

Reconstructing a corrupted Erdős problem on small distances

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Abstract

The statement reproduced as Erdős Problem #662 from Erdős's 1997 paper is inconsistent as written: it compares an n -dependent quantity with a constant shell count from the triangular lattice, and even some of the numerical shell values are wrong. In this note we separate the two natural repairs.

The first repair is a *threshold-count* problem. If $E_t(X)$ denotes the number of pairs in a finite 1-separated set $X \subset \mathbb{R}^2$ whose distance is at most t , then the only coherent asymptotic formulation is to compare the average degree $2E_t(X)/|X|$ with the triangular-lattice shell count $f_\Delta(t)$. We prove that for $1 \leq t < \sqrt{2}$ one has

$$\limsup_{n \rightarrow \infty} \sup_{\substack{X \subset \mathbb{R}^2, |X|=n \\ \min\{|x-y|: x \neq y\} \geq 1}} \frac{2E_t(X)}{n} = 6 = f_\Delta(t),$$

while for every $\sqrt{2} \leq t < \sqrt{3}$ the square lattice gives average degree asymptotic to 8, so the triangular-lattice prediction fails. Thus the printed "in particular" clause with $\sqrt{3} - \varepsilon$ has the wrong breakpoint: the threshold version is true below $\sqrt{2}$ and false on $[\sqrt{2}, \sqrt{3})$.

The second repair is the one supported by the literature around Erdős and Vesztergombi. If $1 = d_1 < d_2 < \dots$ are the distinct distances of a finite set and m_i is the multiplicity of d_i , then Vesztergombi proved in 1987 that $m_1 + m_2 \leq 6n$, explicitly noting that this theorem was suggested by Erdős. Csizmadia later described this as a question of Erdős and Vesztergombi and showed that the triangular lattice is the extremizer when $d_2 = \sqrt{3}$. We conclude that the historically correct reading of the $\sqrt{3}$ -line is the classical theorem on the *two smallest distinct distances*, not a raw bound on all pairs with distance at most $\sqrt{3}$.

1 The printed statement cannot be right as written

Let

$$\Lambda_\Delta := \{a(1, 0) + b(\frac{1}{2}, \frac{\sqrt{3}}{2}) : a, b \in \mathbb{Z}\}$$

be the triangular lattice of minimal distance 1. For $t > 0$, let

$$f_\Delta(t) := \#\{u \in \Lambda_\Delta \setminus \{0\} : |u| \leq t\}.$$

Thus $f_\Delta(1) = 6$, $f_\Delta(\sqrt{3}) = 12$, $f_\Delta(2) = 18$, $f_\Delta(\sqrt{7}) = 30$, and $f_\Delta(3) = 36$.

Bloom's history page for Problem #662 records the statement from Erdős's paper [5, p. 532] essentially verbatim and notes that it does not make sense as written [1]. Besides the missing normalization by n , the numerical data themselves are corrupted: the printed value $f(3) = 18$ is wrong, and the later shell sequence $1, \sqrt{3}, 3, 5, \dots$ omits the genuine shell radii 2 and $\sqrt{7}$.

The missing normalization is already fatal. For example, if $R_m \subset \Lambda_\Delta$ is the rhombus

$$R_m := \{i(1, 0) + j(\frac{1}{2}, \frac{\sqrt{3}}{2}) : 0 \leq i, j \leq m - 1\},$$

then $|R_m| = m^2$ and the number of unit-distance pairs in R_m is

$$m(m-1) + m(m-1) + (m-1)^2 = 3m^2 - 4m + 1. \quad (1)$$

Hence even at $t = 1$ the literal pair count is of order $3n$, not of order 1.

There are therefore two natural repairs:

- (i) a *threshold-count* problem, in which one compares the normalized quantity $2E_t(X)/|X|$ with $f_\Delta(t)$; and
- (ii) a *shell-multiplicity* problem, in which one counts only the first few *distinct* distances d_1, d_2, \dots and their multiplicities.

The first repair is settled in Sections 2–3; the second is the historically correct one and is discussed in Section 4.

