EXTREMAL COPRIME COVERINGS UNDER A RECIPROCAL BUDGET AND THE DICKMAN–DE BRUIJN FUNCTION

PRZEMYSLAW CHOJECKI

Abstract. Fix $C > 0$ and let N be large. Among sets $A \subseteq \{2, \dots, N\}$ of pairwise coprime integers with reciprocal budget $\sum_{a\in A} \frac{1}{a} \le C$, we ask for the choice of A that minimizes the number of integers $m \le N$ not divisible by any $a \in A$. This is Erdős Problem #783 [3].

We prove that the optimal uncovered density equals $\rho(e^C) + o(1)$ as $N \to \infty$, where ρ is the Dickman–de Bruijn function, and that this density is achieved by taking A to be a tail of primes. The upper bound follows from the classical asymptotics for smooth numbers. For the lower bound we follow a reduction to the prime case due to Tao [7], but we supply complete proofs for all reduction steps. The prime-only input is a theorem of Hildebrand [5] (with a simplified proof in Granville–Soundararajan [4]).

1. Introduction

Fix $C > 0$ and let N be large. We consider sets $A \subseteq \{2, \dots, N\}$ satisfying

(i) $\gcd(a, b) = 1$ for all distinct $a, b \in A$ (pairwise coprime), and

(ii) $\mu(A) := \sum_{a\in A} \frac{1}{a} \le C$ (a reciprocal budget).

Such a set A *covers* an integer $m \le N$ if $a \mid m$ for some $a \in A$. Write

$$U(A; N) := \#\{1 \le m \le N : \ \forall a \in A, \ a \nmid m\} \tag{1.1}$$

for the number of *uncovered* integers up to N, and

$$\sigma_N(A) := \frac{U(A; N)}{N}$$

for the uncovered density.

Definition 1.1. For $C > 0$ and $N \in \mathbb{N}$ let $\mathcal{A}_N(C)$ be the family of all pairwise coprime $A \subseteq \{2, \dots, N\}$ with $\mu(A) \le C$. Define the extremal uncovered proportion

$$F_N(C) := \inf_{A\in\mathcal{A}_N(C)} \sigma_N(A). \tag{1.2}$$

Erdős suggested that an extremal (or near-extremal) choice of A should be a tail of primes near N [3]. If one takes

$$A = \{p \in \mathbb{P} : \ y < p \le N\}$$

for a threshold $y = y(N)$, then $U(A; N)$ counts y-smooth integers $\le N$, and is governed by the Dickman–de Bruijn function ρ. The main theorem confirms that this construction is asymptotically optimal.

Date: February 20, 2026.

Dickman–de Bruijn function.

Definition 1.2. The Dickman–de Bruijn function $\rho : [0, \infty) \to \mathbb{R}$ is defined by $\rho(u) = 1$ for $0 \le u \le 1$ and, for $u > 1$, by the delay differential equation

$$u\rho'(u) + \rho(u-1) = 0,$$

together with continuity on $[0, \infty)$.

Main theorem.

Theorem 1.3 (Solution to Erdős #783)**.** *Fix* $C > 0$. *Then, as* $N \to \infty$,

$$F_N(C) = \rho(e^C) + o(1). \tag{1.3}$$

More generally, for every pairwise coprime $A \subseteq \{2, \dots, N\}$ *with* $\mu(A) = O(1)$ *one has*

$$\sigma_N(A) \;\ge\; \rho\big(e^{\mu(A)}\big) + o(1), \tag{1.4}$$

where the error is with respect to $N \to \infty$ *and depends only on an upper bound for* $\mu(A)$.

Remark 1.4 (The regime $0 \le C \le \log 2$)**.** For $1 \le u \le 2$ one has $\rho(u) = 1 - \log u$, hence $\rho(e^C) = 1 - C$ for $0 \le C \le \log 2$. Thus (1.3) recovers the overlap-free formula $F_N(C) = 1 - C + o(1)$. In this range the inequality (1.4) also follows directly from the union bound (Lemma 3.4).

2. Prime tails and the Dickman upper bound

Let $\Psi(N, y)$ denote the number of y-smooth integers $\le N$. A classical theorem of Dickman, de Bruijn and Ramaswami (see, e.g., [8, Ch. III]) asserts that for fixed $u \ge 1$,

$$\Psi\big(N, N^{1/u}\big) \;=\; N\rho(u) \;+\; O\Big(\frac{N}{\log N}\Big) \qquad (N \to \infty). \tag{2.1}$$

Lemma 2.1. *Fix* $2 \le y \le N$ *and set* $A_y(N) := \{p \in \mathbb{P} :\; y < p \le N\}$. *Then an integer* $m \le N$ *is uncovered by* $A_y(N)$ *if and only if* m *is* y*-smooth; in particular*

$$U(A_y(N); N) = \Psi(N, y). \tag{2.2}$$

Proof. The condition "no prime $p \in (y, N]$ divides m" is equivalent to "all prime factors of m are $\le y$". ☐

Theorem 2.2 (Dickman upper bound)**.** *Fix* $C > 0$. *Let* $y = y(N)$ *satisfy* $\log y = (e^{-C} + o(1)) \log N$ *as* $N \to \infty$. *Let* $A \subseteq \{2, \dots, N\}$ *be a prime tail of the form* $\{p :\; y < p \le N\}$ *with finitely many boundary primes removed so that* $\mu(A) \le C$. *Then*

$$\sigma_N(A) = \rho(e^C) + o(1),$$

and hence $F_N(C) \le \rho(e^C) + o(1)$.

