Almost surely half of the zeros of a random Littlewood polynomial lie in the unit disk

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Abstract

Let

$$P_n(z) = \sum_{k=0}^{n} \varepsilon_k z^k, \qquad \mathbb{P}(\varepsilon_k = 1) = \mathbb{P}(\varepsilon_k = -1) = \tfrac{1}{2},$$

with the coefficients independent. Write R_n for the number of zeros of P_n in $\{z \in \mathbb{C} : |z| \le 1\}$, counted with multiplicity. We prove the strong-law statement asked by Erdős:

$$\frac{R_n}{n/2} \longrightarrow 1 \qquad \text{almost surely.}$$

In fact our argument gives the quantitative estimate

$$R_n = \frac{n}{2} + O_\omega\left(n^{149/150}\right).$$

The proof combines three ingredients. First, Jensen's formula reduces the root count to logarithmic integrals at radii 1 and $1 - n^{-1-\alpha}$, with an elementary reciprocal-polynomial squeeze handling the closed disk exactly. Second, a logarithmic-integrability theorem of Nazarov, Nishry and Sodin controls the singularity of $\log|P_n|$ near zero. Third, a quantitative central-limit argument for angle averages of the normalized polynomial, in the spirit of Angst and Poly, yields almost-sure convergence of suitable smooth truncations of the logarithmic integral. This is precisely the route suggested in the discussion of Erdős problem #522.

1 Introduction

The random Littlewood polynomial

$$P_n(z) = \sum_{k=0}^{n} \varepsilon_k z^k, \qquad \varepsilon_k \in \{-1, 1\} \text{ i.i.d.},$$

has its zeros concentrated near the unit circle. Erdős asked whether, almost surely, asymptotically half of the zeros lie in the unit disk; see [4, p. 252] (the original source is [3]). On the Erdős problems site this is problem #522, and Yakir's theorem is recorded there as the known in-probability result [5]. Yakir proved that

$$\mathbb{P}\left(\left|R_n - \frac{n}{2}\right| \ge n^{9/10}\right) \to 0,$$

thereby solving the weaker problem from Hayman's book [8]. The comments on the discussion thread point to the natural route to the strong law: use Jensen at two nearby radii, import logarithmic integrability from Nazarov–Nishry–Sodin, and add an almost-sure quantitative central-limit theorem of Salem–Zygmund type in the spirit of Angst–Poly [6, 1, 7]. The purpose of this note is to write that argument out.

Our main result is the following.

Theorem 1. *With probability one,*

$$R_n = \frac{n}{2} + o(n).$$

In fact, with probability one,

$$R_n = \frac{n}{2} + O\Big(n^{149/150}\Big).$$

Consequently,

$$\frac{R_n}{n/2} \longrightarrow 1 \qquad \textit{almost surely,}$$

and also

$$\mathbb{E}R_n = \frac{n}{2} + o(n).$$

We count zeros with multiplicity throughout. Let

$$\mu(\mathrm{d}\theta) = \frac{\mathrm{d}\theta}{2\pi}$$

be normalized Lebesgue measure on $[-\pi, \pi]$.

Choice of exponents

To keep the bookkeeping clean we fix once and for all

$$\alpha := \frac{1}{100}, \qquad \eta := \frac{1}{40}, \qquad \beta := \frac{1}{60}, \qquad N_j := j^6.$$

None of these choices is optimal. The proof only needs $\alpha < \beta < \eta < 1/12$ and a mild summability inequality; the explicit exponents above were chosen simply to make all estimates transparent.

2 Deterministic preliminaries

Define

$$\rho_n := 1 - n^{-1-\alpha}, \qquad \sigma_n(r)^2 := \sum_{k=0}^{n} r^{2k}, \qquad F_{n,r}(\theta) := \frac{P_n(re^{\mathrm{i}\theta})}{\sigma_n(r)}.$$

We also set

$$a_{n,k}(r) := \begin{cases} \dfrac{r^k}{\sigma_n(r)}, & 0 \le k \le n, \\ 0, & k > n, \end{cases} \qquad B_{n,r}(t) := \sum_{k=0}^{n} a_{n,k}(r)^2 e^{\mathrm{i}kt}.$$

Thus

$$F_{n,r}(\theta) = \sum_{k=0}^{n} a_{n,k}(r)\, \varepsilon_k e^{\mathrm{i}k\theta}.$$

We will use the same notation for the reciprocal polynomial

$$Q_n(z) := z^n P_n(1/z) = \sum_{k=0}^{n} \varepsilon_{n-k} z^k,$$

which has the same one-dimensional law as P_n for each fixed n.

Lemma 2. *Uniformly for $r \in [\rho_n, 1]$ one has*

$$\sigma_n(r)^2 = n + 1 + O\Big(n^{1-\alpha}\Big), \qquad \sigma_n(r) \asymp n^{1/2}, \qquad \max_{0 \le k \le n} a_{n,k}(r) \ll n^{-1/2},$$

and

$$\sum_{k=0}^{n} a_{n,k}(r)^3 \ll n^{-1/2}.$$

The implied constants depend only on α.

