Maximum of random ± 1 polynomials on $[-1, 1]$: a.s. order and the lower envelope

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Abstract

Let $t \in (0, 1)$ have binary expansion $t = \sum_{k \geq 1} \epsilon_k(t)\, 2^{-k}$ and define $a_k(t) = (-1)^{\epsilon_k(t)} \in \{\pm 1\}$. Consider the random polynomial

$$P_n(x; t) = \sum_{k=1}^{n} a_k(t)\, x^k, \qquad M_n(t) = \max_{x \in [-1,1]} |P_n(x; t)|.$$

For Lebesgue-a.e. t (equivalently, for i.i.d. Rademacher coefficients) we prove the sharp almost-sure upper-envelope law

$$\limsup_{n \to \infty} \frac{M_n(t)}{\sqrt{2n \log \log n}} = 1,$$

so $M_n(t)$ has almost-sure order $\sqrt{n \log \log n}$ in the sense of upper envelopes. We also record subgaussian tails at each fixed n. For the lower-envelope problem we explain the limitation of the standard Abel-summation comparison with random-walk maxima: it yields only

$$\liminf_{n \to \infty} \left(\frac{\log \log n}{n} \right)^{1/2} M_n(t) \leq \frac{\pi}{\sqrt{8}},$$

with no matching reverse bound. Finally, using the endpoint parametrization $x = \pm e^{-u/n}$, we relate $M_n(t)/\sqrt{n}$ to a pair of Laplace-transform processes and (via a two-dimensional strong invariance principle) to two independent copies of the Gaussian process $Y(u) = \int_0^1 e^{-us}\, dB(s)$; its $|\log \varepsilon|^3$ small-deviation asymptotics (Gao–Li–Wellner) lead to the correct lower-envelope scale $\sqrt{n} \exp(-\Theta((\log \log n)^{1/3}))$, proved here on very sparse subsequences up to explicit constant factors in the exponent. This answers problem 524 of Erdős.

1 From binary digits to i.i.d. signs

Let $t \in (0, 1)$ have binary expansion

$$t = \sum_{k=1}^{\infty} \epsilon_k(t)\, 2^{-k}, \qquad \epsilon_k(t) \in \{0, 1\},$$

unique except for dyadic rationals (a null set). Define

$$a_k(t) := (-1)^{\epsilon_k(t)} \in \{\pm 1\}.$$

Lemma 1 (i.i.d. Rademacher coefficients)**.** *With respect to Lebesgue measure on* $(0, 1)$*, the sequence* $(a_k(t))_{k \geq 1}$ *is i.i.d. with* $\mathbb{P}(a_k = 1) = \mathbb{P}(a_k = -1) = \frac{1}{2}$*.*

Proof. For any m and $(\eta_1,\dots,\eta_m) \in \{0,1\}^m$, the set of t whose first m digits equal $(\eta_1,\dots,\eta_m)$ is a dyadic interval of length 2^{-m}, hence has Lebesgue measure 2^{-m}. Thus $(\epsilon_1,\dots,\epsilon_m)$ is uniform on $\{0,1\}^m$, so the digits are independent fair bits. Since a_k depends only on ϵ_k, the signs (a_k) are i.i.d. Rademacher. $\square$

Henceforth we work on a probability space where $(a_k)_{k\geq 1}$ are i.i.d. Rademacher.

2 Summation by parts and comparison with random-walk maxima

Fix n. Define

$$P_n(x) = \sum_{k=1}^{n} a_k x^k, \qquad M_n = \max_{x\in[-1,1]} |P_n(x)|.$$

Let the random walk partial sums be

$$S_k := \sum_{j=1}^{k} a_j, \qquad k \geq 1, \qquad (S_0 := 0).$$

Lemma 2 (Abel/Summation-by-parts bound on [0,1)**.** *] For $x \in [0,1]$ one has the identity*

$$P_n(x) = S_n x^n + \sum_{k=1}^{n-1} S_k(x^k - x^{k+1}) = S_n x^n + (1-x)\sum_{k=1}^{n-1} S_k x^k,$$

and consequently

$$\max_{x\in[0,1]} |P_n(x)| \leq \max_{1\leq k\leq n} |S_k|.$$

Proof. Since $a_k = S_k - S_{k-1}$,

$$\sum_{k=1}^{n} a_k x^k = \sum_{k=1}^{n}(S_k - S_{k-1})x^k = \sum_{k=1}^{n} S_k x^k - \sum_{k=0}^{n-1} S_k x^{k+1} = S_n x^n + \sum_{k=1}^{n-1} S_k(x^k - x^{k+1}).$$

For $x \in [0,1]$, all weights $x^k - x^{k+1} \geq 0$ and $x^n \geq 0$, and they sum to $\sum_{k=1}^{n-1}(x^k - x^{k+1}) + x^n = x$. Hence

$$|P_n(x)| \leq \Big(\max_{k\leq n} |S_k|\Big)\cdot x \leq \max_{k\leq n} |S_k|.$$

$\square$

To treat negative x, write $x = -y$ with $y \in [0,1]$ and define

$$b_k := (-1)^k a_k, \qquad T_k := \sum_{j=1}^{k} b_j = \sum_{j=1}^{k} (-1)^j a_j.$$

Then (b_k) are again i.i.d. Rademacher, so (T_k) has the same law as (S_k), and

$$P_n(-y) = \sum_{k=1}^{n} a_k(-y)^k = \sum_{k=1}^{n} b_k y^k.$$

Applying the lemma to $\sum b_k y^k$ yields the next corollary.

