A STRONG LAW FOR THE ROOTS OF RANDOM LITTLEWOOD POLYNOMIALS

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Abstract. Let $(\varepsilon_k)_{k\geqslant 0}$ be independent Rademacher variables and put

$$P_n(z) = \sum_{k=0}^{n} \varepsilon_k z^k, \qquad \mathbb{P}(\varepsilon_k = 1) = \mathbb{P}(\varepsilon_k = -1) = \frac{1}{2}.$$

If R_n denotes the number of zeros of P_n in the closed unit disk, counted with multiplicity, then

$$R_n = \frac{n}{2} + O_\omega(n^{399/400}) \qquad \text{almost surely.}$$

In particular, $R_n/(n/2) \to 1$ almost surely.

1. Introduction

A Littlewood polynomial is a polynomial all of whose coefficients belong to $\{-1, 1\}$. Erdős asked whether, for the random Littlewood polynomial

$$P_n(z) = \sum_{k=0}^{n} \varepsilon_k z^k$$

with independent uniform signs, the number R_n of zeros in $\{z \in \mathbb{C} : |z| \leqslant 1\}$ satisfies $R_n/(n/2) \to 1$ almost surely; see [1–3]. Yakir proved a closely related convergence-in-probability result for random Littlewood polynomials [4]. We prove the almost-sure version, with a power saving.

Theorem 1.1. *Let R_n count zeros of P_n in $\{|z| \leqslant 1\}$, with multiplicity. With probability one,*

$$R_n = \frac{n}{2} + O_\omega\big(n^{399/400}\big).$$

Consequently,

$$\frac{R_n}{n/2} \longrightarrow 1 \qquad \textit{almost surely.}$$

The proof uses Jensen's formula at the two radii $1 - n^{-401/400}$ and $(1 - n^{-401/400})^{-1}$. Apart from the quoted logarithmic-integrability theorem of Nazarov, Nishry and Sodin for Rademacher Fourier series, all estimates needed below are proved explicitly. The required smooth central-limit estimate is a direct Lindeberg replacement argument.

2. Notation and elementary estimates

For $n \geqslant 2$ and $j \geqslant 1$, set

$$\rho_n := 1 - n^{-401/400}, \qquad \tau_n := \rho_n^{-1}, \qquad N_j := \lfloor j^4 \rfloor.$$

All asymptotic assertions below concern sufficiently large n, and we let

$$\eta := \frac{1}{100}, \qquad \beta := \frac{1}{200}.$$

Let

$$\mu(\,\mathrm{d}\theta) = \frac{\mathrm{d}\theta}{2\pi} \qquad (-\pi \leqslant \theta \leqslant \pi).$$

For $r>0$ put

$$\sigma_n(r)^2 := \sum_{k=0}^{n} r^{2k}, \qquad F_{n,r}(\theta) := \frac{P_n(re^{\mathrm{i}\theta})}{\sigma_n(r)}.$$

Thus

$$F_{n,r}(\theta) = \sum_{k=0}^{n} a_{n,k}(r)\varepsilon_k e^{\mathrm{i}k\theta}, \qquad a_{n,k}(r) := \frac{r^k}{\sigma_n(r)},$$

and $\sum_{k=0}^{n} a_{n,k}(r)^2 = 1$. We regard $\mathbb{C}$ as $\mathbb{R}^2$ when taking C^m norms. All implicit constants are deterministic unless the notation O_ω is used.

Lemma 2.1. *Uniformly for* $r \in [\rho_n, \tau_n]$,

$$\sigma_n(r)^2 = n+1+O(n^{399/400}), \tag{2.1}$$

$$\max_{0\leqslant k\leqslant n} a_{n,k}(r) \ll n^{-1/2}, \tag{2.2}$$

$$\sum_{k=0}^{n} a_{n,k}(r)^3 \ll n^{-1/2}. \tag{2.3}$$

Moreover, if

$$B_{n,r}(t) := \sum_{k=0}^{n} a_{n,k}(r)^2 e^{\mathrm{i}kt},$$

then

$$|B_{n,r}(t)| \ll n^{-1/2}$$

whenever $\operatorname{dist}(t, 2\pi\mathbb{Z}) \geqslant n^{-1/2}$.

Proof. Since $|\log r| \ll n^{-401/400}$ for $r \in [\rho_n, \tau_n]$, we have $r^{2k} = 1 + O(kn^{-401/400})$ uniformly for $0 \leqslant k \leqslant n$. Summing over k gives (2.1). Also $r^k \ll 1$ uniformly in the same range, and therefore (2.2) follows. Since $\sum_k a_{n,k}(r)^2 = 1$, this gives (2.3).