Proof. Since $r \in [\rho_n, 1]$ we have $|1 - r| \ll n^{-1-\alpha}$. Hence uniformly in $0 \le k \le n$,

$$r^{2k} = 1 + O\Big(kn^{-1-\alpha}\Big).$$

Summing over k gives

$$\sigma_n(r)^2 = \sum_{k=0}^{n} r^{2k} = (n+1) + O\left(n^{-1-\alpha} \sum_{k=0}^{n} k\right) = (n+1) + O\Big(n^{1-\alpha}\Big),$$

whence $\sigma_n(r) \asymp n^{1/2}$. Since $r \le 1$,

$$\max_k a_{n,k}(r) \le \sigma_n(r)^{-1} \ll n^{-1/2}.$$

Finally,

$$\sum_{k=0}^{n} a_{n,k}(r)^3 \le \Big(\max_k a_{n,k}(r)\Big) \sum_{k=0}^{n} a_{n,k}(r)^2 = \max_k a_{n,k}(r) \ll n^{-1/2}.$$

□

Lemma 3. *Uniformly for $r \in [\rho_n, 1]$ and $t \in \mathbb{R}$,*

$$B_{n,r}(t) = \frac{1 - (r^2 e^{\mathrm{i}t})^{n+1}}{\sigma_n(r)^2 (1 - r^2 e^{\mathrm{i}t})}.$$

In particular, if $\operatorname{dist}(t, 2\pi\mathbb{Z}) \ge n^{-1/2}$, then

$$|B_{n,r}(t)| \ll n^{-1/2}.$$

Proof. The displayed identity is just the geometric-series formula. Since $\sigma_n(r)^2 \asymp n$ by theorem 2,

$$|B_{n,r}(t)| \le \frac{2}{\sigma_n(r)^2 \, |1 - r^2 e^{\mathrm{i}t}|} \ll \frac{1}{n \, |1 - r^2 e^{\mathrm{i}t}|}.$$

For $\operatorname{dist}(t, 2\pi\mathbb{Z}) \le \pi$ and $r \in [\rho_n, 1]$, one has

$$\left|1 - r^2 e^{\mathrm{i}t}\right| \gg |t|,$$

because $1 - r^2 \ll n^{-1-\alpha} \ll n^{-1/2}$ while $|t| \ge n^{-1/2}$. Therefore

$$|B_{n,r}(t)| \ll \frac{1}{n\,|t|} \ll n^{-1/2}.$$

□

Lemma 4. *One has*

$$\frac{\log\sigma_n(1)-\log\sigma_n(\rho_n)}{-\log\rho_n}=\frac{n}{2}+O\Big(n^{1-\alpha}\Big).$$

Proof. Write $\sigma=\sigma_n$. Since

$$\sigma(r)^2=\sum_{k=0}^{n}r^{2k},$$

a direct differentiation gives

$$\frac{\partial}{\partial r}\log\sigma(r)=\frac{\sum_{k=0}^{n}kr^{2k-1}}{\sum_{k=0}^{n}r^{2k}},\qquad \frac{\partial^2}{\partial r^2}\log\sigma(r)=O(n^2)\quad(r\in[\rho_n,1]),$$

because the first numerator is $O(n^2)$, the second derivative numerator is $O(n^3)$, and the denominator is $\asymp n$ by theorem 2. At $r=1$,

$$\frac{\partial}{\partial r}\log\sigma(r)\Big|_{r=1}=\frac{\sum_{k=0}^{n}k}{n+1}=\frac{n}{2}.$$

Therefore Taylor's theorem gives

$$\log\sigma(1)-\log\sigma(\rho_n)=\frac{n}{2}(1-\rho_n)+O\Big(n^2(1-\rho_n)^2\Big).$$

Since $1-\rho_n=n^{-1-\alpha}$ and $-\log\rho_n=(1-\rho_n)+O((1-\rho_n)^2)$,

$$\frac{\log\sigma_n(1)-\log\sigma_n(\rho_n)}{-\log\rho_n}=\frac{n}{2}+O\Big(n^2(1-\rho_n)\Big)=\frac{n}{2}+O\Big(n^{1-\alpha}\Big).\qquad\square$$

Lemma 5. *Let (r_n) be either the constant sequence $r_n\equiv 1$ or the sequence $r_n=\rho_n$. If $N_j\le n\le m<N_{j+1}$, then*

$$\sum_{k\ge 0}|a_{m,k}(r_m)-a_{n,k}(r_n)|^2\ll N_j^{-1/6}.$$

Hence

$$\left(\int_{-\pi}^{\pi}|F_{m,r_m}(\theta)-F_{n,r_n}(\theta)|^2\,\mu(\mathrm{d}\theta)\right)^{1/2}\ll N_j^{-1/12}.$$

Proof. By Parseval,

$$\int|F_{m,r_m}-F_{n,r_n}|^2\,\mathrm{d}\mu=\sum_{k\ge 0}|a_{m,k}(r_m)-a_{n,k}(r_n)|^2.$$

So it suffices to prove the first claim. We may assume $n\le m$.