Proof. By Lemma 2.1, $\sigma_N(A) = \Psi(N, y)/N$. Write $y = N^{1/u}$, so $u = \frac{\log N}{\log y} = e^C + o(1)$. Applying (2.1) (uniform under $u = e^C + o(1)$) gives $\Psi(N, y) = N\rho(e^C) + o(N)$. Removing finitely many primes changes $\mu(A)$ by $o(1)$ and changes $\sigma_N(A)$ by at most $o(1)$ by Lemma 3.4. ☐

3. The lower bound via reduction to primes

The lower bound (1.4) is proved by reducing the general pairwise coprime case to the prime case. This reduction is due to Tao [7]; we follow his scheme, but we give full proofs of all reduction steps. The one external input is the prime-only theorem below.

3.1. **Prime-only input.**

Theorem 3.1 (Prime case: Hildebrand; Granville–Soundararajan)**.** *Let P be a finite set of primes with $\mu(P) = O(1)$. Then, as $N \to \infty$,*

$$\sigma_N(P) \ \geq \ \rho\big(e^{\mu(P)}\big) + o(1), \tag{3.1}$$

where the error depends only on an upper bound for $\mu(P)$.

Remark 3.2. This is a reformulation of Hildebrand [5, Cor. 1]; see also the simplified proof of Granville–Soundararajan [4].

3.2. **Two soft tools.**

Lemma 3.3 (Pairwise coprime sets are not much denser than primes)**.** *Let A be pairwise coprime. Then for any $2 \leq y < x$,*

$$|A \cap (y, x]| \ \leq \ (\pi(x) - \pi(y)) \ + \ \pi(\sqrt{x}).$$

Proof. Every $a \in A \cap (y, x]$ is either prime, or else has a prime factor $\leq \sqrt{x}$. Because A is pairwise coprime, each prime factor can occur for at most one element of A. Hence the number of composites in $A \cap (y, x]$ is at most $\pi(\sqrt{x})$ and the primes contribute at most $\pi(x) - \pi(y)$. ☐

Lemma 3.4 (Lipschitz in reciprocal mass)**.** *For any sets $A, A' \subseteq \{2, \ldots, N\}$,*

$$|\sigma_N(A') - \sigma_N(A)| \ \leq \ \mu(A \triangle A'),$$

where $A \triangle A'$ is the symmetric difference.

Proof. By the triangle inequality it suffices to treat $A' \subseteq A$. If $m \leq N$ is uncovered by A' but not by A, then $a \mid m$ for some $a \in A \setminus A'$. Thus by the union bound,

$$0 \leq \sigma_N(A') - \sigma_N(A) \leq \sum_{a \in A \setminus A'} \frac{1}{N} \lfloor \frac{N}{a} \rfloor \leq \sum_{a \in A \setminus A'} \frac{1}{a} = \mu(A \setminus A').$$

☐

3.3. **Bonferroni truncation.**

For an integer $r \geq 1$ define the truncated inclusion–exclusion approximation

$$\sigma_{N,r}(A) := \sum_{j=0}^{r} \sum_{\substack{a_1 < \cdots < a_j \in A \\ a_1 \cdots a_j \leq N}} \frac{(-1)^j}{a_1 \cdots a_j}. \tag{3.2}$$

Lemma 3.5 (Bonferroni approximation)**.** *Let $A \subseteq \{2, \ldots, N\}$ be pairwise coprime and let $r \geq 1$. Then*

$$\sigma_N(A) = \sigma_{N,r}(A) + O\Big(\frac{\mu(A)^{r+1}}{(r+1)!}\Big) + O\Big(\frac{r e^{\mu(A)}}{\log N}\Big), \tag{3.3}$$

where the implied constants are absolute.

Proof. We may assume r is odd; the even case is similar, with inequalities reversed below. For $a \in A$, set $E_a := \{1 \leq m \leq N : \ a \mid m\}$ and let

$$1_{\mathrm{un}}(m) := 1_{m \notin \cup_{a \in A} E_a}$$

be the indicator of being uncovered. The Bonferroni inequalities give, pointwise in m,

$$\sum_{j=0}^{r} \sum_{a_1 < \cdots < a_j \in A} (-1)^j 1_{E_{a_1} \cap \cdots \cap E_{a_j}}(m) \ \leq \ 1_{\mathrm{un}}(m) \ \leq \ \sum_{j=0}^{r+1} \sum_{a_1 < \cdots < a_j \in A} (-1)^j 1_{E_{a_1} \cap \cdots \cap E_{a_j}}(m).$$

Averaging over $1 \le m \le N$ yields

$$\sum_{j=0}^{r} \sum_{a_1<\cdots<a_j\in A} \frac{(-1)^j}{N} \#(E_{a_1}\cap\cdots\cap E_{a_j}) \le \sigma_N(A) \le \sum_{j=0}^{r+1} \sum_{a_1<\cdots<a_j\in A} \frac{(-1)^j}{N} \#(E_{a_1}\cap\cdots\cap E_{a_j}). \tag{3.4}$$

Because A is pairwise coprime, one has

$$\#(E_{a_1}\cap\cdots\cap E_{a_j}) = \lfloor \frac{N}{a_1\cdots a_j} \rfloor,$$

and this vanishes unless $a_1\cdots a_j \le N$. Thus the $j \le r$ terms in (3.4) agree with $\sigma_{N,r}(A)$ up to the replacement $\lfloor N/d \rfloor = N/d + O(1)$. The $(r+1)$-level term contributes at most

$$\sum_{a_1<\cdots<a_{r+1}\in A} \frac{1}{a_1\cdots a_{r+1}} \le \frac{1}{(r+1)!}\Big(\sum_{a\in A}\frac{1}{a}\Big)^{r+1} = \frac{\mu(A)^{r+1}}{(r+1)!},$$

giving the first error term.