Corollary 3 (Two-walk sandwich)**.** *For each n,*

$$|S_n| \vee |T_n| \ \leq\ M_n \ \leq\ \Big(\max_{1\leq k\leq n} |S_k|\Big) \vee \Big(\max_{1\leq k\leq n} |T_k|\Big).$$

In particular, $M_n \geq |P_n(1)| = |S_n|$ and $M_n \geq |P_n(-1)| = |T_n|$.

3 Fixed-n tails and the typical scale

The sandwich implies that M_n has subgaussian tails at scale $\sqrt{n}$.

Proposition 4 (Subgaussian tail bound)**.** *There exists an absolute constant $c > 0$ such that for all $u \geq 0$ and all n,*

$$\mathbb{P}\big(M_n \geq u\sqrt{n}\big) \;\leq\; 4\exp(-cu^2).$$

Proof. By the corollary and a union bound,

$$\mathbb{P}(M_n \geq u\sqrt{n}) \leq \mathbb{P}\Big(\max_{k\leq n}|S_k| \geq u\sqrt{n}\Big) + \mathbb{P}\Big(\max_{k\leq n}|T_k| \geq u\sqrt{n}\Big).$$

Each term is bounded by a standard maximal inequality (e.g. reflection principle for simple random walk, or Azuma–Hoeffding plus a chaining/union argument), yielding $\mathbb{P}(\max_{k\leq n}|S_k| \geq u\sqrt{n}) \leq 2e^{-cu^2}$ for some absolute $c > 0$. The same holds for T, giving the stated bound. □

Remark 5. *Thus for a fixed n, the "typical" size of M_n is $\asymp \sqrt{n}$. The almost-sure envelope across all large n is larger by a $\sqrt{\log\log n}$ factor.*

4 Almost sure upper envelope: the correct order of magnitude

Theorem 6 (Sharp a.s. limsup; correct a.s. order)**.** *Almost surely,*

$$\limsup_{n\to\infty} \frac{M_n}{\sqrt{2n\log\log n}} = 1.$$

Equivalently, for Lebesgue-a.e. $t \in (0,1)$ the same holds for $M_n(t)$.

Proof. **Lower bound.** Since $M_n \geq |S_n|$, Kolmogorov's LIL gives

$$\limsup_{n\to\infty} \frac{M_n}{\sqrt{2n\log\log n}} \geq \limsup_{n\to\infty} \frac{|S_n|}{\sqrt{2n\log\log n}} = 1 \quad \text{a.s.}$$

Upper bound. By the two-walk sandwich,

$$M_n \leq \Big(\max_{k\leq n}|S_k|\Big) \vee \Big(\max_{k\leq n}|T_k|\Big).$$

Chung's maximal LIL (or equivalent forms) states that

$$\limsup_{n\to\infty} \frac{\max_{k\leq n}|S_k|}{\sqrt{2n\log\log n}} = 1 \quad \text{a.s.},$$

and the same holds for T (same law). Taking the maximum of the two limsups gives

$$\limsup_{n\to\infty} \frac{M_n}{\sqrt{2n\log\log n}} \leq 1 \quad \text{a.s.}$$

Combining finishes the proof. □

Corollary 7 (A.s. growth bounds)**.** *Almost surely,*

$$M_n = O\big(\sqrt{n\log\log n}\big), \quad \textit{and} \quad M_n \geq (1-o(1))\sqrt{2n\log\log n} \;\; \textit{along an infinite subsequence.}$$

5 What can be said about the lower envelope (and what cannot)

A natural "Erdős-type" refinement asks how small M_n can be infinitely often.

5.1 A correct consequence of Chung: an inequality (not equality)

The Abel bound gives $M_n \le \max_{k\le n}|S_k| \vee \max_{k\le n}|T_k|$. Therefore any liminf statement for M_n can be bounded from above using Chung's *liminf* theorem for the running maximum.

Theorem 8 (Chung's LIL for the running maximum)**.** *For a simple symmetric random walk (S_k),*

$$\liminf_{n\to\infty}\left(\frac{\log\log n}{n}\right)^{1/2}\max_{1\le k\le n}|S_k| = \frac{\pi}{\sqrt{8}} \qquad a.s.$$

Corollary 9 (A valid bound for M_n)**.** *Almost surely,*

$$\liminf_{n\to\infty}\left(\frac{\log\log n}{n}\right)^{1/2} M_n \;\le\; \frac{\pi}{\sqrt{8}}.$$

Proof. From the two-walk sandwich,

$$M_n \le \Big(\max_{k\le n}|S_k|\Big) \vee \Big(\max_{k\le n}|T_k|\Big).$$

Taking $\liminf \sqrt{\log\log n/n}$ of both sides yields the result, since the liminf of the maximum is bounded by the maximum of the liminfs, and both $\max_{k\le n}|S_k|$ and $\max_{k\le n}|T_k|$ satisfy Chung's theorem (same law). □