Before turning to the threshold repair, let us also note that the obvious *per-vertex* interpretation is false.

Proposition 1. *Let*

$$\rho_7 := \frac{1}{2 \sin(\pi/7)} \approx 1.15238.$$

For every $t > \rho_7$ there exist arbitrarily large finite sets $X \subset \mathbb{R}^2$ with mutual distances at least 1 such that some point of X has at least 7 neighbors within distance t . In particular, since $f_\Delta(t) = 6$ for every $1 \leq t < \sqrt{3}$, the per-vertex interpretation of Problem #662 is false for many $t < \sqrt{3}$.

Proof. Let H be a regular heptagon of circumradius ρ_7 and side length 1, and let o be its center. Then every distance among the 8 points $\{o\} \cup V(H)$ is at least 1: adjacent vertices of H are at distance 1, all other chords of H are larger than 1, and $|o - v| = \rho_7 > 1$ for every vertex v of H . On the other hand, o has 7 neighbors at distance exactly ρ_7 . Adding any number of points very far away preserves these properties and makes the set arbitrarily large. \square

2 The coherent threshold repair

For a finite set $X \subset \mathbb{R}^2$ and $t > 0$, let

$$E_t(X) := \#\{\{x, y\} \subset X : |x - y| \leq t\}.$$

The only coherent threshold version of the printed statement is to ask for the asymptotic maximal average degree

$$M(t) := \limsup_{n \rightarrow \infty} \sup \left\{ \frac{2E_t(X)}{n} : X \subset \mathbb{R}^2, |X| = n, \min_{x \neq y \in X} |x - y| \geq 1 \right\}. \quad (2)$$

The quantity $M(t)$ is the largest possible asymptotic number of t -neighbors per point in a large 1-separated set.

The following theorem settles this repair throughout the range relevant to the corrupted $\sqrt{3} - \varepsilon$ clause.

Theorem 2. *For the quantity $M(t)$ defined in (2) one has:*

- (a) $M(t) = 0$ for $0 < t < 1$;

- (b) $M(t) = 6 = f_{\Delta}(t)$ for every $1 \leq t < \sqrt{2}$;
(c) $M(t) \geq 8 > f_{\Delta}(t) = 6$ for every $\sqrt{2} \leq t < \sqrt{3}$.

In particular, if one rewrites the printed problem as the threshold question $M(t) \stackrel{?}{=} f_{\Delta}(t)$, then the statement is true for $1 \leq t < \sqrt{2}$ and false for $\sqrt{2} \leq t < \sqrt{3}$.

A convenient geometric lemma is that below the threshold $\sqrt{2}$ the graph of short pairs is automatically plane.

Lemma 3. *Let $A, B, C, D \in \mathbb{R}^2$ be the vertices of a convex quadrilateral listed cyclically, and assume that*

$$|AB|, |BC|, |CD|, |DA| \geq 1.$$

Then at least one diagonal has length at least $\sqrt{2}$.

Proof. Assume that $|AC| = p < \sqrt{2}$; we shall prove that $|BD| > \sqrt{2}$. After a rigid motion we may write

$$A = (0, 0), \quad C = (p, 0),$$

and, because the diagonals cross, we may write

$$B = (x, h), \quad D = (y, -k)$$

with $0 \leq x, y \leq p$ and $h, k > 0$.

From $|AB| \geq 1$ and $|BC| \geq 1$ we obtain

$$h^2 \geq 1 - x^2 \quad \text{and} \quad h^2 \geq 1 - (p - x)^2.$$

Therefore

$$h^2 \geq \max\{1 - x^2, 1 - (p - x)^2\} \geq 1 - \frac{p^2}{4},$$

since the maximum of the two displayed quantities is minimized when $x = p/2$. Exactly the same argument with D in place of B gives

$$k^2 \geq 1 - \frac{p^2}{4}.$$

Hence

$$|BD|^2 = (x - y)^2 + (h + k)^2 \geq (h + k)^2 \geq 4\left(1 - \frac{p^2}{4}\right) = 4 - p^2 > 2,$$

