{s}$. [3, 1] The lower bound $s(m) \ge L(m)$ is given by the standard edge-midpoint construction in a regular simplex. □

The following elementary inequality is the only counting input in the proof of the upper bound.

Lemma 2.4. *Let $a_1, \dots, a_q \ge 2$ be integers and set $N := a_1 + \cdots + a_q$. Then*

$$\sum_{i=1}^{q} U(a_i) \le \begin{cases} U(N), & q = 1, \\ L(N), & q \ge 2. \end{cases}$$

Proof. The case $q = 1$ is tautological. Assume $q \ge 2$. Then

$$\begin{aligned} L(N) - \sum_{i=1}^{q} U(a_i) &= \frac{N(N+1) - \sum_i (a_i^2 + 3a_i)}{2} \\ &= \sum_{1 \le i < j \le q} a_i a_j - N. \end{aligned}$$

So it is enough to show $\sum_{i<j} a_i a_j \ge N$. Since

$$\sum_{1 \le i < j \le q} a_i a_j = \frac{N^2 - \sum_{i=1}^{q} a_i^2}{2},$$

for fixed N this sum is minimized when $\sum_i a_i^2$ is maximized. Under the constraints $a_i \ge 2$ and $q \ge 2$, the maximum of $\sum_i a_i^2$ is attained when $q = 2$ and $(a_1, a_2) = (2, N-2)$, so

$$\sum_{1 \le i < j \le q} a_i a_j \ge 2(N-2) \ge N.$$

Hence $L(N) - \sum_i U(a_i) \ge 0$. □

3. Lower constructions

The first construction contributes to $f(d)$, while the second contributes to both $f(d)$ and $h(d)$.

Proposition 3.1. *For every integer $d \geq 2$,*

$$f(d) \geq s(d) + 1, \qquad h(d) \geq s(d-1) + 3.$$

Consequently,

$$f(d) \geq \max\{g(d),\ s(d)+1,\ s(d-1)+3\}.$$

Proof. Let $X \subset \mathbb{R}^d$ be a spherical 2-distance set of size $s(d)$, and let c be the center of a sphere containing X. Then $X \cup \{c\}$ is isosceles: every triangle contained in X is already isosceles, and every triangle of the form $\{x, y, c\}$ with $x, y \in X$ is isosceles because $|x - c| = |y - c|$. Hence

$$f(d) \geq s(d) + 1.$$

For the second construction, assume first that $d \geq 3$. Let $H \cong \mathbb{R}^{d-1}$ be a hyperplane, let $X \subset H$ be a spherical 2-distance set of size $s(d-1)$, let $c \in H$ be the center of a sphere containing X, and let ℓ be the line through c perpendicular to H. Write $r := |x - c|$ for $x \in X$, choose points $p, q \in \ell$ such that c is the midpoint of pq and $|p - c| = |q - c| = r$, and set

$$Y := X \cup \{p, c, q\}.$$

Then every triangle contained in X is isosceles, every triangle of the form $\{x, y, c\}$ with $x, y \in X$ is isosceles because $|x - c| = |y - c|$, every triangle of the form $\{p, x, y\}$ or $\{q, x, y\}$ with $x, y \in X$ is isosceles because

$$|p - x|^2 = |p - c|^2 + |c - x|^2 = 2r^2 = |p - y|^2$$

and similarly for q, and the remaining mixed triangles are isosceles since $|p - c| = |q - c|$ and $|p - x| = |q - x|$ for every $x \in X$. Thus Y is isosceles.

Moreover, Y has more than two distances: it realizes the three distances

$$r, \qquad \sqrt{2}\, r, \qquad 2r,$$

namely $|x - c| = r$ for $x \in X$, $|p - x| = \sqrt{2}\, r$ for $x \in X$, and $|p - q| = 2r$. Hence $|Y| = s(d-1) + 3$ contributes to $h(d)$.

When $d = 2$, the same construction is obtained by taking X to be the two points of a 0-sphere in a line, so $|X| = s(1) = 2$ by convention. Again the resulting 5-point set has distances r, $\sqrt{2}\, r$, and $2r$. Therefore in all cases

$$h(d) \geq s(d-1) + 3.$$

Finally, every Euclidean 2-distance set is isosceles, so

$$f(d) = \max\{g(d), h(d)\} \geq \max\{g(d), s(d)+1, s(d-1)+3\}. \qquad \square$$

Remark 3.2. The first construction does *not* in general imply $h(d) \geq s(d) + 1$. Consider the three vertices of an equilateral triangle on the unit circle in $\mathbb{R}^2$:

$$X = \left\{(1, 0),\ \left(\tfrac{1}{2}, \tfrac{\sqrt{3}}{2}\right),\ \left(\tfrac{1}{2}, -\tfrac{\sqrt{3}}{2}\right)\right\}.$$

The set X is spherical and its two distances are 1 and $\sqrt{3}$. After adjoining the center $(0, 0)$, the resulting set still realizes only those same two distances. Thus $X \cup \{0\}$ contributes to $f(2)$ but not to $h(2)$.