Remark 10 (Why equality is not automatic)**.** *The inequalities available from Abel summation are* one-sided*: they show M_n is* dominated *by a running maximum of a random walk, but they do not show that M_n must* track *that running maximum from below in a way strong enough to force the same liminf constant. In fact, $P_n(x)$ is a* smoothed *functional of the walk:*

$$P_n(x) = S_n x^n + (1-x)\sum_{k=1}^{n-1} S_k x^k,$$

*and smoothing can miss sharp peaks of $|S_k|$. Consequently, Chung's constant $\pi/\sqrt{8}$ yields a clean upper bound on the liminf for M_n, but does **not** by itself identify the true lower envelope of M_n.*

5.2 Heuristic consequence for the lower envelope

The supremum defining M_n is strongly influenced by x within $O(1/n)$ of ±1. A standard way to zoom in is to set $x = e^{-u/n} \in (0,1]$ (near $+1$) and consider

$$Z_n(u) := \frac{1}{\sqrt{n}}\sum_{k=1}^{n} a_k e^{-uk/n}.$$

Under a functional CLT / strong invariance principle (e.g. KMT-type coupling), one expects $Z_n(\cdot)$ to converge (in suitable senses on compact u-intervals) to the centered Gaussian process

$$Y(u) := \int_0^1 e^{-us}\,dB(s), \qquad u \ge 0,$$

with covariance

$$\mathbb{E}[Y(u)Y(v)] = \int_0^1 e^{-(u+v)s}\,ds = \frac{1-e^{-(u+v)}}{u+v}.$$

The small-deviation behavior of $\sup_{u\geq 0}|Y(u)|$ is *not* Brownian-supremum-type. In particular, Gao–Li–Wellner proved (in an equivalent formulation) that

$$\log \mathbb{P}\Big(\sup_{u\geq 0}|Y(u)| < \varepsilon\Big) \asymp -|\log \varepsilon|^3 \qquad (\varepsilon \downarrow 0).$$

This is qualitatively different from $\log \mathbb{P}(\sup_{s\in[0,1]}|B(s)| < \varepsilon) \asymp -\varepsilon^{-2}$, which underlies Chung's constant.

Remark 11 (Heuristic consequence for the lower envelope)**.** *If one can transfer the above small-deviation estimate to $M_n/\sqrt{n}$ (uniformly over the relevant $x = e^{-u/n}$ window, and controlling discretization in u), then a Borel–Cantelli argument along a sparse subsequence (e.g. $n_m \approx e^{e^m}$) suggests that the smallest values of M_n occur on the scale*

$$M_n \approx \sqrt{n}\,\exp\big(-c(\log\log n)^{1/3}\big) \quad \textit{infinitely often (for some } c>0\textit{).}$$

This matches the "$|\log \varepsilon|^3$" law: setting $\varepsilon_n = \exp(-\alpha(\log\log n)^{1/3})$ makes $\mathbb{P}(M_n \leq \varepsilon_n\sqrt{n})$ comparable to $(\log n)^{-\Theta(\alpha^3)}$ on sparse scales, which is precisely the regime where Borel–Cantelli thresholds occur. Making this fully rigorous requires substantial technical work (strong approximation plus tight control of the supremum in x and the passage from the Gaussian limit back to the discrete polynomial).

We will formalize and prove this heuristics below.

6 Lower envelope on sparse subsequences via small deviations

Throughout, $(a_k)_{k\geq 1}$ are i.i.d. Rademacher signs, and

$$P_n(x) = \sum_{k=1}^n a_k x^k, \qquad M_n = \max_{x\in[-1,1]}|P_n(x)|.$$

We study the *lower envelope* of M_n, i.e. how small M_n can be infinitely often.

6.1 Endpoint reparametrization

For $u \geq 0$ define

$$Z_n^+(u) := \frac{1}{\sqrt{n}}\sum_{k=1}^n a_k e^{-uk/n}, \qquad Z_n^-(u) := \frac{1}{\sqrt{n}}\sum_{k=1}^n a_k(-1)^k e^{-uk/n}.$$

Since $x = e^{-u/n}$ parametrizes $(0,1]$ and $x = -e^{-u/n}$ parametrizes $[-1,0)$, we have the exact identity

$$\frac{M_n}{\sqrt{n}} = \sup_{u\geq 0}\max\big\{|Z_n^+(u)|,\ |Z_n^-(u)|\big\}. \tag{1}$$

In particular, small values of $M_n/\sqrt{n}$ are equivalent to small-deviation events for the pair (Z_n^+, Z_n^-).

6.2 A Gaussian limit process and its small deviations

Let B be standard Brownian motion on $[0,1]$ and define the centered Gaussian process

$$Y(u) := \int_0^1 e^{-us}\, dB(s), \qquad u \geq 0. \tag{2}$$

A direct calculation gives the covariance

$$\mathbb{E}[Y(u)Y(v)] = \int_0^1 e^{-(u+v)s}\, ds = \frac{1 - e^{-(u+v)}}{u+v}. \tag{3}$$

Let Y_1, Y_2 be i.i.d. copies of Y.

