A greedy matching proof of Erdős's two-fold residue-class problem

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Abstract

We prove that, for all sufficiently large n, one can choose one residue class $a_p \pmod p$ for every prime $p \le n$ so that every integer in $[1, n]$ lies in at least two chosen classes. The construction starts with zero classes, switches a finite set of small auxiliary primes, and then uses robust primes above a small threshold to repair the remaining one-fold covered integers. The main point is a greedy transport lemma. A positive proportion of the repairs are paired by robust primes in a short interval. The lower bound for the number of pairing edges is a fixed-modulus Green–Tao prime-triples estimate for the non-degenerate system $q, q', bq' - aq$, while all degree bounds are Selberg upper-bound sieve estimates. The local products at the switched primes are evaluated explicitly; this keeps the edge and degree constants independent of the auxiliary switched-prime set.

1 Statement and analytic inputs

Write $N = n/\log n$. All asymptotic notation is for $n \to \infty$ after the auxiliary constants and finite prime sets have been fixed.

Theorem 1.1. *For all sufficiently large n, there exist residues $a_p \pmod p$, one for each prime $p \le n$, such that every integer $m \in [1, n]$ satisfies at least two of the congruences*

$$m \equiv a_p \pmod p.$$

This is Erdős Problem #689 in Bloom's database [1]. The problem appears in Erdős's paper *Some unconventional problems in number theory* [3, p.79], and is also recorded in his 1980 survey [4, p.108].

We use three standard analytic inputs. The first is the prime number theorem in fixed arithmetic progressions.

Lemma 1.2 (Prime number theorem in fixed progressions)**.** *Let M be fixed. For each reduced residue class $r \pmod M$ and each fixed interval $(\alpha n, \beta n]$, with $0 \le \alpha < \beta \le 1$,*

$$\#\{p \in (\alpha n, \beta n] : p \equiv r \pmod M\} = (\beta - \alpha + o(1))\frac{n}{\varphi(M)\log n}.$$

See Davenport [2, Chapter 20].

The second input is the finite-complexity prime-tuples theorem of Green and Tao, used only in the special ternary linear form needed below.

Lemma 1.3 (Fixed-modulus prime triples)**.** *Fix positive integers* a, b, M *and a bounded convex region* $\Omega \subset \mathbb{R}^2$ *with piecewise smooth boundary and positive area. Let* ρ_1, ρ_2, ρ_3 *be residue classes modulo* M *such that*

$$\rho_1, \rho_2, \rho_3 \in (\mathbb{Z}/M\mathbb{Z})^\times, \qquad b\rho_2 - a\rho_1 \equiv \rho_3 \pmod{M}.$$

Then the number of pairs $(q, q') \in n\Omega \cap \mathbb{Z}^2$ *for which*

$$q, \quad q', \quad bq' - aq$$

are positive primes and

$$q \equiv \rho_1, \quad q' \equiv \rho_2, \quad bq' - aq \equiv \rho_3 \pmod{M}$$

equals

$$(\mathfrak{S}_{a,b,M,\rho} + o(1)) \operatorname{area}(\Omega) \frac{n^2}{(\log n)^3},$$

where $\mathfrak{S}_{a,b,M,\rho} > 0$ *precisely when the three forms are locally admissible. For fixed* a, b, M *the convergence is uniform over the finitely many admissible triples of residue classes.*

This is the case of Green–Tao's linear-equations-in-primes theorem corresponding to the single equation $x_3 - bx_2 + ax_1 = 0$ in three prime variables. The row vector $(a, -b, 1)$ has three nonzero entries, so the non-degeneracy condition in Green–Tao [6, Theorem 1.8] is satisfied; equivalently, the three affine-linear forms $q, q', bq' - aq$ are pairwise non-affinely-dependent. The unconditional input for this complexity is stated in [6, Corollary 1.7]. The later papers [7, 8] remove the higher-complexity hypotheses in general, but the present argument needs only this ternary case.

