LOCAL ODD-CYCLE TRANSVERSALS IN GENERALIZED MYCIELSKI GRAPHS AND AN ERDŐS PROBLEM ON ALMOST-HALF INDEPENDENT SETS

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Abstract. Let $f : \mathbb{N} \to [0, \infty)$ satisfy $f(m) \to \infty$. We prove that there is a graph G of infinite chromatic number such that every finite subgraph $F \subseteq G$ on m vertices satisfies

$$\alpha(F) \geq \frac{m}{2} - f(m).$$

This answers a question of Erdős recorded as Problem #750 on the Erdős Problems site. We prove a stronger local statement. For every nondecreasing unbounded function $g : \mathbb{N} \to \mathbb{N}_0$, there is a graph G of infinite chromatic number such that every finite subgraph $F \subseteq G$ on m vertices can be made bipartite by deleting at most $g(m)$ vertices. The proof is deterministic. Its finite ingredient is a profile theorem for generalized Mycielski graphs: for every r and every such g, some r-chromatic generalized Mycielski graph has odd-cycle-transversal number at most $g(|X|)$ on every nonempty induced subgraph X.

1. Introduction

Erdős asked whether, for every function $f(m) \to \infty$, there exists a graph of infinite chromatic number such that every subgraph on m vertices contains an independent set of size at least

$$\frac{m}{2} - f(m).$$

The problem is listed as open on the Erdős Problems site as Problem #750 [2]. The known linear-error version follows from a theorem of Erdős, Hajnal and Szemerédi [1]: for each fixed $\varepsilon > 0$ there are graphs of infinite chromatic number in which every m-vertex subgraph has an independent set of size at least $(1 - \varepsilon)m/2$. The point of the question is the additive regime, where $f(m)$ may grow arbitrarily slowly.

We prove a positive answer by proving a stronger statement involving odd-cycle transversals. For a finite graph H, let

$$\operatorname{oct}(H) = \min\{|T| : T \subseteq V(H) \text{ and } H - T \text{ is bipartite}\}.$$

If $|V(H)| = m$ and $\operatorname{oct}(H) \leq t$, then H contains an independent set of size at least $(m-t)/2$, because after deleting t vertices one of the two bipartition

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classes has size at least $(m-t)/2$. Thus a local bound on oct implies a local lower bound on independence number.

The main theorem is the following.

Theorem 1.1. *Let $g : \mathbb{N} \to \mathbb{N}_0$ be nondecreasing and satisfy $g(m) \to \infty$. Then there exists a graph G with $\chi(G) = \infty$ such that every finite subgraph $F \subseteq G$ satisfies*

$$\mathrm{oct}(F) \le g(|V(F)|).$$

The desired answer to Erdős's question is an immediate corollary.

Corollary 1.2. *Let $f : \mathbb{N} \to [0, \infty)$ satisfy $f(m) \to \infty$. Then there exists a graph G with $\chi(G) = \infty$ such that every finite subgraph $F \subseteq G$ on m vertices satisfies*

$$\alpha(F) \ge \frac{m}{2} - f(m).$$

The construction is based on generalized Mycielski graphs, also called cones over graphs. Stiebitz determined the chromatic number of the recursively generated generalized Mycielski graphs; the proof uses the topological method of Lovász. We use this result as a black box. See Stiebitz [6], Sachs–Stiebitz [5], Tardif [7], and the later exposition of Müller and Stehlík [3].

The proof can be summarized as follows. The generalized Mycielski operation M_s has s levels and one apex. If a set X of vertices in $M_s(H)$ projects to a set $P \subseteq V(H)$, then deleting all copies of an odd-cycle transversal of $H[P]$, and also deleting the apex if necessary, makes $M_s(H)[X]$ bipartite. Thus

$$\mathrm{oct}(M_s(H)[X]) \le s\ \mathrm{oct}(H[P]) + 1.$$

When the right-hand side is too large for very small $|X|$, choosing s large postpones every odd cycle through the apex to length at least $2s+1$. This gives a finite profile theorem by induction on the chromatic number. A disjoint union of finite graphs with geometrically scaled profiles then gives the infinite graph.