so $|BD| > \sqrt{2}$, as required. □

Corollary 4. *Let $1 \leq t < \sqrt{2}$, and let $G_t(X)$ be the geometric graph on a finite 1-separated set $X \subset \mathbb{R}^2$ in which two points are joined when their distance is at most t . Then $G_t(X)$ is plane. Consequently,*

$$E_t(X) \leq 3|X| - 6 \quad (|X| \geq 3).$$

Proof. If two edges of $G_t(X)$ crossed, their four endpoints would form a convex quadrilateral all of whose sides have length at least 1, while both diagonals would have length at most $t < \sqrt{2}$, contradicting Lemma 3. Therefore $G_t(X)$ is a straight-line plane graph, and the Euler bound gives $E_t(X) \leq 3|X| - 6$. □

3 Proof of Theorem 2

Proof of Theorem 2. Part (a) is immediate: if $t < 1$, no pair has distance at most t in a 1-separated set.

For part (b), let $1 \leq t < \sqrt{2}$ and let X be any finite 1-separated set of size $n \geq 3$. By Corollary 4,

$$E_t(X) \leq 3n - 6,$$

so

$$\frac{2E_t(X)}{n} \leq 6 - \frac{12}{n} < 6.$$

Taking the limsup in (2) gives $M(t) \leq 6$. On the other hand, for every such t the only distances in the triangular lattice below t are the unit distances, so $f_\Delta(t) = 6$. The rhombi R_m from (1) satisfy

$$\frac{2E_t(R_m)}{|R_m|} = \frac{2(3m^2 - 4m + 1)}{m^2} = 6 - \frac{8}{m} + \frac{2}{m^2} \xrightarrow{m \rightarrow \infty} 6.$$

Thus $M(t) \geq 6$, hence $M(t) = 6 = f_\Delta(t)$.

For part (c), let $t \in [\sqrt{2}, \sqrt{3})$. Consider the $m \times m$ square-lattice patch

$$Q_m := \{(i, j) : 0 \leq i, j \leq m - 1\} \subset \mathbb{Z}^2.$$

It is 1-separated. Every horizontal and vertical unit edge contributes to $E_t(Q_m)$, giving $2m(m - 1)$ pairs, and every unit square contributes its two diagonals of length $\sqrt{2}$, giving $2(m - 1)^2$ further pairs. Therefore

$$E_t(Q_m) \geq 2m(m - 1) + 2(m - 1)^2 = 4m^2 - 6m + 2.$$

Since $|Q_m| = m^2$,

$$\frac{2E_t(Q_m)}{|Q_m|} \geq 8 - \frac{12}{m} + \frac{4}{m^2} \xrightarrow{m \rightarrow \infty} 8.$$

Hence $M(t) \geq 8$. But $f_\Delta(t) = 6$ throughout the entire interval $[\sqrt{2}, \sqrt{3})$, because the second shell of the triangular lattice occurs only at distance $\sqrt{3}$. Therefore $M(t) > f_\Delta(t)$ on this interval. \square

The printed ‘‘in particular’’ clause therefore has an exact corrected form.

Corollary 5. *Let $t = \sqrt{3} - \varepsilon$ with $0 < \varepsilon < \sqrt{3} - 1$. In the coherent threshold formulation $M(t) \stackrel{?}{=} f_\Delta(t)$, the answer is:*

- *yes, if $\varepsilon > \sqrt{3} - \sqrt{2}$;*
- *no, if $0 < \varepsilon \leq \sqrt{3} - \sqrt{2}$.*

Equivalently, the true breakpoint is $\sqrt{2}$, not $\sqrt{3}$.

4 What the literature shows the problem really was

The range $[1, \sqrt{3}]$ is special in the triangular lattice because $\sqrt{3}$ is its second distinct distance. This is exactly how the small-distance problem appears in the literature surrounding Erdős and Vesztergombi.

