# An expanded note on Erdős Problem #603

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**Abstract**

We expand the earlier note on Erdős Problem #603 in two directions suggested by LouisD in the discussion thread. First, we isolate a general partition-calculus template: if $k \geq 2$ is finite and $\kappa \to (\lambda)^k_\mu$, then the family $\{[X]^k : X \in [\kappa]^\lambda\}$ has all members of size $\lambda$, has pairwise intersections never equal to $k$, and has chromatic number exceeding $\mu$. This yields quantitative upper bounds for the invariants $f(k, \lambda, \mu)$ and $g(k, \lambda, \mu)$ proposed in the thread. In particular,

$$f(k, \mu^+, \mu) \leq \beth_{k-1}(\mu)^+, \qquad g(k, \mu^+, \mu) \leq \left(\beth_{k-1}(\mu)^+\right)^{\mu^+}.$$

Second, specializing to $k = 2$ and $\lambda = \omega$ settles the current arbitrary-family wording of Problem #603: there is no cardinal bound at all on the chromatic number. We also record the alternative countable-family reading, where the exact answer is 2 and the intersection hypothesis is unnecessary.

## 1. The problem and the quantitative reformulation

The current Erdős Problems page formulates Problem #603 as follows [1]: if $\mathcal{A}$ is a family of countably infinite sets such that $|A \cap B| \neq 2$ for all distinct $A, B \in \mathcal{A}$, determine the smallest cardinal $C$ such that $\bigcup \mathcal{A}$ can be colored with at most $C$ colors and no member of $\mathcal{A}$ is monochromatic. In the forum thread LouisD made two useful observations [2]: the construction should generalize from the forbidden intersection size 2 to an arbitrary forbidden size $k \geq 2$, and it is natural to package the question in terms of quantitative invariants.

For a family $\mathcal{A}$ of nonempty sets, let $\chi(\mathcal{A})$ denote the least cardinal $\tau$ such that $\bigcup \mathcal{A}$ can be colored with $\tau$ colors in such a way that no $A \in \mathcal{A}$ is monochromatic.

Fix an integer $k \geq 2$ and infinite cardinals $\lambda, \mu$. Following LouisD, say that $\mathcal{A}$ is a $(k, \lambda, \mu)$-*family* if:

- $|A| = \lambda$ for every $A \in \mathcal{A}$,
- $|A \cap B| \neq k$ for all distinct $A, B \in \mathcal{A}$,
- $\chi(\mathcal{A}) > \mu$.

Define

$$f(k, \lambda, \mu) = \min\left\{ \left|\bigcup \mathcal{A}\right| : \mathcal{A} \text{ is a } (k, \lambda, \mu)\text{-family} \right\},$$

and

$$g(k, \lambda, \mu) = \min\left\{ |\mathcal{A}| : \mathcal{A} \text{ is a } (k, \lambda, \mu)\text{-family} \right\}.$$

## 2. A general construction from partition relations

The basic construction is the following.

**Theorem 1.** *Let $k \geq 2$ be an integer, let $\lambda$ be an infinite cardinal, let $\mu$ be any cardinal, and assume that*

$$\kappa \to (\lambda)^k_\mu.$$

*Then there exists a $(k, \lambda, \mu)$-family $\mathcal{A}$ such that*

$$\left|\bigcup \mathcal{A}\right| = \kappa \qquad \textit{and} \qquad |\mathcal{A}| = \kappa^\lambda.$$

*Proof.* Let

$$V = [\kappa]^k.$$

Since $k$ is finite and $\kappa$ is infinite, $|V| = \kappa$. For each $X \in [\kappa]^\lambda$, define

$$A_X = [X]^k \subseteq V.$$

Let

$$\mathcal{A} = \{A_X : X \in [\kappa]^\lambda\}.$$

Because $\lambda$ is infinite and $k$ is finite, each $A_X$ has size $\lambda$, and also $|\mathcal{A}| = |[\kappa]^\lambda| = \kappa^\lambda$.

Now take distinct $X, Y \in [\kappa]^\lambda$. Then

$$A_X \cap A_Y = [X]^k \cap [Y]^k = [X \cap Y]^k.$$

If $|X \cap Y| = t < \omega$, then

$$|A_X \cap A_Y| = \binom{t}{k}.$$

For $t < k$ this is 0, for $t = k$ it is 1, and for $t \geq k+1$ it is at least $\binom{k+1}{k} = k+1$. Hence no finite value of $|A_X \cap A_Y|$ can equal $k$. If $X \cap Y$ is infinite, then

$$|A_X \cap A_Y| = \left|[X \cap Y]^k\right| = |X \cap Y|,$$

which is an infinite cardinal and therefore also not equal to $k$. Thus $|A_X \cap A_Y| \neq k$ for all distinct $A_X, A_Y \in \mathcal{A}$.