The third input is the standard Selberg upper-bound sieve for two affine-linear forms.

Lemma 1.4 (Two-form Selberg upper bound)**.** *There is an absolute constant* C_{Sel} *with the following property. Let* $L_1(t) = \alpha_1 t + \beta_1$ *and* $L_2(t) = \alpha_2 t + \beta_2$ *be non-proportional integral affine-linear forms. For* t *in an interval of length* X*, the number of* t *for which* $L_1(t)$ *and* $L_2(t)$ *are prime is at most*

$$C_{\mathrm{Sel}}\, \mathfrak{S}(L_1, L_2) \frac{X + 1}{(\log X)^2},$$

where $\mathfrak{S}(L_1, L_2)$ *is the usual two-form singular product. The same bound holds after imposing fixed congruence restrictions, with the corresponding local factors inserted into the singular product.*

This is the classical upper-bound consequence of Selberg's sieve; see Halberstam–Richert [9, Chapters 2 and 5] or Friedlander–Iwaniec [5, Chapter 7, Section 7.1]. We shall use only the fact that the implied constant is absolute once the local singular product is displayed.

2 Preliminary classes and residual demand

Choose numbers

$$0 < \tau < \lambda < 10^{-2}, \qquad J = \lceil \tau^{-1} \rceil.$$

Later τ will be chosen much smaller than λ. Let S be a finite set of primes, all larger than J and 3, and choose nonzero residues $b_s \pmod{s}$ for $s \in S$. Put

$$W = \prod_{s \in S} s, \qquad H(m) = \#\{s \in S : m \equiv b_s \pmod{s}\}.$$

Initially take

$$a_2 \equiv 1 \pmod 2, \qquad a_3 \equiv 0 \pmod 3, \qquad a_s \equiv b_s \pmod s \quad (s \in S),$$

and take $a_p \equiv 0 \pmod p$ for every other prime $p \le n$. Let $c_0(m)$ be the number of hits that m receives from these preliminary classes, and define the residual demand

$$D(n) = \sum_{m \le n} \max(0, 2 - c_0(m)).$$

Lemma 2.1 (Residual demand)**.** *For fixed S,*

$$D(n) = (1 + o(1)) \frac{n}{\log n}.$$

Apart from $o(n/\log n)$ demand tokens, the residual integers are exactly

$$m = 2^k uq \le n, \qquad k \ge 1,$$

where u is composed only of primes in S, $q \notin S \cup \{2, 3\}$ is prime and occurs to the first power, and $H(m) = 0$. Each such integer has exactly one preliminary hit.

Proof. An odd integer with an outside prime factor is hit by $a_2 \equiv 1 \pmod 2$ and by the zero class of that outside prime. An integer divisible by 3 is hit by the zero class modulo 3. An integer with two distinct outside prime factors is hit by two zero classes. Thus, except for integers supported on the fixed prime set $\{2, 3\} \cup S$ and integers of the form cq^j with $j \ge 2$ and c supported on that fixed set, the only one-hit integers have the displayed form. The exceptional set has size

$$O((\log n)^{|S|+2}) + O\left(\sum_{j \ge 2} n^{1/j} (\log n)^{|S|+2} \right) = o(n/\log n).$$

For fixed k and fixed S-smooth u, the prime number theorem in the fixed reduced classes modulo W gives

$$\#\{q \le n/(2^k u) : H(2^k uq) = 0\} = (1 + o(1)) \frac{n}{2^k u \log n} \prod_{s \in S:\, s \nmid u} \frac{s-2}{s-1}.$$

Summing over $k \ge 1$ and $u = \prod_{s \in S} s^{e_s}$ gives

$$\sum_{k \ge 1} 2^{-k} \prod_{s \in S} \left(\frac{s-2}{s-1} + \sum_{e \ge 1} s^{-e} \right) = 1 \cdot \prod_{s \in S} \left(\frac{s-2}{s-1} + \frac{1}{s-1} \right) = 1.$$

