A note on an Erdős path problem for transcendental entire functions

Abstract

Let f be a transcendental entire function and let

$$M(r,f) = \max_{|z|=r} |f(z)|.$$

Erdős asked whether there exists a path to ∞ on which $|f(z)|$ dominates all powers of $|z|$, whether the length of such a path can be estimated in terms of $M(r,f)$, and whether one can force $|f(z)|$ to grow faster than a fixed function of $M(r,f)$, for example a positive power of $M(r,f)$. Using a theorem of J.-M. Wu on subharmonic functions along paths, we show that the first question has a positive answer and that one may choose a path whose initial segment up to radius R has length $O(M(R,f)^{\varepsilon})$ for every $\varepsilon > 0$. On the other hand, using a construction recorded by Langley, we show that no positive power of $M(r,f)$ can serve as a universal lower bound along such a path. Thus the parenthetical power-type version of Erdős's third question has a negative answer. The broader formulation involving an arbitrary fixed function of $M(r,f)$ remains outside the scope of this note.

1 Introduction

In [1, p. 249] Erdős asked whether every transcendental entire function f admits a path L to infinity such that

$$\left|\frac{f(z)}{z^n}\right| \to \infty \qquad (z \to \infty,\ z \in L)$$

for every fixed $n \in \mathbb{N}$. The same question reappears in [2], where Erdős also asked whether the length of such a path can be estimated in terms of the maximum modulus $M(r,f)$, and whether one can force $|f(z)|$ to grow faster than a fixed function of $M(r,f)$, for example $M(r,f)^{\varepsilon}$.

The existence part is implicit in the work of Lewis, Rossi and Weitsman [4] and is stated in an especially convenient form by Wu [5, Theorem B]. Our first result answers Erdős's first two questions. Since any path to ∞ has infinite total length, the second question must be interpreted in terms of initial segments.

By a *path to infinity* we mean a continuous map $\gamma : [0,\infty) \to \mathbb{C}$ with $|\gamma(t)| \to \infty$ as $t \to \infty$. For $R > 0$ we set

$$t_R := \inf\{t \geq 0 : |\gamma(t)| = R\},$$

which is finite for all sufficiently large R, and we write $\ell_\gamma(R)$ for the length of the initial segment $\gamma([0,t_R])$.

Theorem 1. *Let f be a transcendental entire function. Then there exists a path to infinity γ such that*

$$\frac{\log|f(\gamma(t))|}{\log|\gamma(t)|} \to \infty \qquad (t \to \infty).$$

In particular,

$$\left|\frac{f(\gamma(t))}{\gamma(t)^n}\right| \to \infty \qquad (t \to \infty)$$

for every fixed $n \in \mathbb{N}$. Moreover, the same path satisfies

$$\ell_\gamma(R) = O\big(M(R,f)^\varepsilon\big) \qquad (R \to \infty)$$

for every $\varepsilon > 0$.

Our second result gives a negative answer to the specific power-type comparison asked by Erdős.

Theorem 2. *There exists a transcendental entire function f with the following property: for every unbounded connected set $E \subset \mathbb{C}$ and every $\varepsilon > 0$, there exists a sequence $w_k \in E$ with $|w_k| \to \infty$ such that*

$$|f(w_k)| \leq M(|w_k|,f)^\varepsilon \qquad (k \to \infty).$$

Consequently, there is no fixed exponent $\varepsilon > 0$ for which every transcendental entire function admits a path to infinity on which $|f(z)| \geq M(|z|,f)^\varepsilon$ eventually.

2 Two background lemmas

We first record the standard fact that a transcendental entire function dominates all powers of r on circles.

Lemma 3. *Let F be a transcendental entire function. Then for every $\alpha > 0$,*

$$\frac{M(r,F)}{r^{\alpha}} \to \infty \qquad (r \to \infty).$$

Proof. Write

$$F(z) = \sum_{m=0}^{\infty} a_m z^m.$$

Given $\alpha > 0$, choose an integer $m > \alpha$ such that $a_m \neq 0$; this is possible because F is not a polynomial. By Cauchy's estimate,

$$|a_m| \le \frac{M(r,F)}{r^m} \qquad (r > 0).$$

Hence

$$\frac{M(r,F)}{r^{\alpha}} \ge |a_m|\, r^{m-\alpha} \to \infty \qquad (r \to \infty),$$

as required. □

The key input for Theorem 1 is the following theorem of Wu, which itself builds on work of Lewis, Rossi and Weitsman.

Theorem 4 (Wu, Theorem B [5])**.** *Let u be a subharmonic function in $\mathbb{C}$, and put*

$$B_u(r) := \sup_{|z|=r} u(z).$$

If

$$\frac{B_u(r)}{\log r} \to \infty \qquad (r \to \infty),$$

then there exists a path to infinity γ such that

$$\frac{u(\gamma(t))}{\log|\gamma(t)|} \to \infty \qquad (t \to \infty),$$

and, for every $\delta > 0$,

$$\int_{\gamma} e^{-\delta u(z)}\, |dz| < \infty.$$

The path γ may be chosen independently of δ.