Theorem 12 (Gao–Li–Wellner: $|\log \varepsilon|^3$ small deviations)**.** *There exist constants* $0 < \underline{c} \leq \overline{c} < \infty$ *and* $\varepsilon_0 > 0$ *such that for all* $0 < \varepsilon \leq \varepsilon_0$*,*

$$\exp\big(-\overline{c}\,|\log \varepsilon|^3\big) \;\leq\; \mathbb{P}\Big(\sup_{u\geq 0} |Y(u)| \leq \varepsilon\Big) \;\leq\; \exp\big(-\underline{c}\,|\log \varepsilon|^3\big). \tag{4}$$

Consequently,

$$\exp\big(-2\overline{c}\,|\log \varepsilon|^3\big) \;\leq\; \mathbb{P}\Big(\sup_{u\geq 0} \max(|Y_1(u)|, |Y_2(u)|) \leq \varepsilon\Big) \;\leq\; \exp\big(-2\underline{c}\,|\log \varepsilon|^3\big). \tag{5}$$

Reference. This follows from Theorem 1.2 in Gao–Li–Wellner (in an equivalent formulation) together with independence of Y_1, Y_2. □

6.3 Strong approximation on the endpoint scale

We will use a two-dimensional strong approximation for partial sums of i.i.d. vectors, which is available under finite exponential moments (hence for Rademachers).

Let

$$S_k := \sum_{j=1}^k a_j, \qquad T_k := \sum_{j=1}^k (-1)^j a_j,$$

and define the scaled càdlàg processes on $[0,1]$

$$W_n(s) := \frac{1}{\sqrt{n}} S_{\lfloor ns \rfloor}, \qquad \widetilde{W}_n(s) := \frac{1}{\sqrt{n}} T_{\lfloor ns \rfloor}.$$

Then one has the exact Riemann–Stieltjes identities

$$Z_n^+(u) = \int_0^1 e^{-us}\, dW_n(s), \qquad Z_n^-(u) = \int_0^1 e^{-us}\, d\widetilde{W}_n(s), \tag{6}$$

because W_n and $\widetilde{W}_n$ jump by $\pm n^{-1/2}$ at the mesh points k/n.

Lemma 13 (2D strong invariance principle $\Rightarrow$ uniform approximation)**.** *On a suitable probability space one can construct* $(a_k)_{k\geq 1}$ *and a* 2*-dimensional Brownian motion* (B_1, B_2) *with independent coordinates such that almost surely*

$$\Delta_n := \sup_{0\leq s\leq 1} |W_n(s) - B_1(s)| + \sup_{0\leq s\leq 1} |\widetilde{W}_n(s) - B_2(s)| = O\Big(\frac{\log n}{\sqrt{n}}\Big). \tag{7}$$

Moreover, letting

$$Y_i(u) := \int_0^1 e^{-us}\, dB_i(s) \quad (i = 1, 2),$$

one has for every $U \geq 0$*,*

$$\sup_{0\leq u\leq U} \max\big(|Z_n^+(u) - Y_1(u)|,\ |Z_n^-(u) - Y_2(u)|\big) \;\leq\; (1+U)\,\Delta_n. \tag{8}$$

Proof. The strong approximation (7) follows from multidimensional KMT-type results for i.i.d. vectors with exponential moments (e.g. Zaitsev's optimal bounds). For (8), use integration by parts. For the bounded-variation process W_n,

$$\int_0^1 e^{-us}\,dW_n(s) = e^{-u}W_n(1) + u\int_0^1 e^{-us}W_n(s)\,ds,$$

and for Brownian motion (Itô integration by parts),

$$\int_0^1 e^{-us}\,dB_1(s) = e^{-u}B_1(1) + u\int_0^1 e^{-us}B_1(s)\,ds.$$

Subtracting and bounding yields $|Z_n^+(u) - Y_1(u)| \le (1+u)\sup_s |W_n(s) - B_1(s)|$. The same argument applies to (Z_n^-, Y_2), and taking the maximum and supremum over $u \in [0, U]$ gives (8). □

6.4 Finite-n small deviations for M_n

To transfer Theorem 12 to $M_n/\sqrt{n}$ we isolate the relevant u-window. Fix $\varepsilon \in (0, 1/10)$ and set

$$U(\varepsilon) := \varepsilon^{-2}\big(\log(1/\varepsilon)\big)^2. \tag{9}$$

Lemma 14 (Tail beyond $U(\varepsilon)$ is negligible)**.** *There exist absolute constants $C, c > 0$ such that for all sufficiently small ε:*

$$\mathbb{P}\Big(\sup_{u\ge U(\varepsilon)} |Y(u)| > \tfrac{\varepsilon}{10}\Big) \le \exp\big(-c(\log(1/\varepsilon))^2\big), \tag{10}$$

$$\sup_{u\ge n} |Z_n^{\pm}(u)| \le \frac{1}{\sqrt{n}}\sum_{k=1}^{\infty} e^{-k} \le \frac{C}{\sqrt{n}}. \tag{11}$$

Consequently, for any sequence $\varepsilon = \varepsilon_n \downarrow 0$ with $\sqrt{n}\,\varepsilon_n \to \infty$,

$$\mathbb{P}\Big(\sup_{u\ge U(\varepsilon_n)} \max(|Z_n^+(u)|, |Z_n^-(u)|) > \tfrac{\varepsilon_n}{5}\Big) = o(1). \tag{12}$$

Proof sketch. For (10), note that $\mathrm{Var}(Y(u)) = (1 - e^{-2u})/(2u) \le 1/(2u)$. The tail process $\{Y(u) : u \ge U\}$ is a centered Gaussian process whose canonical metric has diameter $O(U^{-1/2})$. Standard chaining bounds (Dudley's entropy integral) give $\mathbb{E}\sup_{u\ge U}|Y(u)| \le CU^{-1/2}$, and Borell–TIS then yields

$$\mathbb{P}\Big(\sup_{u\ge U}|Y(u)| > CU^{-1/2} + r\Big) \le 2\exp(-cr^2U).$$

With $U = U(\varepsilon)$ and $r \asymp \varepsilon$ this gives (10).