Remark 3.3. The standard midpoint construction does provide a uniform lower bound for $h(d)$. If Λ_d is the set of edge-midpoints of a regular simplex in $\mathbb{R}^d$, then $|\Lambda_d| = L(d) = \binom{d+1}{2}$ and the center c of the simplex is at distance $\sqrt{2(d-1)/(d+1)}$ from every point of Λ_d, whereas the two distances inside Λ_d are $\sqrt{2}$ and 2. Hence

$$h(d) \geq \binom{d+1}{2} + 1$$

for every $d \geq 2$. This lower bound is weaker than the $s(d-1)+3$ construction in some dimensions and incomparable with the term $s(d)+1$ in general.

4. The upper bound for $h(d)$

We now prove the structural estimate behind Theorem 1.1.

Proposition 4.1. *For every integer $d \geq 2$,*

$$h(d) \leq \max\{s(d)+1,\ s(d-1)+3\}.$$

Proof. Let $S \subset \mathbb{R}^d$ be an isosceles set with more than two distances. Choose a complete decomposition

$$S = S_1 \sqcup S_2 \sqcup \cdots \sqcup S_k$$

as in Theorem 2.2. Since S has more than two distances, we have $k \geq 2$.

For $1 \leq i \leq k$, write

$$m_i := \dim S_i.$$

For $1 \leq i \leq k-1$, the block S_i is spherical and has at least 3 points. A sphere in a one-dimensional affine space contains at most two points, so

$$m_i \geq 2 \qquad (1 \leq i \leq k-1).$$

Set

$$N := m_1 + \cdots + m_{k-1}, \qquad e := m_k.$$

Then $N \geq 2$, $e \geq 0$, and Theorem 2.2 gives

$$N + e \leq d.$$

We split into cases according to the number $k-1$ of spherical blocks and the value of e.

Case 1: $k=2$. Then S_1 is spherical of dimension $N \geq 2$ and S_2 has dimension e.

If $e=0$, then $|S_2|=1$, so

$$|S| \leq s(N)+1 \leq s(d)+1.$$

If $e=1$, then S_2 is a 2-distance set on a line, hence $|S_2| \leq 3$. Since $N \leq d-1$,

$$|S| \leq s(N)+3 \leq s(d-1)+3.$$

Assume finally that $e \geq 2$. Then

$$|S| \leq s(N)+g(e) \leq U(N)+G(e).$$

Also

$$L(N+e)+1-\big(U(N)+G(e)\big) = Ne-N-e \geq 0$$

because $N, e \geq 2$. Therefore

$$|S| \leq L(N+e)+1 \leq L(d)+1 \leq s(d)+1.$$

Case 2: $k \geq 3$. Now there are at least two spherical blocks, so by Lemma 2.4,

$$\sum_{i=1}^{k-1} |S_i| \leq \sum_{i=1}^{k-1} s(m_i) \leq \sum_{i=1}^{k-1} U(m_i) \leq L(N).$$

If $e=0$, then $|S_k|=1$, so

$$|S| \leq L(N)+1 \leq L(d)+1 \leq s(d)+1.$$

If $e=1$, then $|S_k| \leq 3$ and $N \leq d-1$, so

$$|S| \leq L(N)+3 \leq L(d-1)+3 \leq s(d-1)+3.$$

Assume finally that $e \geq 2$. Then

$$|S| \leq L(N)+g(e) \leq L(N)+G(e).$$

A direct calculation gives

$$L(N+e)+1-\big(L(N)+G(e)\big)=e(N-1)\geq 0,$$

so again

$$|S|\leq L(N+e)+1\leq L(d)+1\leq s(d)+1.$$

In all cases,

$$|S|\leq \max\{s(d)+1, s(d-1)+3\},$$

which proves the proposition. □

5. Exact formula for $f(d)$

We can now prove the main theorem.

Proof of Theorem 1.1. The case $d=1$ is immediate: any four points on a line contain a triple that is not isosceles, so $f(1)=3$.

Now let $d\geq 2$. By Proposition 3.1,

$$f(d)\geq \max\{g(d), s(d)+1, s(d-1)+3\}.$$

On the other hand,

$$f(d)=\max\{g(d), h(d)\}$$

and Proposition 4.1 gives

$$h(d)\leq \max\{s(d)+1, s(d-1)+3\}.$$

Therefore

$$f(d)\leq \max\{g(d), s(d)+1, s(d-1)+3\}.$$

Combining the lower and upper bounds yields

$$f(d)=\max\{g(d), s(d)+1, s(d-1)+3\}.$$

□

Corollary 5.1. *If $d\geq 2$ and $s(d-1)+3\geq s(d)+1$, then*

$$h(d)=s(d-1)+3.$$

In particular,

$$h(3)=8, \qquad h(23)=278.$$

Proof. The lower bound $h(d)\geq s(d-1)+3$ is Proposition 3.1, and the upper bound $h(d)\leq \max\{s(d)+1, s(d-1)+3\}$ is Proposition 4.1.