Let $X \subset \mathbb{R}^2$ be a finite set with minimal distance $d_1 = 1$, and let

$$1 = d_1 < d_2 < d_3 < \dots$$

be the distinct distances occurring in X . Let $m_i = m_i(X)$ denote the number of pairs of points at distance d_i .

Vesztergombi’s 1987 paper [7] is the crucial source. It studies “small distances” in exactly this sense and proves:

- $m_j \leq 3jn$ for every fixed j ;
- $m_2 \leq 5n$;
- $m_1 + m_2 \leq 6n$.

Moreover, in Section 4 Vesztergombi writes that the theorem $m_1 + m_2 \leq 6n$ was suggested by P. Erdős, and he notes that the triangular lattice is sharp for this theorem.

This is followed by Brass’s sharp bound for the second-smallest distance,

$$m_2 < \frac{24}{7}n,$$

with best possible constant $24/7$ [3], and by Csizmadia’s paper [4], whose introduction explicitly states that it is answering a question of Erdős and Vesztergombi on the combined multiplicity of the two smallest distances. Csizmadia proves that the multiplicity of the two smallest distances is always $< 6n$, that this bound is maximized by the triangular lattice when $d_2 = \sqrt{3}$, and that if $d_2 \neq \sqrt{3}$ then substantially better bounds hold.

This identifies the historically correct reading of the corrupted $\sqrt{3}$ -line:

the relevant quantity is not the number of all pairs with absolute distance at most $\sqrt{3}$, but the combined multiplicity of the first two distinct distances.

In the triangular lattice an interior point has exactly 6 neighbors at distance 1 and 6 at distance $\sqrt{3}$, i.e. 12 neighbors at the first two shells. Dividing the theorem $m_1 + m_2 \leq 6n$ by $n/2$ produces exactly this local shell count $12 = f_\Delta(\sqrt{3})$.

By contrast, the raw threshold formulation is already a different question before the second shell appears: throughout $[\sqrt{2}, \sqrt{3})$ square-lattice patches have average degree asymptotic to 8, whereas the triangular lattice still has $f_\Delta(t) = 6$. For $t \geq \sqrt{3}$ the threshold problem no longer tracks the first two distinct distances at all. So the printed statement conflates two different notions:

- (i) *threshold counts* $E_t(X)$, whose first nontrivial breakpoint is $\sqrt{2}$; and
- (ii) *shell multiplicities* $m_1 + m_2 + \dots$, whose two-shell case is the classical Erdős–Vesztergombi problem solved by Vesztergombi and refined by Csizmadia.

There is also a genuinely open threshold question nearby. Brass formulated the following coherent conjecture: there exists $\eta > 0$ such that any n -point set in the plane with smallest distance 1 determines at most Harborth’s extremal number $\lfloor 3n - \sqrt{12n - 3} \rfloor$ of pairs at distance less than $1 + \eta$ [2, Section 5.7, Conj. 6]. This is the correct “near-unit-threshold” problem; it is different from the corrupted $\sqrt{3} - \varepsilon$ line.

5 Conclusion

Problem #662, in the form reproduced from [5], is not a meaningful statement. The mathematical content splits into two separate questions.

First, if one repairs it as a threshold problem by normalizing the pair count, then the relevant quantity is $M(t)$ from (2). We proved that $M(t) = f_\Delta(t) = 6$ for $1 \leq t < \sqrt{2}$, while $M(t) \geq 8 >$

$6 = f_{\Delta}(t)$ for $\sqrt{2} \leq t < \sqrt{3}$. Thus the specific $\sqrt{3} - \varepsilon$ clause is settled completely: the threshold version is true below $\sqrt{2}$ and false on $[\sqrt{2}, \sqrt{3})$.

Second, the literature shows that the historically correct interpretation of the appearance of $\sqrt{3}$ is the theorem on the first two distinct distances, namely $m_1 + m_2 \leq 6n$, suggested by Erdős and proved by Vesztergombi, with later refinements by Brass and Csizmadia. In this sense the “right interpretation” of the Erdős problem is already classical: it is a statement about the two smallest shells, not about all pairs below an absolute threshold.

References

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