Finally, let $c : V \to \mu$ be any coloring. By the partition relation $\kappa \to (\lambda)^k_\mu$, there exists $H \in [\kappa]^\lambda$ such that all elements of $[H]^k$ receive the same color. Equivalently, $A_H$ is monochromatic. Hence no coloring of $\bigcup \mathcal{A} = V$ with $\mu$ colors can avoid monochromatic members of $\mathcal{A}$, so $\chi(\mathcal{A}) > \mu$. □

A useful special case already lives on a countable ground set.

**Proposition 2.** *For every integer $k \geq 2$, the family*

$$\mathcal{R}_k = \left\{[X]^k : X \in [\omega]^\omega\right\}$$

*satisfies $|A \cap B| \neq k$ for all distinct $A, B \in \mathcal{R}_k$ and*

$$\chi(\mathcal{R}_k) = \omega.$$

*Proof.* The intersection statement is a special case of the proof of Theorem 1. To see that $\chi(\mathcal{R}_k) > n$ for every finite $n$, let $c : [\omega]^k \to n$ be any coloring. By Ramsey's theorem [6], there exists an infinite set $H \subseteq \omega$ such that all members of $[H]^k$ have the same color. Hence $[H]^k \in \mathcal{R}_k$ is monochromatic.

On the other hand, $\chi(\mathcal{R}_k) \leq \omega$: color each $s \in [\omega]^k$ by $\min(s)$. Every member of $\mathcal{R}_k$ contains $k$-subsets with infinitely many different minima, so no member is monochromatic. □

## 3. Quantitative bounds for $f$ and $g$

The same construction answers the quantitative question from the thread.

**Corollary 3.** *Let $k \geq 2$ be an integer, let $\lambda, \mu$ be infinite cardinals, and let $\rho$ be an infinite cardinal with*

$$\mu \leq \rho \qquad \text{and} \qquad \lambda \leq \rho^+.$$

*Then*

$$f(k, \lambda, \mu) \leq \beth_{k-1}(\rho)^+ \qquad \text{and} \qquad g(k, \lambda, \mu) \leq \left(\beth_{k-1}(\rho)^+\right)^{\lambda}.$$

*Here $\beth_0(\rho) = \rho$ and $\beth_{m+1}(\rho) = 2^{\beth_m(\rho)}$.*

*Proof.* By the Erdős–Rado theorem,

$$\beth_{k-1}(\rho)^+ \to (\rho^+)^k_\rho$$

for every infinite cardinal $\rho$ and every finite $k \geq 2$ [4]. Since $\mu \leq \rho$, any coloring with $\mu$ colors is in particular a coloring with at most $\rho$ colors, so we also have

$$\beth_{k-1}(\rho)^+ \to (\rho^+)^k_\mu.$$

As $\lambda \leq \rho^+$, any homogeneous set of size $\rho^+$ contains one of size $\lambda$, and therefore

$$\beth_{k-1}(\rho)^+ \to (\lambda)^k_\mu.$$

Now apply Theorem 1 with $\kappa = \beth_{k-1}(\rho)^+$. □

**Corollary 4.** *For every integer $k \geq 2$ and every infinite cardinal $\mu$,*

$$f(k, \mu^+, \mu) \leq \beth_{k-1}(\mu)^+ \qquad \text{and} \qquad g(k, \mu^+, \mu) \leq \left(\beth_{k-1}(\mu)^+\right)^{\mu^+}.$$

*Proof.* Apply Corollary 3 with $\rho = \mu$ and $\lambda = \mu^+$. □

**Proposition 5.** *For every integer $k \geq 2$ and all infinite cardinals $\lambda, \mu$,*

$$f(k, \lambda, \mu) \geq \max\{\lambda, \mu^+\}.$$

*Proof.* Let $\mathcal{A}$ be a $(k, \lambda, \mu)$-family. Trivially $|\bigcup \mathcal{A}| \geq \lambda$. Also $|\bigcup \mathcal{A}| > \mu$, because if $|\bigcup \mathcal{A}| \leq \mu$, then we may color every point of $\bigcup \mathcal{A}$ injectively with at most $\mu$ colors. Since each member of $\mathcal{A}$ has size $\lambda \geq \omega > 1$, no member can then be monochromatic, contradicting $\chi(\mathcal{A}) > \mu$. □

*Remark* 6. Corollary 3 is enough to recover LouisD's proposed bounds. If one wants a bound stated only in terms of $\lambda$ and $\mu$, one may take $\rho = \max\{\lambda, \mu\}$ and obtain

$$f(k, \lambda, \mu) \leq \beth_{k-1}(\max\{\lambda, \mu\})^+, \qquad g(k, \lambda, \mu) \leq \left(\beth_{k-1}(\max\{\lambda, \mu\})^+\right)^{\lambda}.$$

This is not always sharp, but it is uniform.

## 4. Application to Erdős Problem #603

We now return to the special case from the problem page.