For the last estimate,

$$B_{n,r}(t) = \frac{1-(r^2e^{\mathrm{i}t})^{n+1}}{\sigma_n(r)^2(1-r^2e^{\mathrm{i}t})}.$$

The numerator is $O(1)$ because $r^{2n} = O(1)$ in the present range, and $\sigma_n(r)^2 \asymp n$. If $\operatorname{dist}(t, 2\pi\mathbb{Z}) \geqslant n^{-1/2}$, then

$$\left|1-r^2e^{\mathrm{i}t}\right| \geqslant \left|1-e^{\mathrm{i}t}\right| - \left|r^2-1\right| \gg \operatorname{dist}(t,2\pi\mathbb{Z}) - O(n^{-401/400}) \gg \operatorname{dist}(t,2\pi\mathbb{Z}).$$

This proves $|B_{n,r}(t)| \ll n^{-1}\operatorname{dist}(t,2\pi\mathbb{Z})^{-1} \ll n^{-1/2}$. ∎

Lemma 2.2 (Block comparison)**.** *Let* $m = N_j$ *and* $m \leqslant n < N_{j+1}$. *For each* N *let* r_N *be one of* $1, \rho_N, \tau_N$, *with the same type of choice made for every* N. *Extend coefficient vectors by setting*

$$\widetilde{a}_{N,k}(r_N) := \begin{cases} a_{N,k}(r_N), & 0 \leqslant k \leqslant N, \\ 0, & k > N. \end{cases}$$

Then

$$\sum_{k\geqslant 0} |\widetilde{a}_{n,k}(r_n) - \widetilde{a}_{m,k}(r_m)|^2 \ll m^{-1/4}.$$

Consequently,

$$\int |F_{n,r_n}(\theta) - F_{m,r_m}(\theta)|\, \mu(\mathrm{d}\theta) \ll m^{-1/8}.$$

Proof. Write $\Delta = n - m$. Since $N_{j+1} - N_j = O(N_j^{3/4})$, we have $\Delta = O(m^{3/4})$. For each of the three allowed choices,

$$|\log r_n - \log r_m| \ll m^{-501/400}.$$

Hence, uniformly for $0 \leqslant k \leqslant m$,

$$\left(\frac{r_n}{r_m}\right)^k = 1 + O(m^{-1/4}).$$

Also,

$$\sigma_n(r_n)^2 - \sigma_m(r_m)^2 = \sum_{k=0}^{m} (r_n^{2k} - r_m^{2k}) + \sum_{k=m+1}^{n} r_n^{2k}.$$

The first sum is

$$O\left(\sum_{k=0}^{m} k\,|\log r_n - \log r_m|\right) = O(m^{299/400}),$$

and the second is $O(n-m) = O(m^{3/4})$. Thus

$$\sigma_n(r_n)^2 - \sigma_m(r_m)^2 = O(m^{3/4}).$$

Since $\sigma_m(r_m)^2 \asymp m$, it follows that

$$\frac{\sigma_m(r_m)}{\sigma_n(r_n)} = 1 + O(m^{-1/4}).$$

Therefore, for $0 \leqslant k \leqslant m$,

$$\frac{a_{n,k}(r_n)}{a_{m,k}(r_m)} = 1 + O(m^{-1/4}),$$

and the contribution of $0 \leqslant k \leqslant m$ to the square sum is $O(m^{-1/2})$. The tail contributes

$$\sum_{k=m+1}^{n} a_{n,k}(r_n)^2 \ll \frac{n-m}{m} \ll m^{-1/4}.$$

This proves the square-sum bound. The L^1 bound follows from Cauchy's inequality and Parseval:

$$\int |F_{n,r_n} - F_{m,r_m}|\,\mathrm{d}\mu \leqslant \left(\sum_{k\geqslant 0} |\widetilde{a}_{n,k}(r_n) - \widetilde{a}_{m,k}(r_m)|^2\right)^{1/2}.$$

□

3. A smooth angular central limit theorem

We first record the replacement estimate used below.