It remains to control the total contribution from the floor-errors. Writing $\lfloor N/d \rfloor = N/d + O(1)$ in the jth summand (for $1 \le j \le r$) produces an error bounded by

$$\frac{1}{N} \sum_{a_1<\cdots<a_j\in A:\ a_1\cdots a_j\le N} 1.$$

Fix j and $a_1,\dots,a_{j-1}$, and set $D := a_1\cdots a_{j-1}$. If there is an admissible a_j then necessarily $D \le N^{1-1/j}$ and

$$a_j \le \frac{N}{D}.$$

Moreover, since $a_j \ge N^{1/j}$ in this situation, Lemma 3.3 and the prime number theorem give

$$\#\{a \in A:\ N^{1/j} < a \le N/D\} \ll \frac{N/D}{\log(N^{1/j})} \ll \frac{j}{\log N}\frac{N}{D},$$

for N large enough (uniformly in $j \le r$). Summing over $a_1 < \cdots < a_{j-1}$ yields

$$\frac{1}{N} \sum_{a_1<\cdots<a_j\in A:\ a_1\cdots a_j\le N} 1 \ll \frac{j}{\log N} \sum_{a_1<\cdots<a_{j-1}\in A} \frac{1}{a_1\cdots a_{j-1}} \le \frac{j}{\log N}\frac{\mu(A)^{j-1}}{(j-1)!}.$$

Finally sum this over $1 \le j \le r$ to obtain an error

$$\ll \frac{1}{\log N}\sum_{j=1}^{r} \frac{j\,\mu(A)^{j-1}}{(j-1)!} \ll \frac{re^{\mu(A)}}{\log N},$$

as claimed. □

3.4. **Splitting small and large moduli.** Define the Euler product

$$\sigma_\infty(A) := \prod_{a\in A}\Big(1-\frac{1}{a}\Big) = \sum_{j\ge0}\sum_{a_1<\cdots<a_j\in A}\frac{(-1)^j}{a_1\cdots a_j}.$$

Lemma 3.6 (Splitting lemma)**.** *Let $2 < z_- < z_+ < N$ and let $r \ge 1$. Assume $A = A_1 \cup A_2$ with $A_1 \subseteq (1, z_-]$ and $A_2 \subseteq (z_+^2, N]$. Then*

$$\sigma_{N,r}(A) = \sigma_{N,r}(A_1)\sigma_{N,r}(A_2) + O\Big(\sum_{j=r+1}^{2r}\frac{\mu(A)^j}{j!}\Big) + O\Big(r\frac{\log z_-}{\log z_+}e^{\mu(A)}\Big). \tag{3.5}$$

Proof. Expanding both $\sigma_{N,r}(A)$ and $\sigma_{N,r}(A_1)\sigma_{N,r}(A_2)$ in terms of tuples $(a_1,\dots,a_j)$ produces sums of terms

$$\frac{(-1)^j}{a_1\cdots a_j}$$

with $j\le r$ and $a_1<\cdots<a_j$. In $\sigma_{N,r}(A)$ we impose the single condition $a_1\cdots a_j\le N$. In the product $\sigma_{N,r}(A_1)\sigma_{N,r}(A_2)$ we split the tuple into the subtuple from A_1 and the subtuple from A_2, and require that each subproduct be $\le N$ separately.

A term appearing in exactly one of the two expressions must be of one of the following types:

(i) it has total length j with $r+1\le j<2r$ (arising from multiplying degrees in the product);
(ii) it has length $1\le j\le r$ and $a_1\cdots a_j>N$, but both partial products (from A_1 and from A_2) are $\le N$.

The total contribution of type (i) is bounded in absolute value by

$$\sum_{j=r+1}^{2r}\ \sum_{a_1<\cdots<a_j\in A}\frac{1}{a_1\cdots a_j}\le\sum_{j=r+1}^{2r}\frac{\mu(A)^j}{j!}.$$

For type (ii), fix an integer $1\le j\le r$ and consider a term from the product $\sigma_{N,r}(A_1)\sigma_{N,r}(A_2)$ coming from a tuple $a_1<\cdots<a_j$ in A such that the A_1-subproduct and the A_2-subproduct are each $\le N$, but the global product satisfies $a_1\cdots a_j>N$ (so that this term does not appear in $\sigma_{N,r}(A)$). Since $A_2\subseteq(z_+^2,N]$ and the tuple is increasing, the last element a_j must lie in A_2 and in particular $a_j>z_+^2>z_+$.

Let j_1 be the largest index with $a_{j_1}\le z_-$ (equivalently, the number of elements of the tuple in A_1); then $a_{j_1+1},\dots,a_j\in A_2$. Write

$$D_1:=a_1\cdots a_{j_1}\le z_-^{j_1}\le z_-^r,\qquad D_2:=a_{j_1+1}\cdots a_{j-1}$$

(with the convention $D_2=1$ if $j_1=j-1$).