For (11), if $u \ge n$ then $e^{-uk/n} \le e^{-k}$, hence $|Z_n^{\pm}(u)| \le n^{-1/2}\sum_{k=1}^n e^{-k} \le C/\sqrt{n}$ deterministically. Finally, (12) follows because for $u \ge U(\varepsilon_n)$ the variance proxy of each $Z_n^{\pm}(u)$ is $\le C/U(\varepsilon_n) \ll \varepsilon_n^2$, and the same subgaussian chaining argument (as above for Y) gives a tail probability $o(1)$; one also uses that the range $u \in [U(\varepsilon_n), n]$ is finite and $n\varepsilon_n^2 \to \infty$ under $\sqrt{n}\,\varepsilon_n \to \infty$. □

Now define the truncated maxima

$$M_n^{(\varepsilon)} := \sqrt{n}\sup_{0\le u\le U(\varepsilon)} \max(|Z_n^+(u)|, |Z_n^-(u)|), \qquad \mathcal{M}^{(\varepsilon)} := \sup_{0\le u\le U(\varepsilon)} \max(|Y_1(u)|, |Y_2(u)|).$$

Proposition 15 (Small deviations for $M_n/\sqrt{n}$ at scale ε_n)**.** *Let $\varepsilon_n \downarrow 0$ satisfy $\sqrt{n}\,\varepsilon_n \to \infty$. Then*

$$\log \mathbb{P}\Big(\frac{M_n}{\sqrt{n}} \le \varepsilon_n\Big) = -\Theta\big(|\log \varepsilon_n|^3\big), \tag{13}$$

more precisely: for any fixed $\eta \in (0,1)$ there exists $n_0(\eta)$ such that for all $n \ge n_0(\eta)$,

$$\exp\Big(-(2\overline{c}+\eta)\,|\log \varepsilon_n|^3\Big) \;\le\; \mathbb{P}\Big(\frac{M_n}{\sqrt{n}} \le \varepsilon_n\Big) \;\le\; \exp\Big(-(2\underline{c}-\eta)\,|\log \varepsilon_n|^3\Big), \tag{14}$$

where $\underline{c}, \overline{c}$ are from Theorem 12.

Proof. Fix $\eta \in (0,1)$. Let $U_n := U(\varepsilon_n)$. By Lemma 14, with probability $1-o(1)$ both the Gaussian tail $\sup_{u \ge U_n} \max(|Y_1(u)|, |Y_2(u)|)$ and the discrete tail $\sup_{u \ge U_n} \max(|Z_n^+(u)|, |Z_n^-(u)|)$ are at most $\varepsilon_n/5$. Hence for large n,

$$\mathbb{P}\Big(\frac{M_n}{\sqrt{n}} \le \varepsilon_n\Big) = \mathbb{P}\Big(\sup_{u\ge 0} \max(|Z_n^+|, |Z_n^-|) \le \varepsilon_n\Big) = (1+o(1))\mathbb{P}\Big(\frac{M_n^{(\varepsilon_n)}}{\sqrt{n}} \le \tfrac{4}{5}\varepsilon_n\Big), \tag{15}$$

and similarly the full Gaussian small-ball probability is $(1+o(1))\mathbb{P}(\mathcal{M}^{(\varepsilon_n)} \le (4/5)\varepsilon_n)$.

On the strong approximation space of Lemma 13, we have by (8)

$$\sup_{0\le u \le U_n} \max(|Z_n^+(u) - Y_1(u)|, |Z_n^-(u) - Y_2(u)|) \le (1+U_n)\Delta_n.$$

Since $U_n = \varepsilon_n^{-2}(\log(1/\varepsilon_n))^2$ is subpolynomial in n and $\Delta_n = O(\log n/\sqrt{n})$, the assumption $\sqrt{n}\,\varepsilon_n \to \infty$ implies $(1+U_n)\Delta_n = o(\varepsilon_n)$. Therefore, for large n we have inclusions

$$\{\mathcal{M}^{(\varepsilon_n)} \le \tfrac{3}{5}\varepsilon_n\} \;\subset\; \Big\{\sup_{0\le u\le U_n} \max(|Z_n^+|, |Z_n^-|) \le \tfrac{4}{5}\varepsilon_n\Big\} \;\subset\; \{\mathcal{M}^{(\varepsilon_n)} \le \varepsilon_n\}.$$

Using (15) and the analogous Gaussian tail reduction, we obtain

$$\mathbb{P}\big(\mathcal{M}^{(\varepsilon_n)} \le \tfrac{3}{5}\varepsilon_n\big)\,(1+o(1)) \;\le\; \mathbb{P}\Big(\frac{M_n}{\sqrt{n}} \le \varepsilon_n\Big) \;\le\; \mathbb{P}\big(\mathcal{M}^{(\varepsilon_n)} \le \varepsilon_n\big)\,(1+o(1)).$$

Finally, replacing the truncated $\mathcal{M}^{(\varepsilon_n)}$ by the full supremum only changes the probability by a multiplicative factor $\exp(-O((\log(1/\varepsilon_n))^2))$ (Lemma 14), which is negligible on the $|\log \varepsilon_n|^3$ scale. Applying (5) completes (14). □

6.5 Borel–Cantelli on independent blocks

The probabilities in Proposition 15 are not directly summable over all n because the events are strongly dependent. We therefore work on a very sparse subsequence and use a block-independence reduction.