Lemma 3.1 (Smooth Lindeberg estimate)**.** *Let $X_1, \dots, X_m$ be independent centered random vectors in $\mathbb{R}^d$. Let $Y_1, \dots, Y_m$ be independent centered Gaussian vectors, independent of the X_k, such that*

$$\mathbb{E} Y_k Y_k^T = \mathbb{E} X_k X_k^T \qquad (1 \leqslant k \leqslant m).$$

If $\Phi : \mathbb{R}^d \to \mathbb{R}$ has bounded derivatives up to order 3, then

$$\left| \mathbb{E}\Phi\left(\sum_{k=1}^{m} X_k\right) - \mathbb{E}\Phi\left(\sum_{k=1}^{m} Y_k\right) \right| \leqslant C_d \,\|\Phi\|_{C^3} \sum_{k=1}^{m} \mathbb{E}|X_k|^3.$$

Proof. This is the standard one-by-one Lindeberg replacement argument; compare Chatterjee's generalized Lindeberg principle [5, Theorem 1.1]. We include the proof to fix the precise normalization. Let

$$S_k := Y_1 + \cdots + Y_{k-1} + X_{k+1} + \cdots + X_m.$$

Then

$$\mathbb{E}\Phi\left(\sum_{i=1}^m X_i\right) - \mathbb{E}\Phi\left(\sum_{i=1}^m Y_i\right) = \sum_{k=1}^m \mathbb{E}\{\Phi(S_k + X_k) - \Phi(S_k + Y_k)\}.$$

The vector S_k is independent of both X_k and Y_k. Conditioning on $S_k = s$ and expanding $\Phi(s+u)$ to second order at $u = 0$, the constant and linear terms cancel because X_k and Y_k are centered, and the quadratic terms cancel because their covariance matrices agree. The Taylor remainder is bounded by

$$C_d \|\Phi\|_{C^3} \left(|X_k|^3 + |Y_k|^3\right).$$

Finally, since Y_k is Gaussian,

$$\mathbb{E}|Y_k|^3 \leqslant C_d(\mathbb{E}|Y_k|^2)^{3/2} = C_d(\mathbb{E}|X_k|^2)^{3/2} \leqslant C_d\mathbb{E}|X_k|^3.$$

Summing over k proves the claim. □

Lemma 3.2 (Gaussian stability)**.** *Let d be fixed. Suppose Σ is a real $d \times d$ covariance matrix with*

$$\left\|\Sigma - \tfrac{1}{2} I_d\right\|_{\mathrm{op}} \leqslant \delta \leqslant \frac{1}{10}.$$

If $Z_\Sigma \sim N(0, \Sigma)$ and $Z \sim N(0, \frac{1}{2} I_d)$, then every Lipschitz function $\Phi : \mathbb{R}^d \to \mathbb{R}$ satisfies

$$|\mathbb{E}\Phi(Z_\Sigma) - \mathbb{E}\Phi(Z)| \leqslant C_d \operatorname{Lip}(\Phi)\delta.$$

Proof. Let $W \sim N(0, I_d)$ and couple $Z_\Sigma = \Sigma^{1/2} W$ and $Z = (\frac{1}{2} I_d)^{1/2} W$. Write $\Sigma = \frac{1}{2}(I_d + A)$, where $\|A\|_{\mathrm{op}} \leqslant 2\delta$. By the spectral theorem,

$$\left\|(I_d + A)^{1/2} - I_d\right\|_{\mathrm{op}} \leqslant C \|A\|_{\mathrm{op}} \leqslant C\delta,$$

since the spectrum of A lies in $[-1/2, 1/2]$. Therefore

$$\mathbb{E}|Z_\Sigma - Z| \leqslant C_d \delta \mathbb{E}|W| \leqslant C_d \delta.$$

The Lipschitz bound for Φ gives the result. □

Proposition 3.3 (Smooth angular central limit theorem)**.** *Let $h : \mathbb{C} \to \mathbb{R}$ be bounded and C^3. Uniformly for $r \in [\rho_n, \tau_n]$,*

$$\left|\mathbb{E} \int h(F_{n,r}(\theta))\mu(\,\mathrm{d}\theta) - \mathbb{E}h(G)\right| \ll (\|h\|_{C^3} + \|h\|_\infty) n^{-1/2}, \tag{3.1}$$

and

$$\operatorname{Var}\left(\int h(F_{n,r}(\theta))\mu(\,\mathrm{d}\theta)\right) \ll (\|h\|_{C^3} + \|h\|_\infty)^2 n^{-1/2}. \tag{3.2}$$

Here G is the standard complex Gaussian, with density $\pi^{-1} e^{-|z|^2}$.