The A_2-subproduct constraint gives $D_2\,a_j\le N$, hence $a_j\le N/D_2$. On the other hand, the global constraint $a_1\cdots a_j>N$ gives

$$a_j>\frac{N}{a_1\cdots a_{j-1}}=\frac{N}{D_1D_2}\ \ge\ \frac{N}{z_-^rD_2}.$$

Also we have $a_j>z_+$, so altogether

$$a_j\in\Big(y,\frac{N}{D_2}\Big],\qquad y:=\max\Big(z_+,\frac{N}{z_-^rD_2}\Big).$$

By definition of y, we have $y\ge z_+$ and also $\frac{N}{D_2}\le yz_-^r$, so in fact

$$a_j\in(y,yz_-^r]\qquad\text{for some } y\ge z_+.$$

By Mertens' theorem (or by the prime number theorem plus partial summation), the harmonic sum of primes in such a short multiplicative interval obeys

$$\sum_{y<p\le yz_-^r}\frac{1}{p}\ll\log\frac{\log(yz_-^r)}{\log y}\ll\frac{r\log z_-}{\log y}\ll r\,\frac{\log z_-}{\log z_+}\qquad(y\ge z_+),$$

To pass from primes to a general pairwise coprime set A, note that any composite $a\in A\cap(y,yz_-^r]$ has a prime factor $\le\sqrt{yz_-^r}$, and distinct composites correspond to distinct primes by coprimality. Therefore there are at most $\pi(\sqrt{yz_-^r})$ such composites, and since each satisfies $a \geq y$ their total reciprocal mass is $\ll \pi(\sqrt{yz_-^r})/y \ll 1/\sqrt{y}$. Consequently,

$$\sum_{a \in A \cap (y, yz_-^r]} \frac{1}{a} \leq \sum_{y < p \leq yz_-^r} \frac{1}{p} + O\Big(\frac{1}{\sqrt{y}}\Big) \ll r\, \frac{\log z_-}{\log z_+},$$

since $y \geq z_+ \to \infty$ in our applications. Summing first over the last element a_j and then bounding the remaining $j-1$ variables by

$$\sum_{j \geq 0} \sum_{a_1 < \cdots < a_j \in A} \frac{1}{a_1 \cdots a_j} = e^{\mu(A)}$$

yields a total type (ii) contribution $\ll r\, \frac{\log z_-}{\log z_+}\, e^{\mu(A)}$. ☐

Lemma 3.7 (Pure Brun sieve for small moduli)**.** *Let $2 \leq z \leq N$, let $A \subseteq [2, z]$ be pairwise coprime, and let $r \geq 1$ with $z^r \leq N$. Then*

$$\sigma_{N,r}(A) = \sigma_\infty(A) + O\Big(\sum_{j=r+1}^{\infty} \frac{\mu(A)^j}{j!}\Big). \tag{3.6}$$

Proof. If $j \leq r$ and $a_i \leq z$ then $a_1 \cdots a_j \leq z^r \leq N$, so the constraint $a_1 \cdots a_j \leq N$ in (3.2) is redundant. Thus $\sigma_{N,r}(A)$ is the truncation of the Euler product expansion for $\sigma_\infty(A)$ at level r, and the tail is bounded by the displayed sum. ☐

3.5. **Transport stability for large moduli.**

Lemma 3.8 (Transport lemma)**.** *Let $N \geq 2$, $0 < \delta < 1$. Let A, A' be finite pairwise coprime sets of the same cardinality, and let $\phi : A \to A'$ be the order-preserving bijection. Assume $\phi(a) = e^{O(\delta)} a$ for all $a \in A$. Then*

$$\mu(A') = e^{O(\delta)} \mu(A),$$

and for each $r \geq 1$,

$$\sigma_{N,r}(A') = \sigma_{N,r}(A) + O\Big(\delta r\, e^{\mu(A) + O(\delta r)}\Big(1 + \frac{r}{\log N}\Big)\Big). \tag{3.7}$$

Proof. The mass bound is immediate from $1/\phi(a) = e^{O(\delta)}/a$. For (3.7) we compare term-by-term the expansions (3.2). Fix $1 \leq j \leq r$ and a tuple $a_1 < \cdots < a_j$ from A. Write $a_i' = \phi(a_i)$; then $a_1' < \cdots < a_j'$ and

$$\frac{1}{a_1' \cdots a_j'} = \frac{e^{O(\delta j)}}{a_1 \cdots a_j}.$$

This multiplicative perturbation produces an aggregate error

$$\sum_{j=1}^{r} e^{O(\delta j)} \delta j \sum_{a_1 < \cdots < a_j \in A} \frac{1}{a_1 \cdots a_j} \ll \delta r\, e^{\mu(A) + O(\delta r)}.$$

There is a second source of error: the cutoffs $a_1 \cdots a_j \leq N$ and $a_1' \cdots a_j' \leq N$ may differ. Such a mismatch can only occur when $a_1 \cdots a_j$ lies within a factor $e^{O(\delta j)}$ of N. More precisely, if $a_1 \cdots a_j \leq N < a_1' \cdots a_j'$ (or vice versa), then

$$N e^{-O(\delta j)} < a_1 \cdots a_j \leq N.$$

Fix j and $a_1, \dots, a_{j-1}$ and set $D := a_1 \cdots a_{j-1}$. Then a_j is constrained to lie in an interval of the form $(y, e^{O(\delta j)} y]$ with $y \geq N/D$. By Mertens' theorem (or PNT plus partial summation),

$$\sum_{y<p\leq e^\eta y} \frac{1}{p} \ll \frac{\eta}{\log y} \qquad (y \to \infty,\ \eta \in (0,1]),$$