2. Generalized Mycielski graphs

All graphs in this paper are simple and undirected. A finite subgraph need not be induced unless explicitly stated. We write $H[X]$ for the subgraph of H induced by a vertex set X.

Definition 2.1 (The operation M_s)**.** *Let H be a graph and let $s \ge 1$ be an integer. The graph $M_s(H)$ has vertex set*

$$\{0, 1, \dots, s-1\} \times V(H) \ \cup \ \{z\},$$

where z is a new vertex called the apex. For each edge $uv \in E(H)$ we add the edges

$$(0, u)(0, v)$$

and, for every $0 \le i < s-1$, the cross-level edges

$$(i, u)(i+1, v), \qquad (i, v)(i+1, u).$$

Finally, z is adjacent to every vertex $(s-1, v)$ in the top level.

Thus $M_2(H)$ is the ordinary Mycielskian of H, with the convention that $M_1(H)$ is obtained from H by adding a universal vertex. The following recursive family is the family of generalized Mycielski graphs in the sense needed here.

Definition 2.2. *Let $\mathcal{M}_2 = \{K_2\}$. For $r \geq 2$, define*

$$\mathcal{M}_{r+1} = \{M_s(H) : H \in \mathcal{M}_r \text{ and } s \geq 1\}.$$

We use the following theorem of Stiebitz.

Theorem 2.3 (Stiebitz)**.** *Every graph in $\mathcal{M}_r$ has chromatic number at least r, and hence exactly r.*

The theorem is being used only for the recursively generated class $\mathcal{M}_r$. It is not true that $\chi(M_s(H)) = \chi(H) + 1$ for every graph H and every $s \geq 3$; the recursive hypothesis is essential.

The upper bound follows directly from the definition, since an r-coloring of H extends to an $(r+1)$-coloring of $M_s(H)$ by using the original colors on every level and one new color for the apex. The lower bound is the nontrivial part of Stiebitz's theorem.

3. Odd-cycle transversals under M_s

The key estimate is a projection inequality. If $X \subseteq V(M_s(H))$, define its projection to H by

$$\pi(X) = \{v \in V(H) : (i, v) \in X \text{ for some } 0 \leq i < s\}.$$

The apex, if present, is ignored by the projection.

Lemma 3.1 (Projection inequality)**.** *Let H be a graph, let $s \geq 1$, and let $X \subseteq V(M_s(H))$. Put $P = \pi(X)$. Then*

$$\operatorname{oct}(M_s(H)[X]) \leq s \operatorname{oct}(H[P]) + \mathbf{1}_{\{z \in X\}}.$$

Proof. Let $T \subseteq P$ be an odd-cycle transversal of $H[P]$, so that $H[P \setminus T]$ is bipartite. Delete from X every copy (i, v) with $v \in T$, over all levels $0 \leq i < s$, and also delete z if $z \in X$. This deletes at most $s|T| + \mathbf{1}_{\{z \in X\}}$ vertices.

It remains to see that the surviving graph is bipartite. Let $A \cup B$ be a bipartition of $H[P \setminus T]$. Color every remaining vertex (i, v) by the side containing v. Each edge of $M_s(H) - z$ arises from an edge uv of H, either inside level 0 or between consecutive levels. Hence its endpoints receive opposite colors. Therefore the surviving graph is bipartite. □

For sets whose projection is already bipartite, the only possible obstruction is an odd cycle through the apex. The next lemma shows that such cycles are long.

Lemma 3.2 (Odd cycles through the apex)**.** *Let B be a bipartite graph and let $s \geq 1$. Every odd cycle of $M_s(B)$ containing the apex has length at least $2s+1$.*

Proof. The graph $M_s(B) - z$ is bipartite: use the bipartition of B on every level. Hence every odd cycle in $M_s(B)$ must contain z. Such a cycle consists of z, two top-level vertices, and a path in $M_s(B) - z$ between those two top-level vertices. Since the whole cycle is odd, this path has odd length.