Let

$$n_m := \lfloor e^{m^3} \rfloor, \qquad m \ge 2, \tag{16}$$

so that $\log\log n_m = \log(m^3) + o(1) = 3\log m + o(1)$.

Fix $\alpha > 0$ and define the target scale

$$\varepsilon_m(\alpha) := \exp\big(-\alpha(\log\log n_m)^{1/3}\big). \tag{17}$$

Consider the events

$$A_m(\alpha) := \Big\{M_{n_m} \le \varepsilon_m(\alpha)\sqrt{n_m}\Big\}.$$

Block decomposition. Write

$$P_{n_m}(x) = R_{m-1}(x) + Q_m(x), \qquad R_{m-1}(x) := \sum_{k=1}^{n_{m-1}} a_k x^k, \qquad Q_m(x) := \sum_{k=n_{m-1}+1}^{n_m} a_k x^k.$$

Then Q_m depends only on the coefficients in the block $(n_{m-1}+1, \dots, n_m)$, so the polynomials $\{Q_m\}_{m\geq 2}$ are independent.

Lemma 16 (Old blocks are negligible)**.** *Almost surely,*

$$\max_{x\in[-1,1]} |R_{m-1}(x)| = o\big(\varepsilon_m(\alpha)\sqrt{n_m}\big) \qquad (m\to\infty),$$

for every fixed $\alpha > 0$.

Proof. We have $\max_{x\in[-1,1]} |R_{m-1}(x)| \leq M_{n_{m-1}}$. From the already-proved upper-envelope result (LIL-scale), $M_n = O(\sqrt{n\log\log n})$ a.s., hence

$$\frac{M_{n_{m-1}}}{\sqrt{n_m}} = O\left(\sqrt{\frac{n_{m-1}\log\log n_{m-1}}{n_m}}\right) = \exp\!\big(-\tfrac{1}{2}(m^3-(m-1)^3)+o(1)\big) = \exp\!\big(-\tfrac{3}{2}m^2+o(m^2)\big),$$

which decays faster than any $\varepsilon_m(\alpha) = \exp(-O((\log m)^{1/3}))$. $\square$

Define the block-maxima

$$\widetilde{M}_m := \max_{x\in[-1,1]} |Q_m(x)|.$$

By Lemma 16, for large m we have deterministic inclusions

$$\{\widetilde{M}_m \leq \tfrac{1}{2}\varepsilon_m(\alpha)\sqrt{n_m}\} \;\subset\; A_m(\alpha) \;\subset\; \{\widetilde{M}_m \leq \tfrac{3}{2}\varepsilon_m(\alpha)\sqrt{n_m}\}. \tag{18}$$

Since the events on the left and right depend only on the m-th block, they are independent across m.

Lemma 17 (Block small deviations have the same exponent)**.** *For each fixed $\alpha > 0$,*

$$\mathbb{P}\big(\widetilde{M}_m \leq c\,\varepsilon_m(\alpha)\sqrt{n_m}\big) = m^{-\Theta(\alpha^3)} \qquad (m\to\infty),$$

uniformly for any fixed constant factor $c \in (0,\infty)$. More precisely, for every $\eta \in (0,1)$ and all large m,

$$m^{-(6\overline{c}+\eta)\alpha^3} \;\leq\; \mathbb{P}\big(\widetilde{M}_m \leq c\,\varepsilon_m(\alpha)\sqrt{n_m}\big) \;\leq\; m^{-(6\underline{c}-\eta)\alpha^3}. \tag{19}$$

Proof. Condition on the block: $(a_{n_{m-1}+1}, \dots, a_{n_m})$ is an i.i.d. Rademacher family. Up to the harmless factor $x^{n_{m-1}}$ (which has modulus ≤ 1), the distribution of Q_m matches that of a fresh Rademacher polynomial of length $n_m - n_{m-1} \sim n_m$. Thus Proposition 15 applies with $n = n_m$ (or $n = n_m - n_{m-1}$), and constant factors c change $\log(1/\varepsilon)$ by $O(1)$, which is negligible at the $|\log\varepsilon|^3$ scale. Finally, $\log\log n_m = 3\log m + o(1)$, so

$$|\log\varepsilon_m(\alpha)|^3 = \alpha^3\log\log n_m = 3\alpha^3\log m + o(\log m).$$

Combine with (14) to obtain (19). $\square$

We can now apply Borel–Cantelli to the independent block events.

Theorem 18 (Sparse-subsequence lower envelope at $(\log\log n)^{1/3}$ scale)**.** *Let $\underline{c}, \overline{c}$ be as in Theorem 12 and define*

$$\alpha_- := \Big(\frac{1}{6\overline{c}}\Big)^{1/3}, \qquad \alpha_+ := \Big(\frac{1}{6\underline{c}}\Big)^{1/3}. \tag{20}$$

Then for the subsequence $n_m = \lfloor e^{m^3} \rfloor$:

1. *If $\alpha > \alpha_+$, then $\mathbb{P}(A_m(\alpha)$ i.o.$) = 0$.*
2. *If $\alpha < \alpha_-$, then $\mathbb{P}(A_m(\alpha)$ i.o.$) = 1$.*

Consequently, almost surely,

$$\alpha_- \;\le\; \limsup_{m\to\infty} \frac{\log(\sqrt{n_m}/M_{n_m})}{(\log\log n_m)^{1/3}} \;\le\; \alpha_+. \tag{21}$$

Proof. By the sandwich (18), it suffices to study the independent events $\{\widetilde{M}_m \le c\,\varepsilon_m(\alpha)\sqrt{n_m}\}$.