To estimate the corresponding reciprocal mass for a general pairwise coprime A, note that

$$\sum_{a\in A\cap(y,e^\eta y]} \frac{1}{a} \leq \frac{1}{y} |A \cap (y, e^\eta y]| \leq \frac{1}{y}\big(\pi(e^\eta y) - \pi(y) + \pi(\sqrt{e^\eta y})\big) \ll \frac{\eta}{\log y} + \frac{1}{\sqrt{y}},$$

by Lemma 3.3 and the prime number theorem. Thus the harmonic mass of admissible a_j for this fixed prefix is $\ll \delta j / \log(N/D)$. Since $D \leq N^{1/j}$ in the relevant ranges, we have $\log(N/D) \gg \log N / j$ and so this mass is $\ll \delta j^2 / \log N$. Summing over all prefixes $(a_1, \dots, a_{j-1})$ contributes a factor $\ll e^{\mu(A)}$. Summing over $j \leq r$ yields an additional error

$$\ll \delta r\, e^{\mu(A)+O(\delta r)} \frac{r}{\log N}.$$

Combining the two contributions gives (3.7). □

3.6. Replacing small moduli by small primes.

Lemma 3.9 (Small moduli ⇒ small primes). *Fix $C_0 > 0$. For every $\varepsilon \in (0,1)$ there exists $z_0 = z_0(C_0, \varepsilon)$ such that for all $z \geq z_0$: whenever $A \subseteq [2, z]$ is pairwise coprime and $\mu(A) \leq C_0$, there exists a set of primes $P \subseteq (1/\varepsilon, z]$ with*

$$\sigma_\infty(P) = \sigma_\infty(A) + O(\varepsilon), \tag{3.8}$$

$$\mu(P) \leq \mu(A) + O(\varepsilon). \tag{3.9}$$

Proof. Let $\alpha := \sigma_\infty(A) \in (0,1]$ and set $T := -\log \alpha$. Since $-\log(1 - 1/a) \leq 1/a$, we have $T \leq \mu(A) \leq C_0$. By Mertens' theorem, $\sum_{p\leq z} 1/p = \log\log z + M + o(1)$, and hence for z large we can ensure

$$\sum_{1/\varepsilon<p\leq z} \frac{1}{p} \geq C_0 + 1.$$

Then one can greedily select primes $p_1 < p_2 < \cdots$ in $(1/\varepsilon, z]$ until

$$\sum_{p\in P} \frac{1}{p} \ \leq\ T\ <\ \sum_{p\in P} \frac{1}{p} + \varepsilon, \qquad P := \{p_1, \dots, p_k\}.$$

For $p \geq 1/\varepsilon$ we have $-\log(1 - 1/p) = \frac{1}{p} + O(1/p^2)$ and $\sum_{p\geq 1/\varepsilon} 1/p^2 \ll \varepsilon$. Hence

$$-\log \sigma_\infty(P) = \sum_{p\in P} -\log(1 - 1/p) = \sum_{p\in P} \frac{1}{p} + O(\varepsilon) = T + O(\varepsilon),$$

which implies (3.8). Also $\mu(P) = \sum_{p\in P} 1/p \leq T + \varepsilon \leq \mu(A) + O(\varepsilon)$, giving (3.9). □

3.7. Replacing large moduli by nearby primes.

Lemma 3.10 (Large moduli ⇒ nearby primes). *Fix $C_0 > 0$. For every $\varepsilon \in (0,1)$ and integer $r \geq 1$ there exists $Z = Z(C_0, \varepsilon, r)$ such that: whenever N is large and $A \subseteq \{2, \dots, N\}$ is pairwise coprime with $\mu(A) \leq C_0$, writing $A_2 := A \cap (Z^2, N]$, there exist*

- *a subset $\widetilde{A}_2 \subseteq A_2$ with $\mu(A_2 \setminus \widetilde{A}_2) \ll \varepsilon$,*
- *a set of primes $P_2 \subseteq (Z, \infty)$ with $|P_2| = |\widetilde{A}_2|$,*
- *an order-preserving bijection $\phi : \widetilde{A}_2 \to P_2$,*

such that $\phi(a) = e^{O(1/r^2)}a$ *for all* $a \in \widetilde{A}_2$*.*

Proof. Partition (Z, ∞) into multiplicative intervals $I_x := (x, e^{1/r^2}x]$ with x ranging over a geometric progression. For x large, the prime number theorem implies

$$\#\{p \in I_x\} \asymp \frac{x/r^2}{\log x}.$$

By Lemma 3.3 and the prime number theorem, for x large one has

$$|A \cap I_x| \ \le \ \#\{p \in I_x\} + O(\sqrt{x}).$$

Thus, after discarding at most $O(\sqrt{x})$ elements of $A \cap I_x$, we may inject the remaining elements into the primes of I_x. Do this independently for each I_x, and take $\widetilde{A}_2$ to be the union of all retained elements.

