

A note on Erdős Problem #603

21 April 2026

Abstract

The current formulation of Erdős Problem #603 asks for the smallest cardinal C such that, whenever \mathcal{A} is a family of countably infinite sets with $|A \cap B| \neq 2$ for all distinct $A, B \in \mathcal{A}$, one can color $\bigcup \mathcal{A}$ with at most C colors so that no member of \mathcal{A} is monochromatic. We show that, under this formulation, there is *no* such cardinal: for every cardinal μ there is a family \mathcal{A} of countably infinite sets with pairwise intersections never equal to 2 such that every coloring of $\bigcup \mathcal{A}$ with μ colors makes some $A \in \mathcal{A}$ monochromatic. Since the notation (A_i) is sometimes also used for a countable sequence, we also record the alternative reading: if the family itself is countable, then 2 colors always suffice, even without any intersection hypothesis.

1. Statement and result

The current Erdős Problems page formulates Problem #603 as follows [1]: if \mathcal{A} is a family of countably infinite sets such that $|A \cap B| \neq 2$ for all distinct $A, B \in \mathcal{A}$, determine the smallest cardinal C such that $\bigcup \mathcal{A}$ can always be colored with at most C colors and no member of \mathcal{A} is monochromatic. The page attributes the problem to Erdős [2]; for comparison, it also notes that the analogous “ $\neq 1$ ” problem has countable-color sufficiency by a theorem of Komjáth [1, 4].

The main point of this note is that the current version of Problem #603 has a negative answer.

Theorem 1. *For every cardinal μ there exists a family \mathcal{A} of countably infinite sets such that:*

- $|A \cap B| \neq 2$ for all distinct $A, B \in \mathcal{A}$;
- for every coloring $c : \bigcup \mathcal{A} \rightarrow \mu$, some $A \in \mathcal{A}$ is monochromatic.

Consequently, there is no cardinal C that works for all families in the current formulation of Problem #603.

The proof is a direct application of classical partition relations.

Lemma 2. *For every cardinal μ there is a cardinal κ such that every coloring of $[\kappa]^2$ with μ colors has a monochromatic countably infinite subset. In partition notation,*

$$\kappa \rightarrow (\omega)_\mu^2.$$

Proof. If $\mu < \omega$, we may take $\kappa = \omega$ by the infinite Ramsey theorem [5]. If $\mu \geq \omega$, we take $\kappa = (2^\mu)^+$. By the Erdős–Rado theorem,

$$(2^\mu)^+ \rightarrow (\mu^+)_\mu^2$$

for every infinite cardinal μ [3]. Since μ^+ is infinite, this implies $(2^\mu)^+ \rightarrow (\omega)_\mu^2$. □

Proof of Theorem 1. Fix a cardinal μ , and let κ be given by Lemma 2. Set

$$V = [\kappa]^2,$$

and for every countably infinite $X \subseteq \kappa$ define

$$A_X = [X]^2 \subseteq V.$$

Let

$$\mathcal{A} = \{A_X : X \in [\kappa]^\omega\}.$$

Each A_X is countably infinite because X is countably infinite.

Now take distinct $X, Y \in [\kappa]^\omega$. Then

$$A_X \cap A_Y = [X]^2 \cap [Y]^2 = [X \cap Y]^2.$$

Hence

$$|A_X \cap A_Y| = |[X \cap Y]^2|.$$

If $|X \cap Y| = n < \omega$, then $|[X \cap Y]^2| = \binom{n}{2}$, so the possible finite values are

$$0, 1, 3, 6, 10, \dots,$$

and in particular never 2. If $X \cap Y$ is infinite, then $|[X \cap Y]^2| = \omega$. Thus $|A_X \cap A_Y| \neq 2$ for all distinct $A_X, A_Y \in \mathcal{A}$.

Finally, let $c : V \rightarrow \mu$ be any coloring. This is exactly a coloring of $[\kappa]^2$ with μ colors, so by Lemma 2 there exists a countably infinite set $H \subseteq \kappa$ such that all pairs from H receive the same color. Equivalently, the set

$$A_H = [H]^2$$

is monochromatic. Therefore no coloring with μ colors can avoid monochromatic members of \mathcal{A} . \square

Corollary 3. *The smallest cardinal asked for in the current formulation of Problem #603 does not exist.*

Remark 4. It may be helpful to view the construction graph-theoretically. The ground set $V = [\kappa]^2$ is the edge set of the complete graph on κ , and each A_X is the edge set of the countably infinite complete subgraph on X . Two such edge sets intersect in the edge set of the complete graph on $X \cap Y$, so the intersection size is $\binom{n}{2}$ for some finite n or is countably infinite.

Remark 5. Already the finite proxy is not always 2-colorable. Let $V = [\{1, 2, 3, 4, 5, 6\}]^2$, and for each 3-element set $T \subseteq \{1, \dots, 6\}$ let $B_T = [T]^2$. Then distinct B_T and $B_{T'}$ intersect in 0 or 1 points, never in 2. A 2-coloring of V is the same as a red/blue coloring of the edges of K_6 , and the classical fact $R(3, 3) = 6$ implies that one of the B_T is monochromatic. So even in finite examples the answer need not be 2.

2. The countable-sequence reading

The notation (A_i) is sometimes read as a countable sequence $(A_n)_{n < \omega}$. Under that interpretation the answer is completely different.

Proposition 6. *Let $\mathcal{A} = \{A_n : n < \omega\}$ be a countable family of infinite sets. Then there is a 2-coloring of $\bigcup_{n < \omega} A_n$ such that every A_n contains both colors. In particular, under the countable-sequence interpretation of Problem #603, the minimum number of colors is exactly 2.*

Proof. Construct pairwise distinct points $x_n, y_n \in A_n$ recursively. Suppose that x_m, y_m have been chosen for all $m < n$. The set

$$F_n = \{x_m, y_m : m < n\}$$

is finite, while A_n is infinite, so we can choose distinct points

$$x_n, y_n \in A_n \setminus F_n.$$

After carrying out the recursion for all $n < \omega$, color every x_n red and every y_n blue; color all remaining points arbitrarily. Then for each n the set A_n contains the red point x_n and the blue point y_n , so A_n is not monochromatic. Since one color can never suffice for a nonempty set, the minimum is exactly 2. \square

3. Conclusion

Thus the current arbitrary-family formulation of Erdős Problem #603 has no cardinal answer at all: the required number of colors is unbounded in the class of cardinals. If one instead interprets (A_i) as a countable sequence, then the intersection condition is irrelevant and the exact answer is 2.

References

- [1] T. F. Bloom, *Erdős Problem #603*, <https://www.erdosproblems.com/603>, accessed 2026-04-21.
- [2] P. Erdős, My joint work with Richard Rado, in C. Whitehead (ed.), *Surveys in Combinatorics 1987*, London Mathematical Society Lecture Note Series 123, Cambridge University Press, Cambridge, 1987, pp. 53–80.
- [3] P. Erdős and R. Rado, A partition calculus in set theory, *Bull. Amer. Math. Soc.* **62** (1956), 427–489.
- [4] P. Komjáth, Set systems with finite chromatic number, *European J. Combin.* **10** (1989), no. 6, 543–549.
- [5] F. P. Ramsey, On a problem of formal logic, *Proc. London Math. Soc.* (2) **30** (1930), 264–286.