A note on the formulation of Erdős Problem #584

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Abstract

We observe that the statement currently displayed on Erdős Problem #584 is false as written. The obstruction comes from high-girth graphs: a construction of Lazebnik, Ustimenko and Woldar yields infinitely many n-vertex graphs with $\Omega(n^{6/5})$ edges and girth at least 10. For such graphs no subgraph with more than one edge can have the property that every two edges lie on a cycle of length at most 8, whereas the target size $\delta^2 n^2$ tends to infinity. Thus the literal δ-formulation on the webpage must be repaired by inserting a sparse-range qualifier, as in the original Duke–Erdős–Rödl problem. For context, we also record a short self-contained proof of the classical $\Omega(\delta^3 n^2)$ weak C_6-connected core theorem.

1 Introduction

The webpage for Erdős Problem #584 currently asks whether every graph G on n vertices with δn^2 edges contains subgraphs $H_1, H_2 \subseteq G$ such that

- $e(H_1) \geq c\delta^3 n^2$, every two edges of H_1 lie on a cycle of length at most 6, and adjacent edges of H_1 lie on a C_4; and
- $e(H_2) \geq c\delta^2 n^2$ and every two edges of H_2 lie on a cycle of length at most 8,

for some absolute constant $c > 0$ independent of n and δ [1].

As written, the second bullet is false. The issue is that the webpage suppresses the small-sparsity quantifier that appears in the classical sparse formulation. In particular, Fox and Sudakov formulate the second question in the form: do there exist constants $c, \beta_0 > 0$ such that every graph with n vertices and $n^{2-\beta}$ edges, for all $0 \leq \beta \leq \beta_0$, contains a subgraph with at least $cn^{2-2\beta}$ edges whose every two edges lie on a cycle of length at most 8 [4, Problem 1.1]. They proved this strengthened C_8-statement for $\beta < 1/5$ [4]. Duke, Erdős and Rödl had earlier proved the weak C_6 statement with $cn^{2-3\beta}$ edges, the strong C_6 statement with $cn^{2-5\beta}$ edges, and a C_{12}-connected statement with $cn^{2-2\beta}$ edges [3]. For fixed positive density, Duke and Erdős proved the first bullet in the strong form [2].

The purpose of this note is modest: we settle the *literal* webpage formulation by giving a counterexample to the posted H_2 bullet, and then we record the standard $\delta^3 n^2$ weak C_6 core theorem in a short self-contained form so that the distinction between the false literal statement and the intended small-β problem is completely explicit.

2 A counterexample to the literal H_2 statement

We begin with a convenient consequence of a theorem of Lazebnik, Ustimenko and Woldar.

Theorem 1. *There is an absolute constant $c_0 > 0$ and infinitely many integers n for which there exists an n-vertex graph G with*

$$e(G) \geq c_0 n^{6/5}$$

and girth at least 10*.*

Proof. Lazebnik, Ustimenko and Woldar construct, for every prime power q, a q-regular graph $CD(5,q)$ of girth at least 10 and order $v \leq 2q^5$ [5]. Let $n = v$. Since the graph is q-regular,

$$e(CD(5,q)) = \frac{qv}{2}.$$

From $v \leq 2q^5$ we obtain $q \geq (v/2)^{1/5}$, and therefore

$$e(CD(5,q)) = \frac{qv}{2} \geq \frac{v}{2}\left(\frac{v}{2}\right)^{1/5} = 2^{-6/5}v^{6/5}.$$

Thus we may take $c_0 = 2^{-6/5}$. □

The posted H_2 bullet now falls immediately.

Corollary 2. *The second bullet in the current webpage statement of Erdős Problem #584 is false.*

Proof. Let G be one of the graphs from Theorem 1, and write $e(G) = \delta n^2$. Then

$$\delta \geq c_0 n^{-4/5},$$

so

$$\delta^2 n^2 \geq c_0^2 n^{2/5} \to \infty.$$

On the other hand, G has girth at least 10, so it contains no cycle of length at most 8. Therefore any subgraph $H \subseteq G$ with the property that *every two edges of H lie on a cycle of length at most* 8 can have at most one edge: if $e(H) \geq 2$, then some pair of edges of H would have to lie on such a cycle, impossible in a graph of girth at least 10.

Hence for every fixed constant $c > 0$ and all sufficiently large n,

$$e(H) \leq 1 < c\,\delta^2 n^2.$$

So no statement of the form $e(H_2) \geq c\delta^2 n^2$ can hold for all n and all δ. □

Remark 3. *Corollary 2 does* not *contradict the classical Duke–Erdős–Rödl problem. It only shows that the sparse-range quantifier cannot be omitted. Any correct C_8 formulation must at least restrict attention to $\delta \geq n^{-\beta_0}$ for some absolute $\beta_0 > 0$.*

3 A classical positive baseline: the weak C_6 core

For completeness we record the standard counting proof of the $\Omega(\delta^3 n^2)$ weak C_6 theorem. This result already appears in Duke–Erdős–Rödl [3]; we include the short argument because it makes transparent where the easy part of the first bullet ends and the genuinely open strong-C_6 question begins.