Proof. Fix $r \in [\rho_n, \tau_n]$. For fixed θ, the vector $(\Re F_{n,r}(\theta), \Im F_{n,r}(\theta))$ is a sum of independent centered vectors

$$X_k(\theta) = \varepsilon_k a_{n,k}(r)(\cos k\theta, \sin k\theta).$$

By Lemma 2.1,

$$\sum_{k=0}^n \mathbb{E}|X_k(\theta)|^3 = \sum_{k=0}^n a_{n,k}(r)^3 \ll n^{-1/2}.$$

Its covariance matrix is

$$\Sigma_\theta = \frac{1}{2} I_2 + \frac{1}{2}\begin{pmatrix} \Re B_{n,r}(2\theta) & \Im B_{n,r}(2\theta) \\ \Im B_{n,r}(2\theta) & -\Re B_{n,r}(2\theta) \end{pmatrix}.$$

The set of θ for which $\mathrm{dist}(2\theta, 2\pi\mathbb{Z}) < n^{-1/2}$ has μ-measure $O(n^{-1/2})$. Outside this set, Lemma 2.1 gives $\Sigma_\theta = \frac{1}{2} I_2 + O(n^{-1/2})$. Lemma 3.1, followed by Lemma 3.2, gives

$$|\mathbb{E}h(F_{n,r}(\theta)) - \mathbb{E}h(G)| \ll \|h\|_{C^3}\, n^{-1/2}$$

on the good set. On the exceptional set we use the trivial bound $2\|h\|_\infty$. Integration in θ proves (3.1).

For the variance, apply the same argument to the four-dimensional vector

$$(\Re F_{n,r}(\theta), \Im F_{n,r}(\theta), \Re F_{n,r}(\phi), \Im F_{n,r}(\phi)).$$

The corresponding summands have Euclidean norm at most $\sqrt{2}\, a_{n,k}(r)$, so their third absolute moments again sum to $O(n^{-1/2})$. The covariance matrix has diagonal blocks of the form just computed, and its off-diagonal blocks are linear combinations of

$$B_{n,r}(\theta - \phi), \qquad B_{n,r}(\theta + \phi).$$

The union of the four exceptional strips

$$\mathrm{dist}(2\theta, 2\pi\mathbb{Z}) < n^{-1/2}, \quad \mathrm{dist}(2\phi, 2\pi\mathbb{Z}) < n^{-1/2},$$

$$\mathrm{dist}(\theta - \phi, 2\pi\mathbb{Z}) < n^{-1/2}, \quad \mathrm{dist}(\theta + \phi, 2\pi\mathbb{Z}) < n^{-1/2}$$

has $(\mu\times\mu)$-measure $O(n^{-1/2})$; each strip has this measure by Fubini. Outside this set the covariance matrix is $\frac{1}{2} I_4 + O(n^{-1/2})$ in operator norm. Applying Lemmas 3.1 and 3.2 to

$$H(z, w) := h(z)h(w)$$

uses

$$\|H\|_{C^3} + \mathrm{Lip}(H) \ll (\|h\|_{C^3} + \|h\|_\infty)^2.$$

On the exceptional set we use the trivial bound $|H| \leqslant \|h\|_\infty^2$, and the independent Gaussian comparison term is bounded in the same way. Thus, on the good set, the comparison Gaussian is within $O(n^{-1/2})$ of a pair of independent standard complex Gaussians, and on the exceptional set the total contribution is $O(\|h\|_\infty^2\, n^{-1/2})$. Integrating over (θ, ϕ) gives

$$\mathbb{E}\left(\int h(F_{n,r})\,\mathrm{d}\mu\right)^2 = (\mathbb{E}h(G))^2 + O\left((\|h\|_{C^3} + \|h\|_\infty)^2 n^{-1/2}\right).$$

Together with (3.1), this implies (3.2). □

4. Logarithmic averages

We use the following theorem of Nazarov, Nishry and Sodin.

Theorem 4.1 (Nazarov–Nishry–Sodin). *There is an absolute constant C with the following property. Let $(\varepsilon_k)_{k\geqslant 0}$ be independent Rademacher variables on a probability space $(\Omega, \mathbb{P})$, and let*

$$f(\omega, \theta) = \sum_{k\geqslant 0} c_k \varepsilon_k(\omega) e^{\mathrm{i}k\theta}, \qquad \sum_{k\geqslant 0} |c_k|^2 = 1.$$

Then, for every $p > 1$,

$$\int_\Omega \int_{-\pi}^{\pi} |\log|f(\omega,\theta)||^p\ \mu(\,\mathrm{d}\theta)\,\mathbb{P}(\,\mathrm{d}\omega) \leqslant (Cp)^{6p}.$$

This is the positive-frequency special case of [6, Corollary 1.2], under the change of variables between $[0,1]$ and $[-\pi,\pi]$. We use it only for $p \geqslant 2$. The finite sums $F_{n,r}$ are special cases, with $c_k = a_{n,k}(r)$ for $0 \leqslant k \leqslant n$ and $c_k = 0$ otherwise.