For the tail $k>n$, using theorem 2 and $m-n\ll N_j^{5/6}$, we get

$$\sum_{k=n+1}^{m}a_{m,k}(r_m)^2\le\frac{m-n}{\sigma_m(r_m)^2}\ll\frac{N_j^{5/6}}{N_j}=N_j^{-1/6}.$$

It remains to estimate the common range $0\le k\le n$. First suppose $r_n=r_m=1$. Then

$$a_{m,k}(1)=\frac{1}{\sqrt{m+1}},\qquad a_{n,k}(1)=\frac{1}{\sqrt{n+1}}\qquad(0\le k\le n),$$

so

$$\sum_{k=0}^{n} |a_{m,k}(1) - a_{n,k}(1)|^2 = (n+1)\left|\frac{1}{\sqrt{m+1}} - \frac{1}{\sqrt{n+1}}\right|^2 \ll \frac{(m-n)^2}{n^2} \ll N_j^{-1/3}.$$

This is more than enough.

Now suppose $r_n = \rho_n$ and $r_m = \rho_m$. Since

$$\rho_t = 1 - t^{-1-\alpha},$$

the mean value theorem implies

$$|\rho_m - \rho_n| \ll |m-n|\, n^{-2-\alpha} \ll N_j^{-7/6-\alpha}.$$

Hence for $0 \le k \le n$,

$$\left|\rho_m^k - \rho_n^k\right| \le k\,|\rho_m - \rho_n| \ll n \cdot N_j^{-7/6-\alpha} \ll N_j^{-1/6-\alpha}.$$

Also,

$$\sigma_t(\rho_t)^2 = \sum_{k=0}^{t} \rho_t^{2k} = t + 1 + O(t^{1-\alpha})$$

by theorem 2, so

$$\left|\sigma_m(\rho_m)^2 - \sigma_n(\rho_n)^2\right| \ll (m-n) + n \cdot N_j^{-1/6-\alpha} \ll N_j^{5/6}.$$

Since $\sigma_n(\rho_n), \sigma_m(\rho_m) \asymp N_j^{1/2}$, this gives

$$|\sigma_m(\rho_m) - \sigma_n(\rho_n)| \ll N_j^{1/3}, \qquad \left|\sigma_m(\rho_m)^{-1} - \sigma_n(\rho_n)^{-1}\right| \ll N_j^{-2/3}.$$

Therefore, for $0 \le k \le n$,

$$\left|\frac{\rho_m^k}{\sigma_m(\rho_m)} - \frac{\rho_n^k}{\sigma_n(\rho_n)}\right| \le \frac{\left|\rho_m^k - \rho_n^k\right|}{\sigma_m(\rho_m)} + \rho_n^k \left|\sigma_m(\rho_m)^{-1} - \sigma_n(\rho_n)^{-1}\right| \ll N_j^{-2/3-\alpha} + N_j^{-2/3}.$$

Squaring and summing over $0 \le k \le n$ yields

$$\sum_{k=0}^{n} |a_{m,k}(\rho_m) - a_{n,k}(\rho_n)|^2 \ll N_j \cdot N_j^{-4/3} = N_j^{-1/3}.$$

Combining this with the tail estimate proves the lemma. □

3 A smooth central-limit estimate for angle averages

Let G denote the standard complex Gaussian, i.e.

$$\frac{1}{\pi} e^{-|z|^2}\, \mathrm{d}m(z)$$

on $\mathbb{C} \cong \mathbb{R}^2$. Equivalently, $\Re G$ and $\Im G$ are independent $\mathcal{N}(0, 1/2)$ random variables. In particular,

$$|G|^2 \sim \mathrm{Exp}(1), \qquad \mathbb{E} \log |G| = -\frac{\gamma_{\mathrm{E}}}{2}.$$

For a bounded C^3 function $h : \mathbb{C} \to \mathbb{R}$ we write

$$\|h\|_{C^3} := \sum_{0 \le a+b \le 3} \left\|\partial_x^a \partial_y^b h\right\|_{L^\infty(\mathbb{R}^2)}.$$

We use the following standard smooth Berry–Esseen estimate in fixed dimension; see, for example, Bhattacharya–Rao [2, Ch. 11].

Smooth Berry–Esseen theorem. Let $d \geq 1$ be fixed. If $X_1, \dots, X_M$ are independent centered random vectors in $\mathbb{R}^d$, $S = \sum_{j=1}^M X_j$, and Z is the centered Gaussian vector with the same covariance as S, then for every bounded C^3 function $h : \mathbb{R}^d \to \mathbb{R}$,

$$|\mathbb{E}h(S) - \mathbb{E}h(Z)| \leq C_d \, \|h\|_{C^3} \sum_{j=1}^{M} \mathbb{E}\,|X_j|^3 .$$

We also use the elementary fact that on any compact subset of the cone of positive-definite matrices, the matrix square-root map is Lipschitz. Therefore if Z_Σ is a centered Gaussian in $\mathbb{R}^d$ with covariance Σ, then for every bounded Lipschitz function h,

$$|\mathbb{E}h(Z_\Sigma) - \mathbb{E}h(Z_{\Sigma_0})| \ll_h \|\Sigma - \Sigma_0\| \tag{1}$$

whenever Σ stays in a fixed compact neighborhood of Σ_0.