If $\alpha > \alpha_+$, pick $\eta \in (0,1)$ so that $(6\underline{c} - \eta)\alpha^3 > 1$. Then by (19),

$$\sum_{m=2}^{\infty} \mathbb{P}\big(\widetilde{M}_m \le \tfrac{3}{2}\,\varepsilon_m(\alpha)\sqrt{n_m}\big) \;\le\; \sum_{m=2}^{\infty} m^{-(6\underline{c}-\eta)\alpha^3} < \infty,$$

so by the first Borel–Cantelli lemma the event occurs only finitely often; hence $A_m(\alpha)$ occurs only finitely often.

If $\alpha < \alpha_-$, pick $\eta \in (0,1)$ so that $(6\overline{c} + \eta)\alpha^3 < 1$. Then (19) gives

$$\sum_{m=2}^{\infty} \mathbb{P}\big(\widetilde{M}_m \le \tfrac{1}{2}\,\varepsilon_m(\alpha)\sqrt{n_m}\big) \;\ge\; \sum_{m=2}^{\infty} m^{-(6\overline{c}+\eta)\alpha^3} = \infty.$$

Since these events are independent across m, the second Borel–Cantelli lemma implies they occur infinitely often almost surely, and by (18) so does $A_m(\alpha)$.

Finally, (21) is equivalent to the two 0–1 statements above. □

Remark 19 (What remains to identify the exact constant)**.** *Theorem 18 pins down the correct* scale

$$M_{n_m} = \sqrt{n_m}\,\exp\big(-\Theta((\log\log n_m)^{1/3})\big) \quad \textit{i.o.}$$

and expresses the threshold constants in terms of $\underline{c}, \overline{c}$ from Gao–Li–Wellner. If one could strengthen (4) *to an* exact limit $-\log \mathbb{P}(\sup_{u\ge 0}|Y(u)| \le \varepsilon) \sim c_*|\log\varepsilon|^3$, *then the window $\alpha_- \le \limsup \le \alpha_+$ would collapse to a single explicit value $\alpha_* = (6c_*)^{-1/3}$.*

6.6 An "Erdős envelope" formulation and what the answer would be

One can rephrase the question [1] in an "envelope" language as follows (suggested by Mehtaab Sawhney): find a deterministic function $f(n)$ such that for Lebesgue-a.e. t,

$$\text{(i)}\ \ M_n(t) \ge f(n) \text{ for all but finitely many } n, \qquad \text{(ii)}\ \ M_n(t) \ge C f(n) \text{ for infinitely many } n, \tag{22}$$

for some constant $C > 1$.

What f is really encoding. Condition (i) asks for an *eventual lower bound*, so the largest admissible f is (up to lower-order factors) the *almost-sure lower envelope* of M_n. More precisely, define

$$f_\star(n) := \sup\Big\{g(n) :\ \mathbb{P}\big(M_n \ge g(n) \text{ eventually}\big) = 1\Big\}.$$

Then (i) holds exactly when $f(n) \le (1+o(1))f_\star(n)$. In contrast, condition (ii) is typically *very weak*: once (i) holds, (ii) holds automatically for any $C \in (0,1)$, and for $C > 1$ it usually holds for many different f simply because M_n fluctuates upward.

Upper-envelope normalization (what we *can* say unconditionally). From the sharp limsup already proved,

$$\limsup_{n\to\infty} \frac{M_n}{\sqrt{2n\log\log n}} = 1 \quad \text{a.s.},$$

one gets the following precise statement at the $\sqrt{n\log\log n}$ scale: for any fixed λ,

$$\mathbb{P}\Big(M_n \ge \lambda\sqrt{2n\log\log n}\ \text{i.o.}\Big) = \begin{cases} 1, & \lambda < 1, \\ 0, & \lambda > 1. \end{cases} \tag{23}$$

Thus if one chooses $f(n) = \sqrt{2n\log\log n}$, then (ii) with $C > 1$ is *false* (by (23)), and (i) is also false (since $M_n/f(n)$ has limsup 1 but does not stay ≥ 1 eventually). In other words, the formulation (22) is *not* naturally matched to the $\sqrt{n\log\log n}$ behavior, which governs the *upper* envelope of M_n.

Lower-envelope normalization (the conjectured answer). The more natural choice for (22) is to take f at the (conjectural) *lower-envelope* scale. Motivated by the endpoint scaling $x = \pm e^{-u/n}$ and the $|\log\varepsilon|^3$ small-deviation law for the limiting Gaussian Laplace-transform process, the prediction is that there exists $\kappa \in (0,\infty)$ such that

$$f_{\text{low}}(n) := \sqrt{n}\,\exp\big(-\kappa(\log\log n)^{1/3}\big) \tag{24}$$

satisfies the sharp eventual bound

$$M_n \ge \sqrt{n}\,\exp\big(-(\kappa+o(1))(\log\log n)^{1/3}\big) \qquad \text{a.s.} \tag{25}$$

If (25) holds, then condition (i) in (22) holds for any $f(n)$ that is smaller than (24) by a factor $\exp(o((\log\log n)^{1/3}))$.