For $r > 0$ define

$$L_n(r) := \int \log |F_{n,r}(\theta)| \, \mu(\mathrm{d}\theta).$$

The logarithmic singularities are integrable deterministically, since on each circle a nonzero polynomial has only finitely many zeros. The theorem above is used for the uniform logarithmic moment bound.

Lemma 4.2. *Fix one of the three deterministic choices $r_n \equiv 1$, $r_n = \rho_n$, or $r_n = \tau_n$. Let $h_j : \mathbb{C} \to \mathbb{R}$ be deterministic C^3 functions. Assume that there is a deterministic constant C_0 such that, for every j,*

$$\|h_j\|_\infty + \|h_j\|_{C^3} \leqslant C_0 N_j^{3\eta}, \qquad \mathrm{Lip}(h_j) \leqslant C_0 N_j^{\eta}.$$

Then, almost surely, for all sufficiently large j and all $n \in [N_j, N_{j+1})$,

$$\left| \int h_j(F_{n,r_n}) \, \mathrm{d}\mu - \mathbb{E} h_j(G) \right| \ll_{C_0} N_j^{-\beta}.$$

Proof. At the block endpoint $n = N_j$, Proposition 3.3 gives

$$\left| \mathbb{E} \int h_j(F_{N_j, r_{N_j}}) \, \mathrm{d}\mu - \mathbb{E} h_j(G) \right| \ll_{C_0} N_j^{3\eta - 1/2}$$

and

$$\mathrm{Var}\left(\int h_j(F_{N_j, r_{N_j}}) \, \mathrm{d}\mu \right) \ll_{C_0} N_j^{6\eta - 1/2}.$$

Since $3\eta - 1/2 < -\beta$, Chebyshev's inequality gives

$$\mathbb{P}\left(\left| \int h_j(F_{N_j, r_{N_j}}) \, \mathrm{d}\mu - \mathbb{E} h_j(G) \right| > 2N_j^{-\beta} \right) \ll_{C_0} N_j^{2\beta + 6\eta - 1/2}.$$

Now $N_j \asymp j^4$ and

$$4(2\beta + 6\eta - 1/2) < -1,$$

so these probabilities are summable. Borel–Cantelli proves the desired estimate at the block endpoints.

If $n \in [N_j, N_{j+1})$, Lemma 2.2 gives

$$\int |F_{n,r_n} - F_{N_j, r_{N_j}}| \, \mathrm{d}\mu \ll N_j^{-1/8}.$$

The Lipschitz bound on h_j therefore gives an additional error $O_{C_0}(N_j^{\eta - 1/8}) = o(N_j^{-\beta})$, completing the proof. □

Lemma 4.3. *Fix one of the three deterministic choices $r_n \equiv 1$, $r_n = \rho_n$, or $r_n = \tau_n$. Almost surely, for all sufficiently large n,*

$$\int |\log |F_{n,r_n}(\theta)||^2 \, \mu(\mathrm{d}\theta) \leqslant (\log n)^{20}.$$

Proof. Apply Theorem 4.1 to F_{n,r_n}. For large n put $q_n := \lfloor \log n \rfloor$. Jensen's inequality gives

$$\mathbb{E}\left(\int |\log |F_{n,r_n}||^2 \, \mathrm{d}\mu \right)^{q_n} \leqslant \mathbb{E} \int |\log |F_{n,r_n}||^{2q_n} \, \mathrm{d}\mu \leqslant (Cq_n)^{12q_n}.$$

Hence

$$\mathbb{P}\left(\int |\log |F_{n,r_n}||^2 \, \mathrm{d}\mu > (\log n)^{20} \right) \leqslant \left(\frac{C^{12}}{(\log n)^8} \right)^{q_n},$$

which is summable in n. Borel–Cantelli proves the claim. □

Lemma 4.4. *Fix one of the three deterministic choices $r_n \equiv 1$, $r_n = \rho_n$, or $r_n = \tau_n$. Almost surely, for all sufficiently large j and all $n \in [N_j, N_{j+1})$,*

$$\mu\{\theta : |F_{n,r_n}(\theta)| \leqslant 2N_j^{-\eta}\} \ll N_j^{-2\eta}.$$

Proof. Choose a smooth radial function $\chi_M : \mathbb{C} \to [0,1]$ such that

$$\mathbf{1}_{|z|\leqslant 2M^{-1}} \leqslant \chi_M(z) \leqslant \mathbf{1}_{|z|\leqslant 4M^{-1}}, \qquad \|\chi_M\|_{C^3} \ll M^3, \qquad \mathrm{Lip}(\chi_M) \ll M.$$