For $h : \mathbb{C} \to \mathbb{R}$ define

$$J_{n,r}(h) := \int h(F_{n,r}(\theta)) \, \mu(\mathrm{d}\theta).$$

Proposition 6. *There is an absolute constant $C > 0$ such that the following holds. Let $h : \mathbb{C} \to \mathbb{R}$ be bounded and C^3, and let $r \in [\rho_n, 1]$. Then*

$$|\mathbb{E}J_{n,r}(h) - \mathbb{E}h(G)| \leq C(\|h\|_{C^3} + \|h\|_{L^\infty})n^{-1/2},$$

and

$$\mathrm{Var}(J_{n,r}(h)) \leq C(\|h\|_{C^3} + \|h\|_{L^\infty})^2 n^{-1/2}.$$

The constants are uniform in $r \in [\rho_n, 1]$.

Proof. We begin with the one-point estimate. Write

$$X_{n,r}(\theta) := (\Re F_{n,r}(\theta), \Im F_{n,r}(\theta)) \in \mathbb{R}^2.$$

Then

$$X_{n,r}(\theta) = \sum_{k=0}^{n} \varepsilon_k a_{n,k}(r)(\cos k\theta, \sin k\theta),$$

a sum of independent centered vectors. By theorem 2,

$$\sum_{k=0}^{n} \mathbb{E}\,|\varepsilon_k a_{n,k}(r)(\cos k\theta, \sin k\theta)|^3 \ll \sum_{k=0}^{n} a_{n,k}(r)^3 \ll n^{-1/2}.$$

Let $V_{n,r}(\theta)$ be the covariance matrix of $X_{n,r}(\theta)$. A short computation gives

$$V_{n,r}(\theta) = \frac{1}{2}I_2 + \frac{1}{2}\begin{pmatrix} \Re B_{n,r}(2\theta) & \Im B_{n,r}(2\theta) \\ \Im B_{n,r}(2\theta) & -\Re B_{n,r}(2\theta) \end{pmatrix}.$$

Hence on the good set

$$\mathcal{G}_n^{(1)} := \left\{\theta \in [-\pi, \pi] : \mathrm{dist}(2\theta, 2\pi\mathbb{Z}) \geq n^{-1/2}\right\},$$

theorem 3 yields

$$\left\|V_{n,r}(\theta) - \tfrac{1}{2}I_2\right\| \ll n^{-1/2}.$$

By the smooth Berry–Esseen theorem and (1),

$$|\mathbb{E}h(X_{n,r}(\theta)) - \mathbb{E}h(G)| \ll (\|h\|_{C^3} + \|h\|_{L^\infty})n^{-1/2} \qquad (\theta \in \mathcal{G}_n^{(1)}).$$

The bad set has measure $\mu([-\pi,\pi] \setminus \mathcal{G}_n^{(1)}) \ll n^{-1/2}$, so integrating over θ gives the first estimate.

For the second estimate we write

$$\operatorname{Var}(J_{n,r}(h)) = \iint \operatorname{Cov}\big(h(F_{n,r}(\theta)), h(F_{n,r}(\phi))\big)\, \mu(\mathrm{d}\theta)\mu(\mathrm{d}\phi).$$

Set

$$Y_{n,r}(\theta,\phi) := \big(\Re F_{n,r}(\theta), \Im F_{n,r}(\theta), \Re F_{n,r}(\phi), \Im F_{n,r}(\phi)\big) \in \mathbb{R}^4.$$

Again this is a sum of independent centered vectors,

$$Y_{n,r}(\theta,\phi) = \sum_{k=0}^{n} \varepsilon_k a_{n,k}(r)(\cos k\theta, \sin k\theta, \cos k\phi, \sin k\phi),$$

whose third moments sum to $O(n^{-1/2})$ uniformly in θ, ϕ. Let $\Sigma_{n,r}(\theta,\phi)$ be the covariance matrix of $Y_{n,r}(\theta,\phi)$. The diagonal 2×2 blocks are the matrices above, while the off-diagonal blocks are linear combinations of $B_{n,r}(\theta-\phi)$ and $B_{n,r}(\theta+\phi)$, because

$$\mathbb{E}\big(F_{n,r}(\theta)\overline{F_{n,r}(\phi)}\big) = B_{n,r}(\theta-\phi), \qquad \mathbb{E}\big(F_{n,r}(\theta)F_{n,r}(\phi)\big) = B_{n,r}(\theta+\phi).$$

Therefore on the good set

$$\mathcal{G}_n^{(2)} := \big\{(\theta,\phi) \in [-\pi,\pi]^2 : \operatorname{dist}(2\theta, 2\pi\mathbb{Z}), \operatorname{dist}(2\phi, 2\pi\mathbb{Z}), \operatorname{dist}(\theta-\phi, 2\pi\mathbb{Z}), \operatorname{dist}(\theta+\phi, 2\pi\mathbb{Z}) \geq n^{-1/2}\big\}$$

we have

$$\left\|\Sigma_{n,r}(\theta,\phi) - \tfrac{1}{2}I_4\right\| \ll n^{-1/2}$$

by theorem 3. The complement of $\mathcal{G}_n^{(2)}$ has $\mu \times \mu$ measure $O(n^{-1/2})$.