The interchange of the fixed-progressions asymptotic with the infinite smooth coefficient sum is justified by truncating to $2^k u \le n^{1/2}$ and then using $\sum_{r > Y,\ r \in \langle 2, S \rangle} r^{-1} = O(Y^{-\theta})$ for some $\theta > 0$ depending on the fixed set S. The complementary range contributes $o(n/\log n)$ even by the trivial prime-counting bound. This proves the lemma. $\square$

We use half-coordinates for the main residuals. Put

$$X_n = \{A = uq : 2A \le n,\ u \text{ is } S\text{-smooth},\ q \notin S \cup \{2, 3\} \text{ prime},\ H(2A) = 0\},$$

and

$$Y_n = \{B = 2^j uq : j \ge 1,\ 2B \le n,\ u \text{ is } S\text{-smooth},\ q \notin S \cup \{2, 3\} \text{ prime},\ H(2B) = 0\}.$$

Then $2A$ and $2B$ are the main residual integers.

3 Robust primes

For a prime $P > \tau n$, call P *robust* if

$$H(jP) \geq 2 \qquad (1 \leq j \leq J).$$

Since every $s \in S$ is larger than J, every $j \leq J$ is invertible modulo every $s \in S$. Let $\mathcal{R} \subset (\mathbb{Z}/W\mathbb{Z})^\times$ be the set of robust residue classes, and write

$$\delta_S = \frac{|\mathcal{R}|}{\varphi(W)}.$$

Lemma 3.1 (Many robust classes)**.** *For fixed J, by choosing S sufficiently large among primes $> J$ one can make δ_S arbitrarily close to 1.*

Proof. Choose a random unit $r \pmod W$. For each fixed $j \leq J$,

$$H(jr) = \sum_{s \in S} X_{s,j},$$

where the $X_{s,j}$ are independent Bernoulli variables and $\mathbb{P}(X_{s,j} = 1) = 1/(s-1)$. Their mean is $L_S = \sum_{s \in S} 1/(s-1)$, which can be made arbitrarily large because the sum of reciprocals of the primes diverges. Thus $\mathbb{P}(H(jr) < 2) \to 0$ for each fixed j. A union bound over $j \leq J$ gives the result. □

By Lemma 1.2, for every fixed interval $(\alpha n, \beta n] \subset (\tau n, n]$,

$$\#\{P \in (\alpha n, \beta n] : P \text{ robust}\} = (\delta_S(\beta - \alpha) + o(1)) \frac{n}{\log n}.$$

The definition of robustness makes every later switch safe: if a robust prime P is changed from the zero class to a nonzero class, then every old multiple $jP \leq n$ still has at least two hits from the auxiliary primes in S.

4 The squarefree pairing hypergraph

Set

$$I_n = (\tau n, (\tau + \lambda) n].$$

We use the squarefree coefficient cores

$$\mathcal{A} = \left\{ \prod_{s \in T} s : T \subset S \right\}, \qquad \mathcal{B} = \left\{ 2 \prod_{s \in T} s : T \subset S \right\}.$$

Let $X_n^\square$ be the subset of X_n with $A = aq$ and $a \in \mathcal{A}$, and let $Y_n^\square$ be the subset of Y_n with $B = bq'$ and $b \in \mathcal{B}$.

Define a 3-partite hypergraph $\mathcal{H}_n$ with vertex classes

$$X_n^\square, \qquad Y_n^\square, \qquad Z_n := \{P \in I_n : P \text{ robust}\},$$

and with an edge (A, B, P) whenever

$$|B - A| = P, \qquad P \nmid A.$$

The exclusion $P \nmid A$ removes the diagonal case in which the residue class modulo P would be zero and hence would not add a new hit to the paired residuals.