3 Proof of Theorem 1

Proof. Set

$$u(z) := \max\{\log|f(z)|, -1\}.$$

Since $\log|f|$ is subharmonic and the maximum of two subharmonic functions is subharmonic, the function u is subharmonic on $\mathbb{C}$. Moreover,

$$B_u(r) = \sup_{|z|=r} u(z) = \max\{\log M(r,f), -1\}.$$

By Lemma 3,

$$\frac{\log M(r,f)}{\log r} \to \infty,$$

and therefore

$$\frac{B_u(r)}{\log r} \to \infty \qquad (r \to \infty).$$

Applying Theorem 4, we obtain a path to infinity γ such that

$$\frac{u(\gamma(t))}{\log|\gamma(t)|} \to \infty \qquad (t \to \infty)$$

and

$$\int_\gamma e^{-\delta u(z)}\,|dz| < \infty \qquad \text{for every } \delta > 0.$$

Because $\log|\gamma(t)| \to \infty$, the first limit implies $u(\gamma(t)) \to \infty$. Hence, for all sufficiently large t, we have $u(\gamma(t)) = \log|f(\gamma(t))|$ and so

$$\frac{\log|f(\gamma(t))|}{\log|\gamma(t)|} \to \infty \qquad (t \to \infty).$$

This proves the first assertion of the theorem.

Now fix $n \in \mathbb{N}$. For all sufficiently large t,

$$\frac{\log|f(\gamma(t))|}{\log|\gamma(t)|} > n+1,$$

which gives $|f(\gamma(t))| > |\gamma(t)|^{n+1}$. Therefore

$$\left|\frac{f(\gamma(t))}{\gamma(t)^n}\right| > |\gamma(t)| \to \infty,$$

so the same path works simultaneously for every fixed n.

It remains to estimate the length of the path. Fix $\varepsilon > 0$. Since $\gamma([0,t_R]) \subset \overline{D(0,R)}$ by definition of t_R, subharmonicity gives

$$u(z) \le B_u(R) \qquad (z \in \gamma([0,t_R])).$$

Hence

$$\begin{aligned} \ell_\gamma(R) &= \int_{\gamma([0,t_R])} 1\,|dz| \\ &= \int_{\gamma([0,t_R])} e^{\varepsilon u(z)} e^{-\varepsilon u(z)}\,|dz| \\ &\le e^{\varepsilon B_u(R)} \int_\gamma e^{-\varepsilon u(z)}\,|dz|. \end{aligned}$$

The last integral is finite by Theorem 4. Also,

$$e^{\varepsilon B_u(R)} = e^{\varepsilon \max\{\log M(R,f),-1\}} = O\big(M(R,f)^\varepsilon\big) \qquad (R \to \infty).$$

Therefore

$$\ell_\gamma(R) = O\big(M(R,f)^\varepsilon\big) \qquad (R \to \infty),$$

which proves the theorem. □

Remark 5. Wu's theorem yields more than we have used. Along the same path one has

$$\int_\gamma |f(z)|^{-\delta}\,|dz| < \infty$$

for every $\delta > 0$ after deleting an initial compact subarc. We have used only the consequence needed for the length bound.

4 Power-type lower bounds in terms of $M(r,f)$ fail

For the negative result we need one additional lemma.

Lemma 6. *Let G be a transcendental entire function and put*

$$B_G(r) := \max_{|z|=r} \Re G(z).$$

Then, for every $\alpha > 0$,

$$\frac{B_G(r)}{r^\alpha} \to \infty \qquad (r \to \infty).$$

Proof. Let $M_G(r) = M(r, G) = \max_{|z|=r} |G(z)|$. Applying the classical Borel–Carathéodory theorem to $G - G(0)$ on $|z| < 2r$, we obtain

$$\max_{|z|=r} |G(z) - G(0)| \le 2 \max_{|z|=2r} \Re\big(G(z) - G(0)\big) \le 2B_G(2r) + 2|G(0)|.$$

Hence

$$M_G(r) \le 2B_G(2r) + 3|G(0)| \qquad (r > 0).$$

Fix $\alpha > 0$. If the conclusion were false, there would exist a constant $C > 0$ and a sequence $s_j \to \infty$ such that

$$B_G(s_j) \le C s_j^{\alpha} \qquad (j \in \mathbb{N}).$$

Writing $s_j = 2r_j$, we would get

$$M_G(r_j) \le 2C(2r_j)^{\alpha} + 3|G(0)| = O(r_j^{\alpha}),$$

contradicting Lemma 3 applied to the transcendental entire function G. Hence $B_G(r)/r^{\alpha} \to \infty$. □

Proof of Theorem 2. By Langley's Theorem 1.4 [3], there exists a transcendental entire function G such that every unbounded connected plane set E contains a sequence $w_n \in E$ with $|w_n| \to \infty$ and

$$(-1)^n \Re G(w_n) \le |w_n|^{1/2}.$$

Passing to the even subsequence, we may assume that

$$\Re G(w_n) \le |w_n|^{1/2} \qquad (n \in \mathbb{N}).$$

Now set

$$f(z) := e^{G(z)}.$$

Then f is transcendental entire and

$$M(r, f) = \max_{|z|=r} e^{\Re G(z)} = e^{B_G(r)}.$$

By Lemma 6,

$$\frac{B_G(r)}{r^{1/2}} \to \infty \qquad (r \to \infty).$$

Fix $\varepsilon > 0$. For all sufficiently large n this implies

$$|w_n|^{1/2} \le \varepsilon B_G(|w_n|).$$

Hence

$$|f(w_n)| = e^{\Re G(w_n)} \le e^{|w_n|^{1/2}} \le e^{\varepsilon B_G(|w_n|)} = M(|w_n|, f)^{\varepsilon}.$$

Since the unbounded connected set E was arbitrary, the theorem follows. □

Remark 7. Theorem 2 rules out every universal lower bound of the form $M(r,f)^{\varepsilon}$ with fixed $\varepsilon > 0$. It does *not* settle the broader question of whether some much slower universal comparison function of $M(r,f)$ might still be forced along a suitable path.

References

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