Apply the smooth Berry–Esseen theorem in $\mathbb{R}^4$ to the function

$$H(z,w) := h(z)h(w), \qquad (z,w) \in \mathbb{C}^2 \cong \mathbb{R}^4.$$

Since $\|H\|_{C^3} \ll (\|h\|_{C^3} + \|h\|_{L^\infty})^2$, the good-pair estimate and (1) imply

$$\mathbb{E}H\big(F_{n,r}(\theta), F_{n,r}(\phi)\big) = \mathbb{E}H(G_1,G_2) + O\Big((\|h\|_{C^3} + \|h\|_{L^\infty})^2 n^{-1/2}\Big)$$

for $(\theta,\phi) \in \mathcal{G}_n^{(2)}$, where G_1, G_2 are independent standard complex Gaussians. On the bad set we simply use the trivial bound

$$|H(F_{n,r}(\theta), F_{n,r}(\phi))| \leq \|h\|_{L^\infty}^2 .$$

Hence

$$\mathbb{E}J_{n,r}(h)^2 = (\mathbb{E}h(G))^2 + O\Big((\|h\|_{C^3} + \|h\|_{L^\infty})^2 n^{-1/2}\Big).$$

Together with the first part of the proposition, this yields the stated variance bound. □

4 Almost sure control for blockwise smooth observables

Fix the block decomposition

$$B_j := [N_j, N_{j+1}) \cap \mathbb{N}, \qquad N_j = j^6.$$

The next proposition is the almost-sure central-limit input. It is a direct subsequence-plus-interpolation argument of the same flavor as Angst–Poly [1, Cor. 1].

Proposition 7. *Let (r_n) be either the constant sequence $r_n \equiv 1$ or the sequence $r_n = \rho_n$. For each $j \geq 1$ let $h_j : \mathbb{C} \to \mathbb{R}$ be bounded and C^3, and suppose that*

$$\|h_j\|_{L^\infty} + \|h_j\|_{C^3} \leq N_j^{3\eta}, \qquad \mathrm{Lip}(h_j) \leq N_j^{\eta}.$$

Then there exists an event of probability one such that on that event, for all sufficiently large j and all $n \in B_j$,

$$\left| \int h_j(F_{n,r_n}(\theta))\, \mu(\mathrm{d}\theta) - \mathbb{E}h_j(G) \right| \ll N_j^{-\beta}.$$

The same conclusion holds simultaneously for the four families obtained by taking P_n or Q_n in place of P_n, and $r_n \equiv 1$ or $r_n = \rho_n$.

Proof. We first work with P_n and a fixed choice of (r_n). Set

$$X_n := \int h_j(F_{n,r_n}(\theta))\, \mu(\mathrm{d}\theta) \qquad (n \in B_j).$$

For the left endpoint N_j, theorem 6 gives

$$\left| \mathbb{E}X_{N_j} - \mathbb{E}h_j(G) \right| \ll N_j^{3\eta - 1/2}, \qquad \mathrm{Var}(X_{N_j}) \ll N_j^{6\eta - 1/2}.$$

Since $3\eta - 1/2 < -\beta$, Chebyshev's inequality yields, for large j,

$$\mathbb{P}\Big(\left| X_{N_j} - \mathbb{E}h_j(G) \right| > 2N_j^{-\beta} \Big) \ll N_j^{2\beta}\, \mathrm{Var}(X_{N_j}) \ll N_j^{2\beta + 6\eta - 1/2}.$$

With our choices $\beta = 1/60$ and $\eta = 1/40$, the exponent is $-1/3$, so along $N_j = j^6$ the right-hand side is $\ll j^{-2}$. By Borel–Cantelli,

$$\left| X_{N_j} - \mathbb{E}h_j(G) \right| \ll N_j^{-\beta} \tag{2}$$

almost surely for all sufficiently large j.

Now let $n \in B_j$. By theorem 5,

$$\int \left| F_{n,r_n}(\theta) - F_{N_j, r_{N_j}}(\theta) \right| \mu(\mathrm{d}\theta) \leq \Big(\int \left| F_{n,r_n} - F_{N_j, r_{N_j}} \right|^2 \mathrm{d}\mu \Big)^{1/2} \ll N_j^{-1/12}.$$

Therefore, using the Lipschitz bound on h_j,

$$\left| X_n - X_{N_j} \right| \leq \mathrm{Lip}(h_j) \int \left| F_{n,r_n} - F_{N_j, r_{N_j}} \right| \mathrm{d}\mu \ll N_j^{\eta - 1/12}.$$

Since $\eta - 1/12 < -\beta$, this is $o(N_j^{-\beta})$. Combining with (2) proves the claim for P_n.