5 The switched-prime local products

Fix $s \in S$ and write $c_s = b_s/2 \in \mathbb{F}_s^\times$. Put

$$C_s = \mathbb{F}_s^\times \setminus \{c_s\}.$$

For a unit label residue $\delta \in \mathbb{F}_s^\times$, define

$$K_s(\delta) = \#\{Q \in C_s : Q + \delta \in C_s\} + \frac{s-1}{s}\mathbf{1}_{\delta \neq c_s} + \frac{s-1}{s}\mathbf{1}_{\delta \neq -c_s}.$$

The first term is the contribution when neither squarefree coefficient is divisible by s; the second and third terms are the contributions, with the archimedean weights $1/s$, when s divides exactly one of the two coefficients.

Lemma 5.1 (Squarefree local kernel)**.** *For every $\delta \in \mathbb{F}_s^\times$,*

$$s - 2 - \frac{2}{s} \leq K_s(\delta) \leq s - 2 - \frac{1}{s}.$$

Consequently

$$0 < c_* := \prod_{p>3}\left(1 - \frac{3}{(p-1)^2}\right) \leq \prod_{s \in S} \frac{s \min_\delta K_s(\delta)}{(s-1)^2} \leq 1.$$

Also

$$\frac{s \max_\delta K_s(\delta)}{(s-1)^2} = 1 - \frac{2}{(s-1)^2} \leq 1.$$

Proof. For $\delta \neq 0$,

$$\#\{Q \in C_s : Q + \delta \in C_s\} = \begin{cases} s-3, & \delta = \pm c_s, \\ s-4, & \delta \neq \pm c_s. \end{cases}$$

If $\delta = \pm c_s$, exactly one of the two indicator terms vanishes, so $K_s(\delta) = s-3+(s-1)/s = s-2-1/s$. Otherwise both indicators are present, and $K_s(\delta) = s - 4 + 2(s-1)/s = s - 2 - 2/s$. Finally

$$\frac{s(s-2-2/s)}{(s-1)^2} = 1 - \frac{3}{(s-1)^2}, \qquad \frac{s(s-2-1/s)}{(s-1)^2} = 1 - \frac{2}{(s-1)^2}.$$

The product defining c_* converges to a positive number because $\sum_p (p-1)^{-2} < \infty$. □

We shall also need the corresponding side-degree local sums.

Lemma 5.2 (Switched-prime factors for side degrees)**.** *Fix a side vertex $A = a_0q_0 \in X_n^\square$. At a switched prime $s \in S$, after summing over the two squarefree possibilities $s \nmid b$ and $s \mid b$ in the neighbour coefficient $b \in \mathcal{B}$, the normalized two-form local factor is at most 1. The same statement holds, with the roles of A and B interchanged, for a fixed side vertex $B \in Y_n^\square$.*

Proof. Suppose first that $s \nmid A$. Since A is residual, $A \in C_s$. If $s \nmid b$, then $B = bq'$ ranges through C_s, and the additional prime $P = B - A$ must be nonzero. There are $s - 3$ allowed residues for B. The normalized two-form local factor is $s(s-3)/(s-1)^2$. If $s \mid b$, then $B \equiv 0 \pmod{s}$ and $P \equiv -A \neq 0$; the archimedean weight $1/s$ gives the contribution $1/(s-1)$. Hence the summed factor is

$$\frac{s(s-3)}{(s-1)^2} + \frac{1}{s-1} = 1 - \frac{2}{(s-1)^2} \leq 1.$$

If $s \mid A$, then the case $s \mid b$ gives $P \equiv 0 \pmod{s}$ and contributes nothing, while the case $s \nmid b$ gives

$$\frac{s(s-2)}{(s-1)^2} = 1 - \frac{1}{(s-1)^2} \leq 1.$$

The argument for a fixed B is identical. $\square$

Lemma 5.3 (Uniform coefficient-summed two-form products)**.** *The two-form singular products needed for the degree estimates are uniformly bounded after the coefficient weights are summed. More precisely, for every side vertex $A \in X_n^{\square}$,*

$$\sum_{b \in \mathcal{B}} \frac{1}{b} \mathfrak{S}_A(b) \ll 1,$$

where $\mathfrak{S}_A(b)$ is the singular product for the two forms q' and $bq' - A$, including the residual congruence conditions for $B = bq'$. The analogous estimate holds for every fixed $B \in Y_n^{\square}$. Moreover, for every robust label P,

$$\sum_{\substack{a \in \mathcal{A},\ b \in \mathcal{B} \\ (a,b)=1}} \frac{1}{ab} \mathfrak{S}_P(a,b) \ll 1,$$

where $\mathfrak{S}_P(a,b)$ is the singular product for the two prime forms obtained from $bq' - aq = \pm P$. The implied constants are absolute; in particular they do not depend on S.