Put $M_j = N_j^{\eta}$ and

$$Y_j := \int \chi_{M_j}(F_{N_j, r_{N_j}}(\theta))\mu(\,\mathrm{d}\theta).$$

By Proposition 3.3,

$$\mathbb{E}Y_j \leqslant \mathbb{E}\chi_{M_j}(G) + O(N_j^{3\eta-1/2}) \ll M_j^{-2},$$

because the complex Gaussian has bounded density near zero and $3\eta - 1/2 < -2\eta$. Also

$$\mathrm{Var}(Y_j) \ll N_j^{6\eta-1/2}.$$

Choosing C larger than the implicit constant in the expectation bound, Chebyshev's inequality gives

$$\mathbb{P}(Y_j > CM_j^{-2}) \ll M_j^4 N_j^{6\eta-1/2} = N_j^{10\eta-1/2}.$$

Since $N_j \asymp j^4$ and $4(10\eta - 1/2) < -1$, these probabilities are summable. Thus $Y_j \ll M_j^{-2}$ almost surely for all sufficiently large j.

For $n \in [N_j, N_{j+1})$, Lemma 2.2 gives

$$\int |F_{n,r_n} - F_{N_j, r_{N_j}}|\,\mathrm{d}\mu \ll N_j^{-1/8}.$$

Hence

$$\left|\int \chi_{M_j}(F_{n,r_n})\,\mathrm{d}\mu - Y_j\right| \ll M_j N_j^{-1/8} = N_j^{\eta-1/8} = o(M_j^{-2}).$$

Therefore,

$$\int \chi_{M_j}(F_{n,r_n})\,\mathrm{d}\mu \ll M_j^{-2}$$

for all large j and all $n \in [N_j, N_{j+1})$. Since $\chi_{M_j} = 1$ on $\{|z| \leqslant 2M_j^{-1}\}$, the desired estimate follows. □

Proposition 4.5. *Fix one of the three deterministic choices $r_n \equiv 1$, $r_n = \rho_n$, or $r_n = \tau_n$. Then almost surely*

$$L_n(r_n) = \mathbb{E}\log|G| + O_\omega(n^{-\beta}).$$

Proof. Let $n \in [N_j, N_{j+1})$ and put $M = N_j^{\eta}$. Let $\Lambda_M : \mathbb{C} \to \mathbb{R}$ be a smooth radial truncation of $\log|z|$ such that

$$\Lambda_M(z) = \log|z| \quad (2M^{-1} \leqslant |z| \leqslant M),$$

$$\|\Lambda_M\|_{C^3} \ll M^3, \qquad \mathrm{Lip}(\Lambda_M) \ll M, \qquad \|\Lambda_M\|_\infty \ll \log M.$$

Such a truncation is obtained by smoothing the radial cutoffs at scales M^{-1} and M; the derivative bounds follow from the derivatives of $\log r$ on $r \geqslant M^{-1}$. By Lemma 4.2, almost surely,

$$\int \Lambda_M(F_{n,r_n})\,\mathrm{d}\mu = \mathbb{E}\Lambda_M(G) + O(N_j^{-\beta})$$

for all sufficiently large j and all $n \in [N_j, N_{j+1})$. Since the complex Gaussian density is bounded near zero and decays exponentially at infinity,

$$\mathbb{E}\Lambda_M(G) = \mathbb{E}\log|G| + O(M^{-2}\log M).$$

As $2\eta > \beta$, this error is $O(N_j^{-\beta})$.

It remains to remove the truncation. Let

$$E_{n,M} := \{\theta : \Lambda_M(F_{n,r_n}(\theta)) \neq \log |F_{n,r_n}(\theta)|\}.$$

Then

$$E_{n,M} \subseteq \{|F_{n,r_n}| < 2M^{-1}\} \cup \{|F_{n,r_n}| > M\}.$$

The first set has measure $O(M^{-2})$ by Lemma 4.4, and the second has measure at most M^{-2} by Markov's inequality and the deterministic identity

$$\int |F_{n,r_n}(\theta)|^2 \mu(\,\mathrm{d}\theta) = 1.$$

Thus $\mu(E_{n,M}) \ll M^{-2}$. By Cauchy–Schwarz and Lemma 4.3,

$$\int_{E_{n,M}} |\log |F_{n,r_n}|| \,\mathrm{d}\mu \ll_\omega (\log n)^{10} M^{-1}.$$