The discarded reciprocal mass from I_x is $\ll \sqrt{x} \cdot \frac{1}{x} \ll x^{-1/2}$. Summing this over the geometric progression gives total discarded mass $\ll r^2 Z^{-1/2}$, which is $\ll \varepsilon$ for Z sufficiently large (depending on ε, r). On the remaining set $\widetilde{A}_2$ we obtain an order-preserving transport to primes in the same intervals, hence $\phi(a) \in (a, e^{1/r^2}a]$ and $\phi(a) = e^{O(1/r^2)}a$. ☐

3.8. **Proof of the universal lower bound (1.4).**

Proof of (1.4). Fix $C_0 > 0$ and let $A \subseteq \{2, \dots, N\}$ be pairwise coprime with $\mu(A) \le C_0$. Fix $\varepsilon \in (0, 1)$.

Parameter choices. Choose parameters in the following order (depending only on C_0, ε):

- an integer r so large that $\sum_{j \ge r+1} C_0^j/j! \ll \varepsilon$ and $\sum_{j=r+1}^{2r} C_0^j/j! \ll \varepsilon$ and $e^{C_0}/r \ll \varepsilon$;
- a base point z_0 so large that Lemma 3.9 applies for all $z \ge z_0$ and $r^2/\sqrt{z_0} \ll \varepsilon$;
- N so large that $re^{C_0}/\log N \ll \varepsilon$ and $r^2 e^{C_0}/\log N \ll \varepsilon$ and $z_0^{r^{2r}} \le N$.

Find a sparse "middle" scale. Define the rapidly growing scales

$$z_j := z_0^{r^{2j}} \qquad (j = 0, 1, \dots, r).$$

Since $\sum_{j=1}^{r} \mu\big(A \cap (z_{j-1}, z_j]\big) \le \mu(A) \le C_0$, there exists some $j \in \{1, \dots, r\}$ such that

$$\mu\big(A \cap (z_{j-1}, z_j]\big) \ \le \ C_0/r \ \ll \ \varepsilon.$$

Set

$$z_- := z_{j-1}, \qquad z_+ := \sqrt{z_j},$$

so that $z_+^2 = z_j$ and $\log z_- / \log z_+ = 2/r^2$. Decompose

$$A = A_1 \cup A_{\text{mid}} \cup A_2, \qquad A_1 := A \cap (2, z_-], \quad A_{\text{mid}} := A \cap (z_-, z_+^2], \quad A_2 := A \cap (z_+^2, N].$$

By Lemma 3.4 we may discard the middle part at cost $O(\varepsilon)$:

$$\sigma_N(A) = \sigma_N(A_1 \cup A_2) + O(\varepsilon). \tag{3.10}$$

Factorization. Apply Lemma 3.5 to $A_1 \cup A_2$, and then apply Lemma 3.6 (with z_-, z_+) and Lemma 3.7 (to $A_1 \subseteq [2, z_-]$; note $z_-^r \le z_+^2 \le N$ by construction and N large). With our parameter choices, all resulting errors are $O(\varepsilon)$, yielding

$$\sigma_N(A) = \sigma_\infty(A_1)\, \sigma_{N,r}(A_2) + O(\varepsilon). \tag{3.11}$$

Replace A_1 by small primes. By Lemma 3.9 (applied with $z = z_-$) there exists a prime set $P_1 \subseteq (1/\varepsilon, z_-]$ such that $\sigma_\infty(P_1) = \sigma_\infty(A_1) + O(\varepsilon)$ and $\mu(P_1) \le \mu(A_1) + O(\varepsilon)$. Since $0 \le \sigma_{N,r}(A_2) \le 1$, inserting this into (3.11) yields

$$\sigma_N(A) = \sigma_\infty(P_1)\, \sigma_{N,r}(A_2) + O(\varepsilon). \tag{3.12}$$

Replace A_2 by nearby primes. Apply Lemma 3.10 with $Z = z_+$ to obtain $\widetilde{A}_2 \subseteq A_2$, a prime set P_2, and an order-preserving $\phi : \widetilde{A}_2 \to P_2$ with $\phi(a) = e^{O(1/r^2)}a$ and $\mu(A_2 \setminus \widetilde{A}_2) \ll \varepsilon$. Lemma 3.4 gives $\sigma_{N,r}(\widetilde{A}_2) = \sigma_{N,r}(A_2) + O(\varepsilon)$. Then Lemma 3.8 with $\delta = 1/r^2$ gives

$$\sigma_{N,r}(P_2) = \sigma_{N,r}(\widetilde{A}_2) + O(\varepsilon).$$

Insert this into (3.12) to obtain

$$\sigma_N(A) = \sigma_\infty(P_1)\,\sigma_{N,r}(P_2) + O(\varepsilon). \tag{3.13}$$

Reassemble and compare with a prime set. Let $P := P_1 \cup P_2$, a set of primes. Repeating the argument that led to (3.11) (now for P in place of A) shows that

$$\sigma_N(P) = \sigma_\infty(P_1)\,\sigma_{N,r}(P_2) + O(\varepsilon),$$

and comparing with (3.13) yields

$$\sigma_N(A) = \sigma_N(P) + O(\varepsilon). \tag{3.14}$$

Moreover, by construction,

$$\mu(P) = \mu(P_1) + \mu(P_2) \le \mu(A_1) + \mu(A_2) + O(\varepsilon) \le \mu(A) + O(\varepsilon).$$