Proof. We record the local factors after the natural archimedean coefficient weights $1/b$ or $1/(ab)$ have been included. For a fixed side vertex $A = a_0 q_0 \in X_n^{\square}$, the relevant forms are q' and $bq' - A$. At primes $\ell \notin \{2,3\} \cup S$ with $\ell \nmid A$, the two forbidden residue classes are distinct and the local factor is

$$\frac{1 - 2/\ell}{(1 - 1/\ell)^2} = 1 - \frac{1}{(\ell - 1)^2} \leq 1.$$

At the unique outside prime divisor q_0 of A the two forbidden classes coincide, giving the factor $q_0/(q_0 - 1) \leq 2$. The primes 2 and 3 contribute only an absolute constant. At switched primes $s \in S$, Lemma 5.2 shows that the two local choices $s \nmid b$ and $s \mid b$, with the coefficient weight 1 and $1/s$ respectively, have total normalized factor at most 1. The Euler product for the coefficient-summed side contribution is therefore bounded by an absolute constant. The proof for a fixed $B \in Y_n^{\square}$ is the same.

For a fixed robust label P, and coprime a, b, the equation $bq' - aq = \pm P$ gives two affine-linear prime forms in one integer variable. At primes outside $\{2,3\} \cup S \cup \{P\}$ the local factor is again $1 - (\ell - 1)^{-2}$. The prime P contributes at most $P/(P-1) \leq 2$, and the primes 2 and 3 contribute only an absolute constant. At switched primes the coefficient-weighted local sum over the three admissible choices $(s \nmid a, s \nmid b)$, $(s \mid a, s \nmid b)$, and $(s \nmid a, s \mid b)$ is bounded by

$$\frac{s \max_\delta K_s(\delta)}{(s-1)^2} \leq 1$$

by Lemma 5.1; the fourth choice, $s \mid a$ and $s \mid b$, is excluded by $(a, b) = 1$ and in any case would force $P \equiv 0 \pmod{s}$. Hence the coefficient-summed label contribution is also absolutely bounded. $\square$

6 Edge and degree estimates

Proposition 6.1 (Many pairing edges)**.** *There is an absolute constant $c_E > 0$ such that, for every fixed choice of τ, λ, S,*

$$|E(\mathcal{H}_n)| \geq (c_E \delta_S \lambda + o(1)) \frac{n^2}{(\log n)^3}.$$

Proof. We count only edges with $B - A = P$; the other sign can only help. Fix squarefree coefficients $a \in \mathcal{A}$ and $b \in \mathcal{B}$. The variables q, q' range over primes in

$$2aq \leq n, \qquad 2bq' \leq n, \qquad bq' - aq \in I_n.$$

In scaled variables $x = aq/n$ and $y = bq'/n$, restrict further to $1/10 \leq x \leq 1/5$ and $y - x \in (\tau, \tau + \lambda]$. Since $\tau + \lambda < 1/10$, this rectangle-strip lies inside $[0, 1/2]^2$ and has area $\gg \lambda$. Thus its area in the (q, q')-plane is at least $c_{\mathrm{ar}} \lambda n^2/(ab)$, with an absolute $c_{\mathrm{ar}} > 0$. This interior restriction also keeps all three prime forms on scales comparable to n/a, n/b, and n, respectively, so the passage from von Mangoldt weights to prime indicators in Lemma 1.3 is harmless.