Consider any odd path in $M_s(B) - z$ whose endpoints are in the top level $s-1$. Edges outside level 0 change the level by exactly one, while the only level-preserving edges lie inside level 0. A path from level $s-1$ to level $s-1$ that uses no level-0 edge has even length. Therefore an odd such path must use a level-0 edge. To use a level-0 edge, the path must descend from level $s-1$ to level 0, use at least one edge in level 0, and then ascend from level 0 to level $s-1$. This requires at least

$$(s-1) + 1 + (s-1) = 2s - 1$$

edges. Adding the two edges incident with z gives length at least $2s+1$. □

4. A finite profile theorem

The finite result below is the engine of the construction.

Theorem 4.1 (Finite local oct profile)**.** *Let $g : \mathbb{N} \to \mathbb{N}_0$ be nondecreasing and satisfy $g(m) \to \infty$. For every integer $r \geq 2$, there exists a graph $H \in \mathcal{M}_r$ such that every nonempty induced subgraph $H[X]$ satisfies*

$$\operatorname{oct}(H[X]) \leq g(|X|).$$

In particular, $\chi(H) = r$.

Proof. We prove the theorem by induction on r. For $r = 2$, take $H = K_2$. Since K_2 is bipartite, $\operatorname{oct}(K_2[X]) = 0 \leq g(|X|)$ for every nonempty $X \subseteq V(K_2)$.

Assume the result known for $r \geq 2$, and let $g : \mathbb{N} \to \mathbb{N}_0$ be nondecreasing and unbounded. Let

$$m_0 = \max\{m \in \mathbb{N} : g(m) = 0\},$$

with $m_0 = 0$ if the set is empty. Choose $s \geq 2$ so that

$$2s + 1 > m_0.$$

Define

$$h(m) = \max\left\{0, \left\lfloor \frac{g(m) - 1}{s} \right\rfloor\right\} \qquad (m \in \mathbb{N}).$$

Then h is nondecreasing and $h(m) \to \infty$. By the induction hypothesis, there is a graph $G \in \mathcal{M}_r$ such that

$$\operatorname{oct}(G[Y]) \leq h(|Y|) \qquad \text{for every nonempty } Y \subseteq V(G).$$

Set

$$H = M_s(G).$$

Then $H \in \mathcal{M}_{r+1}$, and hence $\chi(H) = r + 1$ by Theorem 2.3.

Let $X \subseteq V(H)$ be nonempty, and put $m = |X|$ and $P = \pi(X)$. If $P = \emptyset$, then $X \subseteq \{z\}$ and the desired bound is immediate. Hence assume $P \neq \emptyset$. Since $|P| \leq m$ and h is nondecreasing, the projection inequality gives

$$\begin{aligned}\operatorname{oct}(H[X]) &\leq s \operatorname{oct}(G[P]) + \mathbf{1}_{\{z \in X\}} \\ &\leq s\, h(|P|) + \mathbf{1}_{\{z \in X\}} \\ &\leq s\, h(m) + \mathbf{1}_{\{z \in X\}}.\end{aligned}$$

If $g(m) \geq 1$, then $h(m) = \lfloor (g(m) - 1)/s \rfloor$, and therefore

$$\operatorname{oct}(H[X]) \leq g(m) - 1 + \mathbf{1}_{\{z \in X\}} \leq g(m).$$

It remains to consider the case $g(m) = 0$. Then $m \leq m_0 < 2s + 1$, and $h(m) = 0$. Hence $h(|P|) = 0$, so $G[P]$ is bipartite. Consequently $M_s(G[P]) - z$ is bipartite. If $z \notin X$, this already implies that $H[X]$ is bipartite. If $z \in X$, then every odd cycle in $H[X]$ would be an odd cycle through the apex in $M_s(G[P])$, and by Lemma 3.2 it would have length at least $2s + 1$. This is impossible because $|X| = m < 2s + 1$. Thus $H[X]$ is bipartite and $\operatorname{oct}(H[X]) = 0 = g(m)$.