**Lemma 7.** *For every cardinal $\mu$ there exists a cardinal $\kappa$ such that*

$$\kappa \to (\omega)^2_\mu.$$

*One may take $\kappa = \omega$ when $\mu < \omega$, and $\kappa = (2^\mu)^+$ when $\mu \geq \omega$.*

*Proof.* If $\mu < \omega$, this is Ramsey's theorem for pairs [6]. If $\mu \geq \omega$, the Erdős–Rado theorem gives

$$(2^\mu)^+ \to (\mu^+)^2_\mu$$

[4], and since $\mu^+ \geq \omega$, this implies

$$(2^\mu)^+ \to (\omega)^2_\mu.$$

□

**Theorem 8.** *For every cardinal $\mu$ there exists a family $\mathcal{A}$ of countably infinite sets such that:*

- $|A \cap B| \neq 2$ *for all distinct* $A, B \in \mathcal{A}$,
- $\chi(\mathcal{A}) > \mu$.

*Consequently, there is no cardinal bound on the chromatic number in the current arbitrary-family formulation of Erdős Problem #603.*

*Proof.* Choose $\kappa$ as in Lemma 7. Then $\kappa \to (\omega)^2_\mu$, so Theorem 1 applies with $k = 2$ and $\lambda = \omega$. □

**Corollary 9.** *Under the current arbitrary-family reading of Problem #603, the answer is negative even to the weaker question "Is there some bound on the chromatic number?" Hence the stronger request to determine the smallest such cardinal also fails.*

*Remark* 10. Proposition 2 already gives a concrete family with forbidden intersection size 2 and chromatic number exactly $\omega$ on a countable ground set. Theorem 8 shows that as one varies the family, the chromatic number is in fact unbounded in the class of all cardinals.

## 5. The countable-family reading

The notation $(A_i)$ is sometimes read as a countable sequence $(A_n)_{n<\omega}$. Under that interpretation the answer is completely different.

**Proposition 11.** *Let $\mathcal{A} = \{A_n : n < \omega\}$ be a countable family of infinite sets. Then there exists a 2-coloring of $\bigcup_{n<\omega} A_n$ such that every $A_n$ contains both colors. In particular, if Problem #603 is interpreted as asking about countable families, then the exact answer is 2.*

*Proof.* Construct pairwise distinct points $x_n, y_n \in A_n$ recursively. Suppose that $x_m, y_m$ have been chosen for all $m < n$. Then

$$F_n = \{x_m, y_m : m < n\}$$

is finite, while $A_n$ is infinite. So we may choose distinct points

$$x_n, y_n \in A_n \setminus F_n.$$

After performing the recursion for all $n < \omega$, color every $x_n$ red and every $y_n$ blue, and color all remaining points arbitrarily. Then each $A_n$ contains both a red point and a blue point, so no $A_n$ is monochromatic. One color can never suffice for a nonempty family of nonempty sets, so the minimum is exactly 2. □

**Corollary 12.** *If $\lambda$ is infinite and $\mu \geq 2$, then every $(k, \lambda, \mu)$-family has size at least $\omega_1$. Equivalently,*

$$g(k, \lambda, \mu) \geq \omega_1.$$

*Proof.* A countable family of infinite sets is 2-colorable by Proposition 11, and therefore also $\mu$-colorable for every $\mu \geq 2$. So no countable family can be a $(k, \lambda, \mu)$-family. □

## 6. Summary

The map $X \mapsto [X]^k$ answers the two mathematical comments of LouisD from the discussion thread:

- it generalizes the original counterexample from forbidden intersection size 2 to every fixed forbidden size $k \geq 2$;
- it yields the quantitative upper bounds for $f(k, \lambda, \mu)$ and $g(k, \lambda, \mu)$ suggested in the thread, including Corollary 4.

Specializing to $k = 2$ and $\lambda = \omega$ settles the current arbitrary-family formulation of Problem #603 negatively. Under the alternative countable-family reading, the exact answer is 2.

## References


[1] T. F. Bloom, *Erdős Problem #603*, `https://www.erdosproblems.com/603`, accessed 2026-04-21.

[2] T. F. Bloom, *Discussion thread for Erdős Problem #603*, `https://www.erdosproblems.com/forum/thread/603`, accessed 2026-04-21.

[3] P. Erdős, My joint work with Richard Rado, in C. Whitehead (ed.), *Surveys in Combinatorics 1987*, London Mathematical Society Lecture Note Series 123, Cambridge University Press, Cambridge, 1987, pp. 53–80.

[4] P. Erdős and R. Rado, A partition calculus in set theory, *Bull. Amer. Math. Soc.* **62** (1956), 427–489.

[5] P. Komjáth, Set systems with finite chromatic number, *European J. Combin.* **10** (1989), no. 6, 543–549.

[6] F. P. Ramsey, On a problem of formal logic, *Proc. London Math. Soc.* (2) **30** (1930), 264–286.