For fixed a, b, apply Lemma 1.3 after splitting into residue classes modulo $6W$ and requiring the third form $bq' - aq$ to lie in the robust residue set $\mathcal{R} \pmod W$. At all primes $\ell \notin \{2, 3\} \cup S$, the three forms have local factor

$$\frac{\ell(\ell - 2)}{(\ell - 1)^2} = 1 - \frac{1}{(\ell - 1)^2},$$

unless ℓ divides one of the fixed coefficients, which cannot happen outside S; the product over these primes is bounded below by the absolute positive constant

$$c_0 := \prod_{\ell > 2} \left(1 - \frac{1}{(\ell - 1)^2}\right) > 0,$$

up to the harmless fixed factors at 2 and 3.

It remains to compute the finite local contribution from the switched primes. Fix a robust residue vector $\pi = (\pi_s)_{s \in S} \in \mathcal{R}$. For a switched prime s, a valid pair of residues $(q, q') \pmod s$ contributes

$$\frac{1}{s^2} \left(\frac{s}{s - 1}\right)^3 = \frac{s}{(s - 1)^3}$$

relative to the archimedean factor $1/(\log n)^3$. If neither $s \mid a$ nor $s \mid b$, the number of valid local choices is $\#\{Q \in C_s : Q + \pi_s \in C_s\}$. If $s \mid a$ and $s \nmid b$, the local equation is $B \equiv \pi_s$, so there are $s - 1$ choices for the unused prime residue and the condition is $\pi_s \neq c_s$; the archimedean coefficient weight contributes $1/s$. This gives $(s - 1)s^{-1}\mathbf{1}_{\pi_s \neq c_s}$. The case $s \mid b$ and $s \nmid a$ gives $(s - 1)s^{-1}\mathbf{1}_{\pi_s \neq -c_s}$. If both coefficients are divisible by s, then $P \equiv 0 \pmod s$ and there is no contribution. Hence the coefficient-summed local factor at s is exactly

$$\frac{sK_s(\pi_s)}{(s - 1)^3}.$$

Summing over all robust residue vectors gives

$$\sum_{\pi \in \mathcal{R}} \prod_{s \in S} \frac{sK_s(\pi_s)}{(s - 1)^3} \geq |\mathcal{R}| \prod_{s \in S} \frac{s \min_\delta K_s(\delta)}{(s - 1)^3} = \delta_S \prod_{s \in S} \frac{s \min_\delta K_s(\delta)}{(s - 1)^2} \geq c_* \delta_S.$$

Combining the archimedean area lower bound, the outside singular-product lower bound, and this switched-prime product proves the asserted edge lower bound. The condition $P \nmid A$ removes only the lower-dimensional case $q = P$, hence $O(n/\log n) = o(n^2/\log^3 n)$ candidate edges. ∎

Proposition 6.2 (Uniform maximum degree)**.** *There is an absolute constant C_D such that every vertex of $\mathcal{H}_n$ has degree at most*

$$C_D \frac{n}{(\log n)^2}.$$

Proof. Fix first a side vertex $A \in X_n^{\square}$. For a coefficient $b \in \mathcal{B}$ and a sign $\sigma \in \{\pm 1\}$, neighbours $B = bq'$ are counted by simultaneous primality of

$$q', \qquad \sigma(bq' - A),$$

with $\sigma(bq' - A) \in I_n$ and with the residual congruence conditions defining $Y_n^{\square}$. The interval for q' has length $O(\lambda n/b + 1)$. By Lemma 1.4, followed by the coefficient-summed singular-product bound of Lemma 5.3, the main contribution summed over all squarefree $b \in \mathcal{B}$ is bounded by

$$\ll \frac{\lambda n}{(\log n)^2} \sum_{b \in \mathcal{B}} \frac{1}{b} \prod_{s \in S} \Theta_s(A, b),$$

where the switched-prime local factors Θ_s are those computed in Lemma 5.2. That lemma shows that the local two-choice sum over $s \mid b$ or $s \nmid b$ is at most 1, so the product sum over b is $O(1)$ with an absolute constant. The additive $+1$ terms in Lemma 1.4 contribute $O_S(1/\log^2 n)$ and are absorbed for sufficiently large n after S has been fixed. Therefore the degree of A is $O(n/\log^2 n)$, uniformly in S and A. The proof for a fixed $B \in Y_n^{\square}$ is the same.