Exactly the same estimates apply to the reciprocal polynomial Q_n, because for each n it has the same coefficient law as P_n. Repeating the argument for the four families mentioned in the statement and intersecting the resulting probability-one events completes the proof. □

5 Logarithmic integrability and the logarithmic integral

We now control the singularity of the logarithm. The key input is the logarithmic-integrability theorem of Nazarov–Nishry–Sodin [7, Cor. 1.2]: if

$$f(\theta) = \sum_{k\geq 0} c_k \varepsilon_k e^{ik\theta} \qquad \text{with} \qquad \sum_k |c_k|^2 = 1,$$

then for every $p \geq 1$,

$$\mathbb{E}\int |\log|f(\theta)||^p \ \mu(\mathrm{d}\theta) \leq (Cp)^{6p} \tag{3}$$

for an absolute constant C.

Lemma 8. *Let (r_n) be either $r_n \equiv 1$ or $r_n = \rho_n$, and let H_n be either P_n or Q_n. Then there exists $A > 0$ such that, almost surely,*

$$\int \left|\log\left|H_n(r_n e^{\mathrm{i}\theta})/\sigma_n(r_n)\right|\right|^2 \mu(\mathrm{d}\theta) \leq (\log n)^A$$

for all sufficiently large n.

Proof. Apply (3) to the normalized Fourier series

$$\theta \mapsto \frac{H_n(r_n e^{\mathrm{i}\theta})}{\sigma_n(r_n)}.$$

For any integer $q \geq 1$, Jensen's inequality gives

$$\mathbb{E}\Big[\Big(\int \left|\log\left|H_n(r_n e^{\mathrm{i}\theta})/\sigma_n(r_n)\right|\right|^2 \mu(\mathrm{d}\theta)\Big)^q\Big] \leq \mathbb{E}\int \left|\log\left|H_n(r_n e^{\mathrm{i}\theta})/\sigma_n(r_n)\right|\right|^{2q} \mu(\mathrm{d}\theta) \leq (Cq)^{12q}.$$

Choose $q_n = \lfloor \log n \rfloor$ and a large integer $A > 13$. Then by Markov's inequality,

$$\mathbb{P}\Big(\int \left|\log\left|H_n(r_n e^{\mathrm{i}\theta})/\sigma_n(r_n)\right|\right|^2 \mathrm{d}\mu > (\log n)^A\Big) \leq \left(\frac{Cq_n^{12}}{(\log n)^A}\right)^{q_n}.$$

The right-hand side is summable in n, because $q_n \asymp \log n$ and $A > 12$. Another application of Borel–Cantelli finishes the proof. □

We now construct the truncations. For $M \geq 2$ let $\chi_M : \mathbb{C} \to [0,1]$ be a smooth radial function such that

$$\mathbf{1}_{|z|\leq M^{-1}} \leq \chi_M(z) \leq \mathbf{1}_{|z|\leq 2M^{-1}}, \qquad \|\chi_M\|_{C^3} \ll M^3, \qquad \mathrm{Lip}(\chi_M) \ll M.$$

Similarly, let $\Lambda_M : \mathbb{C} \to \mathbb{R}$ be a smooth radial function such that

$$\begin{aligned} &\Lambda_M(z) = \log|z|, && M^{-1} \leq |z| \leq M, \\ &\Lambda_M(z) = \text{constant}, && |z| \leq (2M)^{-1} \text{ or } |z| \geq 2M, \end{aligned}$$

and

$$|\Lambda_M(z)| \ll \log M, \qquad \|\Lambda_M\|_{C^3} \ll M^3, \qquad \mathrm{Lip}(\Lambda_M) \ll M.$$

Such functions are obtained by smoothing the obvious piecewise-radial truncations.

Set

$$M_j := N_j^\eta = j^{6\eta}.$$

Lemma 9. *Let (r_n) be either $r_n \equiv 1$ or $r_n = \rho_n$, and let H_n be either P_n or Q_n. With probability one, for all sufficiently large j and all $n \in B_j$,*

$$\int \chi_{M_j}\left(\frac{H_n(r_n e^{\mathrm{i}\theta})}{\sigma_n(r_n)}\right) \mu(\mathrm{d}\theta) \ll M_j^{-2}.$$

In particular,

$$\mu\Big(\Big\{\theta : \Big|H_n(r_n e^{\mathrm{i}\theta})/\sigma_n(r_n)\Big| \le M_j^{-1}\Big\}\Big) \ll M_j^{-2}.$$