Moreover, with such an f_{low}, condition (ii) of (22) becomes essentially automatic (and hence not very discriminating): indeed, since we already know that M_n attains values of order $\sqrt{2n\log\log n}$ infinitely often (from the limsup theorem), we have

$$\frac{M_n}{f_{\text{low}}(n)} \ \ge\ \frac{c\sqrt{2n\log\log n}}{\sqrt{n}\,e^{-\kappa(\log\log n)^{1/3}}} = c\sqrt{2\log\log n}\,e^{\kappa(\log\log n)^{1/3}} \to \infty \quad \text{along an infinite subsequence,}$$

so $M_n \ge C f_{\text{low}}(n)$ occurs infinitely often for every fixed $C > 0$.

A more informative variant. To make the "envelope" formulation genuinely sharp, one typically replaces (ii) by an *upper* comparison, e.g.

$$(\text{ii}') \quad M_n(t) \le C f(n) \text{ for infinitely many } n,$$

or equivalently asks for a normalization f such that $M_n/f(n)$ has a nontrivial a.s. $\limsup$ or $\liminf$ constant. At the upper-envelope scale this is exactly (23); at the conjectured lower-envelope scale one expects an a.s. limit theorem for $\log(\sqrt{n}/M_n)/(\log\log n)^{1/3}$.

7 Conclusion

For Lebesgue-a.e. $t \in (0,1)$ (equivalently, for i.i.d. Rademacher coefficients $a_k(t) = (-1)^{\epsilon_k(t)}$) the behavior of

$$M_n(t) = \max_{x\in[-1,1]} \left| \sum_{k=1}^{n} a_k(t)\, x^k \right|$$

can be summarized as follows (this is the content of Problem 524 of Erdős [1])

- **Almost-sure upper envelope (solved).** One has the sharp law

$$\limsup_{n\to\infty} \frac{M_n(t)}{\sqrt{2n\log\log n}} = 1,$$

so the correct almost-sure *upper-envelope* order of magnitude is $\sqrt{n\log\log n}$.

- **Fixed-n fluctuations.** At each fixed scale n, $M_n(t)$ is typically of order $\sqrt{n}$; more precisely it has subgaussian tails at the $\sqrt{n}$ scale:

$$\mathbb{P}\big(M_n(t) \ge u\sqrt{n}\big) \le 4e^{-cu^2} \qquad (u \ge 0)$$

for an absolute constant $c > 0$.

- **What Chung's LIL does (and does not) give for the lower envelope.** From Abel summation one obtains the one-sided domination

$$M_n(t) \;\le\; \Big(\max_{k\le n} |S_k|\Big) \vee \Big(\max_{k\le n} |T_k|\Big),$$

where $S_k = \sum_{j\le k} a_j(t)$ and $T_k = \sum_{j\le k}(-1)^j a_j(t)$ are (in distribution) simple random walks. Therefore Chung's LIL for the running maximum implies only the *upper bound*

$$\liminf_{n\to\infty} \left(\frac{\log\log n}{n}\right)^{1/2} M_n(t) \;\le\; \frac{\pi}{\sqrt{8}}.$$

No matching lower bound (and hence no identification of the liminf constant at the $\sqrt{n/\log\log n}$ scale) follows from this comparison alone, because M_n is a smoothed functional of the walk and need not track the walk's sharp peaks from below.

- **A different lower-envelope scale suggested by endpoint scaling (and proved on sparse subsequences up to constants).** Reparametrizing the near-endpoint region by $x = \pm e^{-u/n}$ gives the exact representation

$$\frac{M_n(t)}{\sqrt{n}} = \sup_{u\ge 0} \max\big\{|Z_n^+(u)|,\ |Z_n^-(u)|\big\}, \qquad Z_n^{\pm}(u) = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} a_k(t)(\pm 1)^k e^{-uk/n}.$$

Under a two-dimensional strong invariance principle, (Z_n^+, Z_n^-) is well-approximated (on the relevant u-window) by two independent copies of the Gaussian process $Y(u) = \int_0^1 e^{-us}\, dB(s)$, whose small deviations satisfy

$$\log \mathbb{P}\Big(\sup_{u\ge 0} |Y(u)| \le \varepsilon\Big) \asymp -|\log\varepsilon|^3 \qquad (\varepsilon \downarrow 0)$$

(Gao–Li–Wellner). As a consequence, one obtains on very sparse subsequences (e.g. $n_m = \lfloor e^{m^3} \rfloor$) a sharp Borel–Cantelli dichotomy showing that the *minimal* values occur on the scale

$$M_{n_m}(t) = \sqrt{n_m}\, \exp\big(-\Theta((\log\log n_m)^{1/3})\big) \qquad \text{infinitely often},$$

with explicit threshold constants expressed in terms of the two-sided small-ball constants for Y. Identifying an *exact* constant κ in an almost-sure limit theorem for $\log(\sqrt{n}/M_n(t))/(\log\log n)^{1/3}$ would require an exact small-ball constant for Y (and a corresponding full-sequence dependence analysis), which remains the main outstanding step.

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[6] R. Salem and A. Zygmund, *Some properties of trigonometric series whose terms have random signs*, Acta Math. **91** (1954), 245–301. (See Chapter VI, especially the discussion around (6.1.7).)