Now fix a robust label $P \in Z_n$. For fixed $a \in \mathcal{A}$, $b \in \mathcal{B}$, and sign σ, the equation

$$bq' - aq = \sigma P$$

parametrizes q, q' by one integer variable after imposing one congruence modulo $b/(a,b)$. If $(a,b) > 1$, there are no solutions because P is coprime to every prime in S. Otherwise Lemma 1.4 applies to the two resulting prime forms. Lemma 5.3 gives the required coefficient-summed singular-product bound. Summing the Selberg main terms over the coprime coefficient pairs therefore gives $O(n/\log^2 n)$ with an absolute constant. The finitely many additive $+1$ terms from the Selberg bound are $O_S(1)$ and are absorbed for large n. Hence

$$d(P) \ll \frac{n}{(\log n)^2}$$

with an absolute implied constant. This proves the proposition. □

Corollary 6.3 (A linear-size matching)**.** *There is an absolute constant $\eta > 0$ such that $\mathcal{H}_n$ contains a matching of size at least*

$$(\eta \delta_S \lambda + o(1)) \frac{n}{\log n}.$$

Proof. A greedy algorithm in a 3-uniform hypergraph with maximum degree Δ finds a matching of size at least $|E|/(3\Delta)$, because each selected edge deletes at most 3Δ edges. Apply Propositions 6.1 and 6.2, and take $\eta = c_E/(3C_D)$. □

7 Choice of parameters and completion of the cover

Let η be the constant in Corollary 6.3. Choose $0 < \lambda < 10^{-2}$ and then choose

$$0 < \tau < \min\{\lambda, \eta\lambda/10\}.$$

Put $J = \lceil \tau^{-1} \rceil$. By Lemma 3.1, choose S with all primes larger than J such that

$$\delta_S > 1 - \frac{\eta\lambda}{10}.$$

Then

$$\delta_S(1-\tau) + \eta\delta_S\lambda > 1.$$

Let M_n be a matching supplied by Corollary 6.3. For every edge $(A, B, P) \in M_n$, set

$$a_P \equiv 2A \pmod{P}.$$

Since $B - A = \pm P$ and $P \nmid A$, this is a nonzero residue modulo P, and it covers both residual integers $2A$ and $2B$. The matching property ensures that no prime or residual token is used twice.

The number of robust primes in $(\tau n, n]$ is

$$R(n) = (\delta_S(1-\tau) + o(1))\frac{n}{\log n}.$$

By Lemma 2.1, the total residual demand is $(1 + o(1))n/\log n$. The matching uses $|M_n|$ robust primes and removes $2|M_n|$ demand tokens. Hence the number of unused robust primes minus the remaining demand is

$$\begin{aligned}(R(n) - |M_n|) - (D(n) - 2|M_n|) &= R(n) + |M_n| - D(n) \\ &\geq (\delta_S(1-\tau) + \eta\delta_S\lambda - 1 + o(1))\,\frac{n}{\log n} > 0.\end{aligned}$$

Thus there are enough unused robust primes to cover all remaining demand tokens one by one. When assigning such a prime P to a token m, avoid the robust prime divisors of m; there are at most $\lfloor \tau^{-1} \rfloor$ of them, while the surplus above is $\gg n/\log n$, so the greedy assignment is possible. Set $a_P \equiv m \pmod{P}$ for the assigned prime.

Every used robust prime has been switched to a nonzero residue. Robustness guarantees that all of its old multiples $jP \leq n$ still have at least two hits from S. Every residual demand token has received the required additional hit, and no other integer loses below two hits. This proves Theorem 1.1.

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