A note on the $n^{1-\varepsilon}$ version of an Erdős problem

Abstract

Erdős, Hajnal, and Szemerédi asked whether there is a graph G with $|G| = \chi(G) = \aleph_1$ and $f_G^1(n) \gg n$, where

$$f_G^1(n) = \min\{\alpha(G[A]) : A \subseteq V(G),\ |A| = n\}.$$

A later weaker fallback asks whether one can at least have $f_G^1(n) > n^{1-\varepsilon}$ for every $\varepsilon > 0$ and all sufficiently large n. We record the short observation that the classical Specker graph $\mathcal{G}_1(\omega_1, 3)$ already gives a positive answer to this weaker question. More precisely, every finite subgraph $H \subseteq \mathcal{G}_1(\omega_1, 3)$ on n vertices satisfies $\alpha(H) \gg n/\log n$.

1 Statement

In [2, p. 120], Erdős, Hajnal, and Szemerédi ask whether there exists an uncountably chromatic graph G of size $\aleph_1$ with $f_G^1(n) \gg n$. Their notation is

$$f_G^1(n) = \min\{\max\{|Z| : Z \subseteq A \text{ is independent}\} : A \subseteq V(G),\ |A| = n\},$$

which is the same as

$$f_G^1(n) = \min\{\alpha(G[A]) : A \subseteq V(G),\ |A| = n\}.$$

Since deleting edges cannot decrease the independence number, any lower bound for all induced subgraphs on n vertices also holds for all subgraphs on n vertices.

The purpose of this note is to isolate the following observation.

Theorem 1. *Let*

$$S = \mathcal{G}_1(\omega_1, 3).$$

Then $|V(S)| = \chi(S) = \aleph_1$. Moreover, there is a constant $C > 0$ such that for every finite subgraph $H \subseteq S$ on $n \geq 2$ vertices,

$$\alpha(H) \geq \frac{n}{C \log_2 n}.$$

Consequently, for every $\varepsilon > 0$ and all sufficiently large n, every subgraph $H \subseteq S$ on n vertices satisfies

$$\alpha(H) > n^{1-\varepsilon}.$$

2 Proof

We first identify the finite Specker graph with a finite type-graph.

Lemma 1. *For every finite ordinal $r \geq 3$, the Specker graph $\mathcal{G}_1(r, 3)$ coincides with the type-graph $G(r, 112122)$.*

Proof. Recall that the vertices of $\mathcal{G}_1(r,3)$ are the 3-subsets of $[r]$. Let

$$X = \{x_0 < x_1 < x_2\}, \qquad Y = \{y_0 < y_1 < y_2\}$$

be distinct vertices. By the definition of the 1-Specker graph in [2, Definition 1.2], after interchanging X and Y if necessary so that $x_0 < y_0$, the pair $\{X, Y\}$ is an edge exactly when

$$x_1 < y_0 < x_2 < y_1.$$

Together with the internal orderings of X and Y, this is equivalent to the union order

$$x_0 < x_1 < y_0 < x_2 < y_1 < y_2.$$

Hence the order type of (X, Y) is exactly 112122. Conversely, if $\tau(X, Y) = 112122$, then the displayed chain holds, so $\{X, Y\}$ is an edge of the Specker graph. Since the type-graph $G(r, 112122)$ declares X and Y adjacent when either $\tau(X, Y) = 112122$ or $\tau(Y, X) = 112122$, the two graphs are the same. □

Lemma 2. *The type* 112122 *is irreducible, and its block decomposition is*

$$112122 = 11 \mid 212 \mid 2.$$

In particular, it has 3 *blocks.*

Proof. A nonempty type is irreducible exactly when no proper prefix has equally many 1's and 2's. For the proper prefixes of 112122, the differences $\#\{1\text{'s}\} - \#\{2's\}$ are

$$1,\ 2,\ 1,\ 2,\ 1,$$

so the type is irreducible.

Since the type starts with 1, the block algorithm of [1, Section 2] begins with the maximal initial block of 1's, namely $B_1 = 11$. The next block must have as many 2's and 3's as B_1 has 1's and 3's, so it must satisfy $2(B_2) = 2$. Reading from left to right in the remaining string 2122, the maximal such block is $B_2 = 212$. The remaining final block is $B_3 = 2$. Thus the block decomposition is $11 \mid 212 \mid 2$. □

Proof of theorem 1. By [2, Lemma 1.1(b)],

$$\chi(\mathcal{G}_1(\omega_1, 3)) = \omega_1.$$

Also,

$$|V(S)| = \left|[\omega_1]^3\right| = \aleph_1.$$

Thus $|V(S)| = \chi(S) = \aleph_1$.

Now let $H \subseteq S$ be a finite subgraph on n vertices. Let

$$U = \bigcup_{X \in V(H)} X,$$

so $|U| \le 3n$. Write $r = |U|$, and let $\pi : U \to [r]$ be the unique order-isomorphism. Applying π to each vertex $X \in V(H)$ identifies H with a subgraph of the finite Specker graph $\mathcal{G}_1(r,3)$. By lemma 1,

$$\mathcal{G}_1(r,3) = G(r, 112122).$$

By lemma 2, the type 112122 is irreducible with $b = 3$ blocks. Therefore, Theorem 1.7 of [1] gives

$$\chi(G(r, 112122)) = \Theta(\log_{(1)} r) = \Theta(\log_2 r).$$

Hence there exists a constant $C_0 > 0$ such that for all $r \geq 2$,

$$\chi(\mathcal{G}_1(r, 3)) \leq C_0 \log_2 r.$$

Since $r \leq 3n$, after increasing the constant if necessary we obtain

$$\chi(H) \leq \chi(\mathcal{G}_1(r, 3)) \leq C \log_2 n$$

for every $n \geq 2$.

Finally, every proper coloring of H has a color class of size at least $n/\chi(H)$, so

$$\alpha(H) \geq \frac{n}{\chi(H)} \geq \frac{n}{C \log_2 n}.$$

This proves the first claim. Since $\log_2 n = o(n^{\varepsilon})$ for every fixed $\varepsilon > 0$, we have

$$\frac{n}{C \log_2 n} > n^{1-\varepsilon}$$

for all sufficiently large n, proving the second claim. □

3 Remarks

Remark 1. *The stronger linear question from [2, p. 120] remains untouched by the argument above. In fact, the same paper proves that for the countable Specker graph $\mathcal{G}_1(\omega, 3)$ one has*

$$f^1_{\mathcal{G}_1(\omega,3)}(m) = O\left(\frac{m \log \log m}{\log m}\right)$$

(see [2, Theorem 2]), so the Specker graph does not *satisfy the linear lower bound $f^1_G(n) \gg n$.*

Remark 2. *At the time of writing, public problem lists still record the $n^{1-\varepsilon}$-version as open. The point of this note is simply to isolate the short Specker-graph argument in a form convenient for checking.*

References

[1] C. Avart, B. Kay, C. Reiher, and V. Rödl, *The chromatic number of finite type-graphs*, J. Combin. Theory Ser. B **122** (2017), 877–896.

[2] P. Erdős, A. Hajnal, and E. Szemerédi, *On almost bipartite large chromatic graphs*, Ann. Discrete Math. **12** (1982), 117–123.