Invoke the prime case. Apply Theorem 3.1 to P:

$$\sigma_N(P) \ge \rho(e^{\mu(P)}) + o(1).$$

Since $\mu(P) = \mu(A) + O(\varepsilon)$ and $\rho(e^t)$ is uniformly continuous for $t \in [0, C_0 + 1]$, we have $\rho(e^{\mu(P)}) = \rho(e^{\mu(A)}) + O(\varepsilon)$. Insert into (3.14) to get

$$\sigma_N(A) \ge \rho(e^{\mu(A)}) + o(1) - O(\varepsilon).$$

As $\varepsilon > 0$ was arbitrary, this proves (1.4). □

Remark 3.11 (Finite-N perturbations)**.** For fixed (N, C) the literal "largest primes until the budget is exhausted" construction need not be exactly optimal: small boundary swaps can improve coverage because of floor effects $\lfloor N/p \rfloor$. Theorem 1.3 implies that such finite-N effects do not change the asymptotic optimum: they affect $F_N(C)$ only at the $o(1)$ level as $N \to \infty$ with C fixed.

Proof of Theorem 1.3. Combine (1.4) with the upper bound from Theorem 2.2. □

Appendix A. A Selberg-sieve weighted alternative to Bonferroni truncation

The proof of the universal lower bound used the sharp depth-r Bonferroni/Brun truncation $\sigma_{N,r}(A)$ (cf. Lemma 3.5). This is completely adequate for the $o(1)$ asymptotics as $N \to \infty$ with $\mu(A) = O(1)$. However, for sharper *explicit* finite-N error terms one can replace that truncation by a Selberg-sieve (or more generally Rosser–Iwaniec β-sieve) *weighted* approximation of level D, which avoids factorial tails and produces errors depending on the sieve level D and on a single "sieve parameter" $s = \frac{\log D}{\log z}$ (where z is the size of the largest modulus being sieved).

We record a formulation adapted to the present "pairwise coprime moduli" setting. Standard references include Selberg's original paper [12] and the books [11, 10, 9].

A.1. **Selberg quadratic weights.** For a finite pairwise coprime set A define the (squarefree) multiplicative semigroup generated by A,

$$\mathcal{D}(A) \ := \ \Big\{ \prod_{a\in S} a \ : \ S \subseteq A \Big\}.$$

(Thus each $d \in \mathcal{D}(A)$ corresponds to a subset $S \subseteq A$, and $\omega_A(d) := |S|$ is well-defined.)

Fix a sieve level $D \geq 1$. Given coefficients $\lambda = (\lambda_d)_{d\in\mathcal{D}(A)}$ supported on $\{d \leq D\}$ with $\lambda_1 = 1$, define Selberg's quadratic weight

$$w_\lambda(n) \ := \ \Big(\sum_{\substack{d\in\mathcal{D}(A)\\ d|n\\ d\leq D}} \lambda_d \Big)^2 \qquad (n \in \mathbb{N}).$$

Lemma A.1 (Selberg majorant)**.** *For every* $n \in \mathbb{N}$ *one has*

$$\mathbf{1}_{\forall a\in A,\ a\nmid n} \ \leq \ w_\lambda(n).$$

Consequently,

$$\sigma_N(A) \ \leq \ \Sigma_{N,D}(A;\lambda) \ := \ \frac{1}{N}\sum_{n\leq N} w_\lambda(n). \tag{A.1}$$

Proof. If n is uncovered by A, then no $a \in A$ divides n, hence the only element of $\mathcal{D}(A)$ dividing n is 1. Therefore the inner sum equals $\lambda_1 = 1$ and $w_\lambda(n) = 1$. If n is covered by A, the left-hand side is 0 while $w_\lambda(n) \geq 0$. Averaging over $n \leq N$ gives (A.1). ☐

Expanding the square yields the explicit quadratic form

$$\Sigma_{N,D}(A;\lambda) = \sum_{\substack{d_1,d_2\in\mathcal{D}(A)\\ d_1,d_2\leq D}} \lambda_{d_1}\lambda_{d_2}\,\frac{1}{N}\Big\lfloor \frac{N}{\operatorname{lcm}(d_1,d_2)} \Big\rfloor. \tag{A.2}$$

The key point for "transport" arguments (as in Lemma 3.8) is that the kernel $\operatorname{lcm}(d_1,d_2)^{-1}$ is smooth under multiplicative perturbations of the moduli, and the condition $d \leq D$ is a *product size* cutoff rather than a *depth* cutoff.

A.2. **The Selberg optimum and the small-sieve fundamental lemma.** Define the multiplicative local density $g(a) := 1/a$ for $a \in A$ and set

$$h(a) := \frac{g(a)}{1-g(a)} = \frac{1/a}{1-1/a} = \frac{1}{a-1}.$$

Extend h multiplicatively to $\mathcal{D}(A)$:

$$h(d) := \prod_{a|d} \frac{1}{a-1} \qquad (d \in \mathcal{D}(A)).$$

Define Selberg's sieve sum at level D by

$$G_D(A) := \sum_{\substack{d\in\mathcal{D}(A)\\ d\leq D}} h(d) = \sum_{\substack{d\in\mathcal{D}(A)\\ d\leq D}} \prod_{a|d} \frac{1}{a-1}. \tag{A.3}$$

Note that $G_D(A)$ increases with D and

$$\lim_{D\to\infty} G_D(A) = \prod_{a\in A}\Big(1 + \frac{1}{a-1}\Big) = \prod_{a\in A} \frac{a}{a-1} = \frac{1}{\sigma_\infty(A)}.$$

Proposition A.2 (Selberg upper bound sieve (coprime moduli form))**.** *For every $D \le N$ one has*

$$\sigma_N(A) \;\le\; \frac{1}{G_D(A)} \;+\; O\Big(\frac{D}{N}\Big), \tag{A.4}$$

with an absolute implied constant.