The induction is complete. □

5. The infinite graph

We now deduce the stronger infinite theorem.

Proof of Theorem 1.1. For each integer $r \geq 2$, define

$$g_r(m) = \left\lfloor \frac{g(m)}{2^r} \right\rfloor \qquad (m \in \mathbb{N}).$$

Each g_r is nondecreasing and unbounded. By Theorem 4.1, choose a graph $H_r \in \mathcal{M}_r$ such that

$$\operatorname{oct}(H_r[X]) \leq g_r(|X|)$$

for every nonempty $X \subseteq V(H_r)$. Let

$$G = \bigsqcup_{r=2}^{\infty} H_r$$

be the disjoint union of these graphs. Since $\chi(H_r) = r$ for each r, we have $\chi(G) = \infty$.

Let $F \subseteq G$ be a finite subgraph, and let $X = V(F)$ with $|X| = m$. Write

$$X_r = X \cap V(H_r).$$

Only finitely many X_r are nonempty. Since F is a subgraph of the induced graph $G[X]$, it is enough to bound $\operatorname{oct}(G[X])$. Odd-cycle transversals may be chosen componentwise, so

$$\begin{aligned}\operatorname{oct}(G[X]) &\le \sum_{r:X_r\neq\emptyset} \operatorname{oct}(H_r[X_r]) \\ &\le \sum_{r:X_r\neq\emptyset} g_r(|X_r|) \\ &\le \sum_{r:X_r\neq\emptyset} g_r(m) \\ &\le \sum_{r\ge 2} \frac{g(m)}{2^r} \le g(m).\end{aligned}$$

Therefore $\operatorname{oct}(F) \le \operatorname{oct}(G[X]) \le g(m)$, as required. □

Proof of Corollary 1.2. Let $f : \mathbb{N} \to [0,\infty)$ satisfy $f(m) \to \infty$. Define a nondecreasing integer-valued minorant of $2f$ by

$$g(m) = \max\left\{0, \left\lfloor \inf_{n\ge m} 2f(n) \right\rfloor\right\}.$$

Then $g(m) \to \infty$ and $g(m) \le 2f(m)$ for every m. Apply Theorem 1.1 to obtain a graph G of infinite chromatic number such that every finite subgraph $F \subseteq G$ on m vertices satisfies $\operatorname{oct}(F) \le g(m)$.

Deleting at most $g(m)$ vertices from F leaves a bipartite graph on at least $m - g(m)$ vertices. One side of a bipartition is an independent set of F of size at least $(m - g(m))/2$. Hence

$$\alpha(F) \ge \frac{m - g(m)}{2} \ge \frac{m}{2} - f(m),$$

as claimed. □

6. Remarks

Remark 6.1 (Finite initial values of f)**.** *Corollary 1.2 is stated for $f \ge 0$. This is the natural form of the question. If arbitrary real-valued functions are allowed, then finite initial values can create trivial obstructions. For example, every graph with an edge has a two-vertex subgraph with independence number 1, so the inequality at $m = 2$ requires $f(2) \ge 0$ for any graph containing an edge. Since the problem is asymptotic, one usually assumes f is nonnegative, or modifies finitely many initial values.*

Remark 6.2 (Why generalized Mycielski graphs appear)**.** *Graphs of large odd girth are locally bipartite in the ordinary sense, but odd girth alone controls only the first odd cycle. The proof above controls a stronger local parameter, the odd-cycle-transversal number. The operation M_s is useful because its projection to the previous graph multiplies transversal size by at most s, while choosing s large postpones every new apex-created odd cycle to length at least $2s+1$. This gives the scale separation needed for an arbitrary prescribed unbounded error function.*

Remark 6.3 (Relation to probabilistic intuition)**.** *The argument is deterministic, but it follows the same logic as a probabilistic scale-separation strategy: a witness to non-bipartiteness beyond a permitted budget is forced to occur only after the ambient size has crossed a prescribed threshold. Generalized Mycielski cones provide a rigid way to implement this idea.*

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