Proposition 4. *There is an absolute constant $c_1 > 0$ with the following property. Let G be a graph on n vertices with*

$$e(G) = \delta n^2$$

and assume $\delta^2 n \geq 1$. Then G contains a subgraph H with

$$e(H) \geq c_1 \delta^3 n^2$$

such that every two edges of H lie on a cycle of length at most 6.

Proof. For distinct vertices $u, v \in V(G)$, let

$$c(u,v) = |N(u) \cap N(v)|$$

be their codegree. Set

$$\Sigma := \sum_{\{u,v\} \subseteq V(G)} c(u,v).$$

Counting cherries by their center gives

$$\Sigma = \sum_{x \in V(G)} \binom{d(x)}{2}.$$

Since $\binom{x}{2}$ is convex for $x \geq 0$, Jensen's inequality yields

$$\Sigma \geq n \binom{\frac{1}{n}\sum_x d(x)}{2} = n \binom{2e(G)/n}{2} = n \binom{2\delta n}{2}.$$

Because $\delta^2 n \geq 1$, we have $2\delta n \geq 2$, and hence $\binom{2\delta n}{2} \geq \delta^2 n^2$. Therefore

$$\Sigma \geq \delta^2 n^3.$$

Let $N = \binom{n}{2}$. The number of copies of C_4 in G is

$$\#C_4(G) = \frac{1}{2} \sum_{\{u,v\}} \binom{c(u,v)}{2},$$

because every C_4 has exactly two pairs of opposite vertices. Another application of Jensen's inequality gives

$$\sum_{\{u,v\}} \binom{c(u,v)}{2} \geq N \binom{\Sigma/N}{2}.$$

Now $N \leq n^2/2$, so

$$\frac{\Sigma}{N} \geq \frac{\delta^2 n^3}{n^2/2} = 2\delta^2 n \geq 2.$$

Thus $\binom{\Sigma/N}{2} \geq (\Sigma/N)^2/4$, and consequently

$$\#C_4(G) \geq \frac{1}{2} \cdot N \cdot \frac{1}{4} \left(\frac{\Sigma}{N}\right)^2 = \frac{\Sigma^2}{8N} \geq \frac{\delta^4 n^6}{8(n^2/2)} = \frac{1}{4}\delta^4 n^4.$$

Hence some edge $xy \in E(G)$ lies in at least

$$\frac{4\#C_4(G)}{e(G)} \geq \delta^3 n^2$$

copies of C_4.

Let H be the union of all copies of C_4 containing the edge xy. Distinct C_4's through xy have distinct opposite edges, so

$$e(H) \geq \delta^3 n^2.$$

It remains to show that every two edges of H lie on a cycle of length at most 6.

Let

$$A = N_H(x) \setminus \{y\}, \qquad B = N_H(y) \setminus \{x\}.$$

Every copy of C_4 through xy has the form $xabyx$ with $a \in A$, $b \in B$, and $ab \in E(H)$. Thus every edge of H is of one of the four types

$$xy, \qquad xa\ (a \in A), \qquad by\ (b \in B), \qquad ab\ (ab \in E(H) \cap (A \times B)).$$

We check that any two edges lie on a cycle of length at most 6.

If one of the two edges is xy, then the claim is immediate because every other edge of H lies on some C_4 containing xy.

Assume now that neither edge is xy.

- If the two edges are ab and $a'b'$ with $ab, a'b' \in E(H) \cap (A \times B)$, then they lie on a C_4 when $a = a'$ or $b = b'$, and otherwise on the C_6 $xabyb'a'x$.
- If the two edges are xa and xa', choose $b, b' \in B$ with $ab, a'b' \in E(H)$. If $b = b'$, then xa and xa' lie on the C_4 $xaba'x$. If $b \neq b'$, then they lie on the C_6 $xabyb'a'x$. By symmetry the same holds for two edges of the form by and $b'y$.
- If the two edges are xa and by, then either $ab \in E(H)$, in which case they lie on the C_4 $xabyx$, or $ab \notin E(H)$. In the latter case choose $b' \in B$ and $a' \in A$ with $ab', a'b \in E(H)$. Then $b' \neq b$ and $a' \neq a$, and the two chosen edges lie on the C_6 $xabyba'x$.
- If the two edges are xa and $a'b$ with $a'b \in E(H)$, then either $ab \in E(H)$, in which case they lie on the C_4 $xaba'x$, or $ab \notin E(H)$. Choose $b' \in B$ with $ab' \in E(H)$. Then $b' \neq b$, and the two chosen edges lie on the C_6 $xab'yba'x$. The remaining mixed case ab and $b'y$ is symmetric.

In every case the two edges lie on a cycle of length at most 6, completing the proof. □

Remark 5. *Proposition 4 is the easy part of the first bullet. The genuinely difficult issue is the* strong C_6 *requirement that adjacent edges must lie on a* C_4*. Duke, Erdős and Rödl obtained this stronger conclusion with* $\delta^5 n^2$ *edges [3], and it remains open to reach the conjectured* $\delta^3 n^2$ *scale in the sparse regime.*

4 Concluding remark

The conclusion is simple. The exact webpage statement of Erdős Problem #584 is false because the H_2 bullet fails for high-girth graphs of superlinear size. The intended mathematical problem is the small-β sparse version studied by Duke, Erdős and Rödl and by Fox and Sudakov. In that corrected form the problem remains open.

References

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