Also,

$$\int_{E_{n,M}} |\Lambda_M(F_{n,r_n})| \,\mathrm{d}\mu \ll (\log M) M^{-2}.$$

Consequently,

$$\left| L_n(r_n) - \int \Lambda_M(F_{n,r_n}) \,\mathrm{d}\mu \right| \ll_\omega (\log n)^{10} M^{-1} + (\log M) M^{-2} = o(N_j^{-\beta}),$$

because $\eta > \beta$. Since $N_j \asymp n$ on the block, the proposition follows. □

5. Jensen's formula and the proof of the theorem

We use the following integrated form of Jensen's formula; compare, for example, Conway [7, Chapter XI, Section 1].

Lemma 5.1 (Integrated Jensen formula)**.** *Let H be a nonzero polynomial, and let $N_H(t)$ denote the number of zeros of H in $\{|z| \leqslant t\}$, counted with multiplicity. If $0 < \rho < s$, then the logarithms below are integrable on the two circles and*

$$\int_\rho^s \frac{N_H(t)}{t} \,\mathrm{d}t = \int \log |H(se^{\mathrm{i}\theta})| \,\mathrm{d}\mu - \int \log |H(\rho e^{\mathrm{i}\theta})| \,\mathrm{d}\mu.$$

Proof. Write $H(z) = c \prod_\alpha (z - \alpha)$, with zeros repeated according to multiplicity. The constant cancels. For $\alpha \neq 0$, the elementary Jensen identity says that

$$\int \log |re^{\mathrm{i}\theta} - \alpha| \,\mathrm{d}\mu = \max\{\log r, \log |\alpha|\} \qquad (r > 0).$$

For $\alpha = 0$, the same integral is simply $\log r$. Therefore, for every zero α,

$$\int \log |se^{\mathrm{i}\theta} - \alpha| \,\mathrm{d}\mu - \int \log |\rho e^{\mathrm{i}\theta} - \alpha| \,\mathrm{d}\mu = \int_\rho^s \mathbf{1}_{|\alpha| \leqslant t} \frac{\mathrm{d}t}{t}.$$

Summing over the zeros proves the formula. Boundary zeros cause no difficulty: the logarithmic singularities are integrable, and changes of $N_H(t)$ at a single value of t do not affect the integral. □

Lemma 5.2. *As $n \to \infty$,*

$$\frac{\log \sigma_n(1) - \log \sigma_n(\rho_n)}{-\log \rho_n} = \frac{n}{2} + O(n^{399/400}), \tag{5.1}$$

$$\frac{\log \sigma_n(\tau_n) - \log \sigma_n(1)}{\log \tau_n} = \frac{n}{2} + O(n^{399/400}). \tag{5.2}$$

Proof. Let $S_n(r) := \sigma_n(r)^2 = \sum_{k=0}^n r^{2k}$. Uniformly for $r \in [\rho_n, \tau_n]$,

$$S_n(r) \asymp n, \qquad S_n'(r) = \sum_{k=1}^n 2kr^{2k-1} = O(n^2), \qquad S_n''(r) = O(n^3).$$

Since $\log \sigma_n(r) = \frac{1}{2} \log S_n(r)$, this implies

$$\frac{\mathrm{d}^2}{\mathrm{d}r^2} \log \sigma_n(r) = \frac{1}{2} \left(\frac{S_n''(r)}{S_n(r)} - \left(\frac{S_n'(r)}{S_n(r)} \right)^2 \right) = O(n^2)$$

uniformly in the same range. Also

$$\frac{\mathrm{d}}{\mathrm{d}r} \log \sigma_n(r) \Big|_{r=1} = \frac{\sum_{k=0}^n k}{n+1} = \frac{n}{2}.$$

Put $\delta_n := 1 - \rho_n = n^{-401/400}$. Taylor's formula gives

$$\log \sigma_n(1) - \log \sigma_n(\rho_n) = \frac{n}{2} \delta_n + O(n^2 \delta_n^2).$$

Since $-\log \rho_n = \delta_n + O(\delta_n^2)$, division by $-\log \rho_n$ gives

$$\frac{\log \sigma_n(1) - \log \sigma_n(\rho_n)}{-\log \rho_n} = \frac{n}{2} + O(n\delta_n) + O(n^2 \delta_n) = \frac{n}{2} + O(n^{399/400}).$$