For $d=3$, one has $s(2)=5$ and $s(3)=6$, so $s(2)+3=8\geq 7=s(3)+1$. For $d=23$, Barg and Yu proved $s(23)=276$, while Musin proved $s(22)=275$, so $s(22)+3=278\geq 277=s(23)+1$. [2, 14] □

6. Consequences for Erdős Problem 503

The first consequence is the exact reduction promised in the title.

Corollary 6.1. *For every integer $d\geq 2$,*

$$f(d)=\max\{g(d),\ s(d)+1,\ s(d-1)+3\}.$$

Hence Erdős Problem #503 is reduced exactly to the Euclidean and spherical 2-distance extremal functions.

Remark 6.2. The forum page currently records the lower bounds $\binom{d+1}{2}$ and $\binom{d+1}{2}+1$. Our theorem shows that the full answer is obtained by comparing the Euclidean 2-distance maximum $g(d)$ with the two spherical templates of sizes $s(d)+1$ and $s(d-1)+3$. In this sense the problem is resolved structurally, even though the numerical values of $g(d)$ and $s(d)$ are not known in all dimensions. [6]

The theorem also recovers the known low-dimensional values.

Corollary 6.3. *Using the known exact values of $g(d)$ for $1 \leq d \leq 8$ and of $s(d)$ for $2 \leq d \leq 8$, one recovers*

$$f(1) = 3, \quad f(2) = 6, \quad f(3) = 8, \quad f(4) = 11, \quad f(5) = 17, \quad f(6) = 28, \quad f(7) = 30, \quad f(8) = 45.$$

Proof. Substitute the known exact values of $g(d)$ and $s(d)$ into Theorem 1.1. For $g(d)$ one may use Lisoněk's classification of maximal Euclidean 2-distance sets in dimensions at most 8. [13] For the corresponding spherical values in low dimensions one may use Musin's summary and the references cited there. [14] The resulting values agree with Ionin's table. [9, p. 2] For dimensions 3 and 4, these values are also consistent with the earlier direct classifications of Kido. [11, 12] □

The dimensions 22 and 23 are especially relevant to the present state of the 2-distance literature.

Corollary 6.4. *One has*

$$f(22) = 276, \qquad h(23) = 278, \qquad f(23) = \max\{g(23), 278\}.$$

In particular, the natural lower bound in dimension 23 *is* 278*.*

Proof. Musin proved that $s(22) = 275$. Therefore

$$f(22) \geq s(22) + 1 = 276$$

by Proposition 3.1. Blokhuis's general upper bound gives

$$f(22) \leq \binom{24}{2} = 276,$$

so $f(22) = 276$.

Next, Barg and Yu proved that $s(23) = 276$. Since $s(22) = 275$, Corollary 5.1 gives

$$h(23) = 278.$$

Hence Theorem 1.1 yields

$$f(23) = \max\{g(23), 278\}.$$

The lower bound $f(23) \geq 278$ comes from the second construction in Proposition 3.1. Separately, Ge, Koolen, and Munemasa constructed a 277-point Euclidean 2-distance set in $\mathbb{R}^{23}$, so $g(23) \geq 277$. [7] □

We also recover the quadratic growth recorded in the literature.

Corollary 6.5. *For every integer $d \geq 2$,*

$$\binom{d+1}{2} + 1 \leq f(d) \leq \binom{d+2}{2}.$$

Consequently,

$$f(d) = \frac{1}{2}d^2 + O(d), \qquad \lim_{d\to\infty} \frac{f(d)}{d^2} = \frac{1}{2}.$$

Proof. The lower bound follows from Remark 3.3, and the upper bound is Blokhuis's theorem. [3] The asymptotic statement is immediate. □

Remark 6.6. If both

$$s(d) = L(d) = \binom{d+1}{2} \qquad \text{and} \qquad s(d-1) = L(d-1) = \binom{d}{2},$$

then Theorem 1.1 simplifies to

$$f(d) = \max\left\{g(d), \binom{d+1}{2} + 1\right\}.$$

Indeed, under these two equalities one has

$$s(d) + 1 = \binom{d+1}{2} + 1$$

and

$$s(d-1) + 3 = \binom{d}{2} + 3 \leq \binom{d+1}{2} + 1$$

for every $d \geq 2$.

Glazyrin and Yu proved that $s(n) = \binom{n+1}{2}$ for all $n \geq 7$ with possible exceptions only when $n = (2k+1)^2 - 3$, and Barg and Yu resolved the case $n = 23$ together with all dimensions $40 \leq n \leq 93$, $n \notin \{46, 78\}$. [8, 2] Consequently, the simplified formula above holds for every d for which both neighboring spherical dimensions $d-1$ and d are known to satisfy the simplex-midpoint equality.

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