Proof (reference-based, with the specialisation spelled out). This is the classical Selberg upper bound sieve (see [11, §§ 4–5], [10, §§ 2.1–2.3], or [9, Ch. 6]), applied to the trivial sequence $a_n \equiv 1$ for $1 \le n \le N$ and the pairwise coprime sieving moduli A.

Concretely: by Lemma A.1 it suffices to choose λ supported on $d \le D$ to minimise the quadratic form (A.2). Selberg's optimisation yields a choice of λ (depending on A and D) for which the main term in (A.2) equals $1/G_D(A)$, where $G_D(A)$ is defined in (A.3). The difference between $\lfloor N/\mathrm{lcm}(d_1, d_2)\rfloor/N$ and $1/\mathrm{lcm}(d_1, d_2)$ produces the standard remainder term in Selberg's sieve, which for the sequence $a_n \equiv 1$ is bounded by $O(D/N)$ (since the relevant divisor sums involve at most D integers $d \le D$, and each remainder term $\lfloor N/d\rfloor - N/d$ has size ≤ 1). This gives (A.4). □

Remark A.3 (Lower-bound weights and replacing Bonferroni)**.** There is also a companion *lower-bound* sieve (Rosser–Iwaniec "β-sieve") that constructs coefficients λ_d^- (supported on $d \le D$) giving a lower estimate for $\sigma_N(A)$. In the standard "dimension 1" situation (sieving by moduli $A \subseteq [2, z]$) one introduces

$$s := \frac{\log D}{\log z},$$

and the β-sieve supplies explicit functions $0 \le f(s) \le 1 \le F(s)$ with $f(s), F(s) \to 1$ as $s \to \infty$ such that

$$\sigma_\infty(A)\, f(s) \;\lesssim\; \sigma_N(A) \;\lesssim\; \sigma_\infty(A)\, F(s),$$

up to an error $O(D/N)$ (and mild uniformity hypotheses that hold in our "coprime moduli" model); see [11, §§ 11–13], [10, §§ 3–5], or [9, Chs. 9–11]. Taking $s \gg \log(1/\varepsilon)$ then gives

$$\sigma_N(A) = \sigma_\infty(A) + O(\varepsilon) + O(D/N),$$

which can be viewed as a weighted-sieve replacement of Lemma 3.5 that avoids the factorial tail $\mu(A)^{r+1}/(r+1)!$ coming from depth truncation.

Remark A.4 (Where this helps in the present argument)**.** If one wants sharper explicit finite-N errors (or a cleaner separation of parameters), one can rerun the "discard the middle, split small/large, replace by primes, transport" pipeline using Selberg/ -sieve weights in place of the truncated inclusion–exclusion sums $\sigma_{N,r}$. The stability under small multiplicative perturbations $a \mapsto a' = e^{o(1)}a$ is particularly natural at the level of (A.2). This refinement does *not* change the limiting constant $\rho(e^C)$, but it can improve the dependence of the error terms on the auxiliary parameters.

REFERENCES

[1] N. G. de Bruijn, *On the number of positive integers $\le x$ and free of prime factors $> y$*, Indagationes Mathematicae **13** (1951), 50–60.

[2] K. Dickman, *On the frequency of numbers containing prime factors of a certain relative magnitude*, Arkiv för Matematik, Astronomi och Fysik **22A** (10) (1930), 1–14.

[3] P. Erdős, *Problems and results on combinatorial number theory*, in *A survey of combinatorial theory* (Proc. Internat. Sympos., Colorado State Univ., Fort Collins, Colo., 1971), North-Holland, Amsterdam, 1973, pp. 117–138.

[4] A. Granville and K. Soundararajan, *The number of unsieved integers up to x*, Acta Arith. **115** (2004), no. 4, 305–328.

[5] A. Hildebrand, *Quantitative mean value theorems for nonnegative multiplicative functions. II*, Acta Arith. **48** (1987), no. 3, 209–260.

[6] V. Ramaswami, *On the number of positive integers less than x and free of prime divisors greater than x^c*, Bull. Amer. Math. Soc. **55** (1949), 1122–1127.

[7] T. Tao, *Sieving by coprime numbers*, note (February 19, 2026), available at `https://terrytao.wordpress.com/wp-content/uploads/2026/02/erdos783-1.pdf`.

[8] G. Tenenbaum, *Introduction to analytic and probabilistic number theory*, Cambridge Studies in Advanced Mathematics, 46, Cambridge University Press, Cambridge, 1995.

[9] J. Friedlander and H. Iwaniec, *Opera de Cribro*, American Mathematical Society Colloquium Publications, 57, American Mathematical Society, Providence, RI, 2010.

[10] G. Greaves, *Sieves in Number Theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge, vol. 43, Springer-Verlag, Berlin, 2001.

[11] H. Halberstam and H.-E. Richert, *Sieve Methods*, London Mathematical Society Monographs, 4, Academic Press, London–New York, 1974.

[12] A. Selberg, *Sieve methods*, in *Proceedings of Symposia in Pure Mathematics, Vol. 20* (Number Theory Institute, 1969), Amer. Math. Soc., Providence, RI, 1971, pp. 311–351.