For the second estimate, use $\tau_n = \rho_n^{-1}$. Then

$$\sigma_n(\tau_n)^2 = \sum_{k=0}^n \tau_n^{2k} = \tau_n^{2n} \sum_{k=0}^n \rho_n^{2k} = \tau_n^{2n} \sigma_n(\rho_n)^2.$$

Thus

$$\log \sigma_n(\tau_n) = n \log \tau_n + \log \sigma_n(\rho_n),$$

and hence

$$\frac{\log \sigma_n(\tau_n) - \log \sigma_n(1)}{\log \tau_n} = n - \frac{\log \sigma_n(1) - \log \sigma_n(\rho_n)}{-\log \rho_n} = \frac{n}{2} + O(n^{399/400}).$$

□

Proof of Theorem 1.1. Work on the probability-one event on which Proposition 4.5 holds for the three choices $r_n = 1, \rho_n, \tau_n$.

First apply Lemma 5.1 to P_n with $(\rho, s) = (\rho_n, 1)$. Since $N_{P_n}(t) \leqslant R_n$ for every $t < 1$,

$$R_n(-\log \rho_n) \geqslant \int \log |P_n(e^{\mathrm{i}\theta})| \, \mathrm{d}\mu - \int \log |P_n(\rho_n e^{\mathrm{i}\theta})| \, \mathrm{d}\mu.$$

Using $P_n(re^{\mathrm{i}\theta}) = \sigma_n(r) F_{n,r}(\theta)$, this gives

$$R_n \geqslant \frac{\log \sigma_n(1) - \log \sigma_n(\rho_n)}{-\log \rho_n} + \frac{L_n(1) - L_n(\rho_n)}{-\log \rho_n}.$$

Because $-\log \rho_n = \log \tau_n \asymp n^{-401/400}$ and $\beta = 1/200$, the $O_\omega(n^{-\beta})$ errors from Proposition 4.5 become $O_\omega(n^{399/400})$ after division by the logarithmic radii. Hence Lemma 5.2 and Proposition 4.5 give

$$R_n \geqslant \frac{n}{2} - O_\omega(n^{399/400}).$$

For the upper bound define the reciprocal polynomial

$$Q_n(z) := z^n P_n(1/z) = \sum_{k=0}^n \varepsilon_{n-k} z^k.$$

Since $\varepsilon_n \neq 0$, the polynomial P_n has exact degree n, so it has exactly n complex zeros counted with multiplicity, and $Q_n(0) = \varepsilon_n \neq 0$. Also $P_n(0) = \varepsilon_0 \neq 0$, so every zero of P_n is nonzero. Thus the reciprocal map gives a multiplicity-preserving bijection between zeros of Q_n in $|z| < 1$ and zeros of P_n in $|z| > 1$. Because R_n counts the zeros of P_n in $|z| \leqslant 1$, we have

$$N_{Q_n}(t) \leqslant n - R_n \qquad (0 < t < 1).$$

Applying Lemma 5.1 to Q_n with $(\rho, s) = (\rho_n, 1)$ gives

$$(n - R_n)(-\log \rho_n) \geqslant \int \log |Q_n(e^{\mathrm{i}\theta})| \,\mathrm{d}\mu - \int \log |Q_n(\rho_n e^{\mathrm{i}\theta})| \,\mathrm{d}\mu.$$

Now

$$Q_n(e^{\mathrm{i}\theta}) = e^{\mathrm{i}n\theta} P_n(e^{-\mathrm{i}\theta}), \qquad Q_n(\rho_n e^{\mathrm{i}\theta}) = \rho_n^n e^{\mathrm{i}n\theta} P_n(\tau_n e^{-\mathrm{i}\theta}).$$

Using the invariance of μ under $\theta \mapsto -\theta$, we obtain

$$n - R_n \geqslant n + \frac{\log \sigma_n(1) - \log \sigma_n(\tau_n)}{\log \tau_n} + \frac{L_n(1) - L_n(\tau_n)}{\log \tau_n}.$$

Since

$$n + \frac{\log \sigma_n(1) - \log \sigma_n(\tau_n)}{\log \tau_n} = n - \frac{\log \sigma_n(\tau_n) - \log \sigma_n(1)}{\log \tau_n},$$

Lemma 5.2 and Proposition 4.5 give

$$n - R_n \geqslant \frac{n}{2} - O_\omega(n^{399/400}).$$

Equivalently,

$$R_n \leqslant \frac{n}{2} + O_\omega(n^{399/400}).$$

Combining this with the lower bound proves the theorem. □

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