A note on the clique-transversal number

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Abstract

Let $\tau(G)$ denote the minimum size of a set of vertices meeting every maximal clique of size at least 2 in a graph G. Erdős, Gallai, and Tuza proved that every n-vertex graph satisfies

$$\tau(G) \le n - \sqrt{2n} + O(1).$$

A question recorded as Erdős Problem #610 asks whether this can be improved to

$$\tau(G) \le n - \omega(n)\sqrt{n} \qquad \text{or even} \qquad \tau(G) \le n - c\sqrt{n \log n}$$

for some absolute constant $c > 0$. We show that the answer is yes. In fact, if

$$T(n) := \max\{\tau(G) : |V(G)| = n\},$$

then

$$T(n) = n - \Theta(\sqrt{n \log n}).$$

The upper bound is an immediate consequence of the asymptotically sharp bound on the clique chromatic number due to Joret, Micek, Reed, and Smid, while the lower bound comes from Kim's construction of triangle-free Ramsey graphs.

1 Introduction

Throughout, a *clique transversal* of a graph G means a set of vertices meeting every inclusion-wise maximal clique of size at least 2. This is the standard convention in the clique-transversal literature; otherwise an edgeless graph would have clique-transversal number n.

The purpose of this note is to record the following consequence of known results.

Theorem 1. *There exist absolute constants $c, C > 0$ such that for all sufficiently large n,*

$$n - C\sqrt{n \log n} \le T(n) \le n - c\sqrt{n \log n}.$$

Consequently,

$$T(n) = n - \Theta(\sqrt{n \log n}),$$

and in particular Erdős Problem #610 has a positive answer to both displayed questions.

The proof of the upper bound is a one-line reduction from clique transversals to clique colourings.

2 The upper bound via clique colourings

Recall that a *clique colouring* of a graph G is a colouring of $V(G)$ such that no maximal clique of size at least 2 is monochromatic. The minimum number of colours in a clique colouring is the *clique chromatic number*, denoted $\chi_c(G)$.

Lemma 2. *For every graph G on n vertices,*

$$\tau(G) \leq n - \left\lceil \frac{n}{\chi_c(G)} \right\rceil.$$

Proof. Let $q = \chi_c(G)$ and let $V(G) = V_1 \cup \cdots \cup V_q$ be a clique colouring with q colour classes. One class, say V_1, has size at least $\lceil n/q \rceil$. Since no maximal clique of size at least 2 is monochromatic, no such maximal clique is contained in V_1. Hence every maximal clique of size at least 2 meets $V(G) \setminus V_1$, so $V(G) \setminus V_1$ is a clique transversal. Therefore

$$\tau(G) \leq n - |V_1| \leq n - \left\lceil \frac{n}{q} \right\rceil,$$

as claimed. □

We now invoke the following theorem of Joret, Micek, Reed, and Smid.

Theorem 3 (Joret–Micek–Reed–Smid [2])**.** *There exists an absolute constant $A > 0$ such that every n-vertex graph satisfies*

$$\chi_c(G) \leq A\sqrt{\frac{n}{\log n}}$$

for all sufficiently large n.

Corollary 4. *There exists an absolute constant $c > 0$ such that every n-vertex graph G satisfies*

$$\tau(G) \leq n - c\sqrt{n \log n}$$

for all sufficiently large n.

Proof. By Lemma 2 and Theorem 3,

$$\tau(G) \leq n - \frac{n}{\chi_c(G)} \leq n - \frac{1}{A}\sqrt{n \log n}.$$

Thus the claim holds with any $c < 1/A$ for all sufficiently large n. □

Since $\sqrt{\log n} \to \infty$, Corollary 4 immediately implies the weaker statement

$$\tau(G) \leq n - \omega(n)\sqrt{n} \qquad \text{with} \qquad \omega(n) = c\sqrt{\log n}.$$

3 The lower bound from triangle-free graphs

We next show that the order $\sqrt{n \log n}$ is best possible.

Lemma 5. *If G is triangle-free on n vertices, then*

$$\tau(G) = n - \alpha(G),$$

where $\alpha(G)$ denotes the independence number of G.

Proof. If G is triangle-free, then every maximal clique of size at least 2 is just an edge. Therefore a clique transversal in G is exactly a vertex cover. Let $\tau_v(G)$ denote the minimum size of a vertex cover of G. We have shown that $\tau(G) = \tau_v(G)$.

Now the complement of any vertex cover is an independent set, so $\tau_v(G) \geq n - \alpha(G)$. Conversely, if I is a maximum independent set, then $V(G) \setminus I$ meets every edge, hence is a vertex cover; therefore $\tau_v(G) \leq n - |I| = n - \alpha(G)$. Thus $\tau_v(G) = n - \alpha(G)$, and hence $\tau(G) = n - \alpha(G)$. □

We use the classical Ramsey-theoretic construction of Kim.

Theorem 6 (Kim [3]). *There exists an absolute constant $B > 0$ such that for all sufficiently large n there exists a triangle-free graph G on n vertices with*

$$\alpha(G) \leq B\sqrt{n \log n}.$$

Proof. Kim proved that there exists an absolute constant $a > 0$ such that

$$R(3,t) \geq a\frac{t^2}{\log t}$$

for all sufficiently large t, where $R(3,t)$ is the Ramsey number. Fix a sufficiently large n, and set

$$t = \left\lceil B\sqrt{n \log n} \right\rceil,$$

where $B > 0$ is chosen so large that

$$a\frac{t^2}{\log t} > n$$

for all sufficiently large n. This is possible because $t^2/\log t \asymp n$ when $t \asymp \sqrt{n \log n}$.

Now take a triangle-free graph H on $R(3,t) - 1$ vertices with no independent set of size t; such a graph exists by the definition of $R(3,t)$. Since $n < R(3,t)$, we may take any induced subgraph G of H on exactly n vertices. Then G is triangle-free and every independent set in G is also independent in H, so

$$\alpha(G) < t \leq B\sqrt{n \log n} + 1.$$

Absorbing the additive 1 into the constant gives the result. □

Corollary 7. *There exists an absolute constant $C > 0$ such that for all sufficiently large n,*

$$T(n) \geq n - C\sqrt{n \log n}.$$

Proof. Apply Theorem 6 and then Lemma 5. □

4 Conclusion

Combining Corollaries 4 and 7 proves Theorem 1.

Remark 8. *The stronger Erdős–Gallai–Tuza speculation is that if $f(n)$ denotes the minimum possible independence number in an n-vertex triangle-free graph, then every n-vertex graph G should satisfy*

$$\tau(G) \leq n - f(n).$$

The argument above does not settle that conjecture; it only shows that the worst-case order of magnitude of $n - \tau(G)$ is $\Theta(\sqrt{n \log n})$.

References

[1] P. Erdős, T. Gallai, and Zs. Tuza, *Covering the cliques of a graph with vertices*, Discrete Math. **108** (1992), 279–289.

[2] G. Joret, P. Micek, B. Reed, and M. Smid, *Tight bounds on the clique chromatic number*, Electron. J. Combin. **28** (2021), Paper No. 3.51, 8 pp.

[3] J. H. Kim, *The Ramsey number $R(3,t)$ has order of magnitude $t^2/\log t$*, Random Structures Algorithms **7** (1995), 173–207.