Proof. Apply theorem 7 with $h_j = \chi_{M_j}$. Since $\left\|\chi_{M_j}\right\|_{L^\infty} \le 1$, $\left\|\chi_{M_j}\right\|_{C^3} \ll M_j^3 = N_j^{3\eta}$, and $\mathrm{Lip}(\chi_{M_j}) \ll M_j = N_j^{\eta}$, the hypotheses are satisfied. Moreover,

$$\mathbb{E}\chi_{M_j}(G) \ll \mathbb{P}(|G| \le 2M_j^{-1}) \ll M_j^{-2},$$

because the complex Gaussian has bounded density near the origin. We now repeat the proof of theorem 7, but with threshold $M_j^{-2} = N_j^{-2\eta}$ instead of $N_j^{-\beta}$. Along the subsequence N_j Chebyshev gives

$$\mathbb{P}\Big(\Big|J_{N_j, r_{N_j}}(\chi_{M_j}) - \mathbb{E}\chi_{M_j}(G)\Big| > N_j^{-2\eta}\Big) \ll N_j^{4\eta+6\eta-1/2} = N_j^{10\eta-1/2}.$$

Since $10\eta - 1/2 = -1/4$, this is summable along $N_j = j^6$. The interpolation error is still

$$\ll M_j N_j^{-1/12} = N_j^{\eta-1/12} = o(N_j^{-2\eta}),$$

because $\eta = 1/40 < 1/36$. Hence almost surely, for all large j and all $n \in B_j$,

$$\int \chi_{M_j}(H_n(r_n e^{\mathrm{i}\theta})/\sigma_n(r_n))\, \mathrm{d}\mu \ll M_j^{-2}.$$

The final claim follows from $\mathbf{1}_{|z| \le M_j^{-1}} \le \chi_{M_j}(z)$. □

Proposition 10. *Let H_n be either P_n or Q_n, and let (r_n) be either $r_n \equiv 1$ or $r_n = \rho_n$. Then with probability one,*

$$\int \log\left|\frac{H_n(r_n e^{\mathrm{i}\theta})}{\sigma_n(r_n)}\right| \mu(\mathrm{d}\theta) = -\frac{\gamma_{\mathrm{E}}}{2} + O\Big(n^{-\beta}\Big). \tag{4}$$

Proof. Fix one of the four families. For $n \in B_j$ write

$$L_n := \int \log\left|\frac{H_n(r_n e^{\mathrm{i}\theta})}{\sigma_n(r_n)}\right| \mu(\mathrm{d}\theta), \qquad T_n := \int \Lambda_{M_j}\left(\frac{H_n(r_n e^{\mathrm{i}\theta})}{\sigma_n(r_n)}\right) \mu(\mathrm{d}\theta).$$

By theorem 7 applied to $h_j = \Lambda_{M_j}$, and using

$$\left\|\Lambda_{M_j}\right\|_{L^\infty} \ll \log M_j \ll M_j^3, \qquad \left\|\Lambda_{M_j}\right\|_{C^3} \ll M_j^3 = N_j^{3\eta}, \qquad \mathrm{Lip}(\Lambda_{M_j}) \ll M_j = N_j^{\eta},$$

we get, almost surely,

$$T_n = \mathbb{E}\Lambda_{M_j}(G) + O\Big(N_j^{-\beta}\Big) \qquad (n \in B_j,\ j \text{ large}). \tag{5}$$

We next compare L_n and T_n. Since $\Lambda_{M_j}(z) = \log|z|$ whenever $M_j^{-1} \le |z| \le M_j$,

$$|L_n - T_n| \le C \int \Big|\log\Big|H_n(r_n e^{\mathrm{i}\theta})/\sigma_n(r_n)\Big|\Big|\, \mathbf{1}_{E_{n,j}}(\theta)\, \mu(\mathrm{d}\theta),$$

where

$$E_{n,j} := \Big\{\theta :\ \Big|H_n(r_n e^{\mathrm{i}\theta})/\sigma_n(r_n)\Big| < 2M_j^{-1} \text{ or } \Big|H_n(r_n e^{\mathrm{i}\theta})/\sigma_n(r_n)\Big| > M_j\Big\}.$$

By Cauchy–Schwarz,

$$|L_n - T_n| \le C\biggl(\int \Big|\log\Big|H_n(r_n e^{\mathrm{i}\theta})/\sigma_n(r_n)\Big|\Big|^2 \,\mathrm{d}\mu\biggr)^{1/2} \mu(E_{n,j})^{1/2}.$$

By theorem 8, the first factor is $\ll_\omega (\log n)^{A/2}$ for some A. By theorem 9, the small-value part of $E_{n,j}$ has measure $\ll M_j^{-2}$. The large-value part satisfies the deterministic Markov bound

$$\mu\Big(\Big\{\theta :\ \Big|H_n(r_n e^{\mathrm{i}\theta})/\sigma_n(r_n)\Big| > M_j\Big\}\Big) \le M_j^{-2} \int \Big|H_n(r_n e^{\mathrm{i}\theta})/\sigma_n(r_n)\Big|^2 \mu(\mathrm{d}\theta) = M_j^{-2},$$

by Parseval. Hence $\mu(E_{n,j}) \ll M_j^{-2}$ and therefore

$$|L_n - T_n| \ll_\omega (\log n)^{A/2} M_j^{-1}.$$

Since $M_j = N_j^\eta$ and $n \asymp N_j$ on B_j, this is $o(n^{-\beta})$ because $\eta > \beta$.

Finally, $\Lambda_{M_j}(G) \to \log|G|$ pointwise and in L^1, because $|\log|G||$ is integrable and the truncation only alters the events $\Big\{|G| < M_j^{-1}\Big\}$ and $\{|G| > M_j\}$, whose Gaussian probabilities are $O(M_j^{-2})$ and $O(e^{-M_j^2})$, respectively. Thus

$$\mathbb{E}\Lambda_{M_j}(G) = \mathbb{E}\log|G| + o(1) = -\frac{\gamma_{\mathrm{E}}}{2} + o(1).$$

Combining this with (5) and the estimate on $|L_n - T_n|$ yields (4). □

6 Proof of the main theorem

Let

$$R_n := \#\{z \in \mathbb{C} :\ P_n(z) = 0,\ |z| \le 1\}$$

(counted with multiplicity). Define also

$$J_n(H) := \frac{1}{-\log \rho_n} \int_{\rho_n}^1 \frac{N_H(t)}{t}\,\mathrm{d}t,$$

where $N_H(t)$ is the number of zeros of a polynomial H in $\{|z| \le t\}$, counted with multiplicity.

For every polynomial H with $H(0) \neq 0$, Jensen's formula gives

$$\int_{\rho_n}^1 \frac{N_H(t)}{t}\,\mathrm{d}t = \int \log\Big|H(e^{\mathrm{i}\theta})\Big|\,\mu(\mathrm{d}\theta) - \int \log\Big|H(\rho_n e^{\mathrm{i}\theta})\Big|\,\mu(\mathrm{d}\theta).$$

Applying this to P_n and writing

$$\ell_n^P(r) := \int \log\left|\frac{P_n(re^{\mathrm{i}\theta})}{\sigma_n(r)}\right| \mu(\mathrm{d}\theta),$$

we obtain

$$J_n(P_n) = \frac{\log\sigma_n(1) - \log\sigma_n(\rho_n)}{-\log\rho_n} + \frac{\ell_n^P(1) - \ell_n^P(\rho_n)}{-\log\rho_n}. \tag{6}$$

By theorem 4 and theorem 10,

$$J_n(P_n) = \frac{n}{2} + O_\omega\left(n^{1-\alpha}\right) + O_\omega\left(n^{1+\alpha-\beta}\right) = \frac{n}{2} + O_\omega\left(n^{149/150}\right),$$

because $1 + \alpha - \beta = 149/150 > 1 - \alpha = 99/100$.

Now $N_{P_n}(t) \leq R_n$ for every $t < 1$, hence $J_n(P_n) \leq R_n$. Therefore

$$R_n \geq \frac{n}{2} + O_\omega\left(n^{149/150}\right). \tag{7}$$

For the upper bound consider the reciprocal polynomial

$$Q_n(z) = z^n P_n(1/z).$$

If a is a zero of P_n with $|a| > 1$, then $1/a$ is a zero of Q_n with $|1/a| < 1$. Consequently, for every $t < 1$,

$$N_{Q_n}(t) \leq n - R_n.$$

Exactly as above,

$$J_n(Q_n) = \frac{n}{2} + O_\omega\left(n^{149/150}\right)$$

by theorems 4 and 10. Since $J_n(Q_n) \leq n - R_n$, we get

$$R_n \leq n - J_n(Q_n) = \frac{n}{2} + O_\omega\left(n^{149/150}\right). \tag{8}$$

Combining (7) and (8) proves

$$R_n = \frac{n}{2} + O_\omega\left(n^{149/150}\right).$$

This proves theorem 1. The expectation statement follows from bounded convergence applied to $R_n/n \to 1/2$ almost surely and $0 \leq R_n/n \leq 1$.

7 Remarks

Remark 11. *The proof above addresses the closed disk exactly. No separate estimate for roots on $|z| = 1$ is needed: the lower bound comes from Jensen applied to P_n, and the upper bound comes from Jensen applied to the reciprocal polynomial Q_n.*

Remark 12. *The exponent 149/150 is not meant to be sharp. It comes from one convenient choice of auxiliary exponents. Any power $n^{1-\delta}$ with some unspecified $\delta > 0$ is enough for the strong law, and one can certainly optimize the bookkeeping. The point of the argument is qualitative: the ingredients already in the literature suffice to upgrade Yakir's convergence in probability to almost sure convergence.*

Remark 13. *Nothing in the proof is special to the precise choice of the inner radius $\rho_n = 1 - n^{-1-\alpha}$ other than convenience. The discussion thread for Erdős problem #522 suggests that one may work with radii at distance about $1/n$ from the unit circle; our proof indeed only needs a radius $1 - o(1)/n$ and a compatible almost-sure rate for the logarithmic integral.*

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