A Power Lower Bound for Runs of Distinct Prime Gaps

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Abstract

Let p_n denote the nth prime and put $d_n = p_{n+1} - p_n$. Let $h(x)$ be the largest integer H for which there is some $n < x$ such that

$$d_n, d_{n+1}, \ldots, d_{n+H-1}$$

are pairwise distinct. Erdős asked for estimates on $h(x)$, noting that Brun's sieve implies only that $h(x) \to \infty$. We prove the unconditional lower bound

$$h(x) \gg (\log x)^{1/3}.$$

Consequently, for every fixed $c < 1/3$, one has $h(x) > (\log x)^c$ for all sufficiently large x. The proof combines a first-moment bound for the total length of a block of consecutive prime gaps with the Selberg upper-bound sieve and an average bound for singular series over fixed affine-linear families.

1 Introduction

Let $p_1 = 2 < p_2 = 3 < \cdots$ be the sequence of primes, and define

$$d_n = p_{n+1} - p_n.$$

For real $x \geq 1$, define $h(x)$ to be the largest integer H for which there exists an index $n < x$ such that

$$d_n, d_{n+1}, \ldots, d_{n+H-1}$$

are pairwise distinct. Erdős asked for estimates for $h(x)$, and in particular asked whether $h(x) > (\log x)^c$ for some fixed $c > 0$; see [2] and the modern problem record [1].

The purpose of this note is to record the following power lower bound.

Theorem 1. *There is an absolute constant $c_0 > 0$ such that, for all sufficiently large x,*

$$h(x) \geq c_0 (\log x)^{1/3}.$$

In particular, for every fixed $c < 1/3$,

$$h(x) > (\log x)^c$$

for all sufficiently large x.

The idea is simple. We look at blocks of H consecutive prime gaps whose total length is at most a constant multiple of $H \log N$. A positive proportion of starting positions have this property. If such a block has two equal gaps, then five explicitly related numbers must be prime:

$$m, \quad m+a, \quad m+a+q, \quad m+b, \quad m+b+q.$$

For fixed a, b, q, the Selberg sieve gives an upper bound of the expected order $N(\log N)^{-5}$, up to the singular series. Averaging the singular series over the three parameters a, b, q then shows that the number of bad blocks is

$$\ll \frac{NY^3}{(\log N)^5}$$

when all offsets are at most Y. Since in our application $Y \asymp H \log N$, this is smaller than the number of available starting positions as soon as H is a sufficiently small multiple of $(\log N)^{1/3}$.

The proof uses only upper-bound sieve input. It does not use the full condition that the two equal gaps are consecutive prime gaps; it only uses the weaker fact that the corresponding endpoints are prime. This is the source of the exponent $1/3$ in the present argument.

2 Sieve inputs

For a finite set $\mathcal{H}$ of integers, write

$$\nu_p(\mathcal{H}) = \#\{h \bmod p : h \in \mathcal{H}\}$$

and define the singular series

$$\mathfrak{S}(\mathcal{H}) = \prod_p \left(1 - \frac{\nu_p(\mathcal{H})}{p}\right)\left(1 - \frac{1}{p}\right)^{-\#\mathcal{H}}.$$

If $\mathcal{H}$ is inadmissible, this product is zero.

We shall use the following standard upper-bound sieve estimate.

Lemma 2 (Selberg upper-bound sieve)**.** *Fix $r \geq 1$. Uniformly for $N \geq 3$ and for distinct integers $h_1, \ldots, h_r$ with $|h_i| \leq N$, one has*

$$\#\{m \leq N : m+h_1, \ldots, m+h_r \in \mathbb{P}\} \ll_r 1 + \frac{\mathfrak{S}(\{h_1, \ldots, h_r\})N}{(\log N)^r}.$$

Proof. This is the usual Selberg upper-bound sieve for prime r-tuples; for example, see Friedlander–Iwaniec [3, Theorem 7.16] or Lichtman–Teräväinen [4, Lemma 2.3]. These references give the displayed bound, without the harmless additive constant, for admissible tuples. If the tuple is inadmissible, then some prime $p \leq r$ occupies all residue classes among the shifts. Any solution must therefore have $m + h_i = p$ for at least one i, giving $O_r(1)$ exceptional solutions. This accounts for the additive constant. □

We shall also use an average bound for singular series over affine-linear families. The form below is a direct special case of Lichtman–Teräväinen [4, Corollary 2.6].

Lemma 3 (Average singular series for affine families)**.** *Let* $r, m \geq 1$ *and* $A \geq 1$ *be fixed. Let*

$$L_j(\mathbf{u}) = c_j + \sum_{i=1}^{m} a_{ij} u_i \qquad (1 \leq j \leq r)$$

be affine-linear forms with integer coefficients satisfying $|c_j|, |a_{ij}| \leq A$. *Then, uniformly for* $Y \geq 1$,

$$\sum_{\mathbf{u} \in [0,Y]^m \cap \mathbb{Z}^m} \mathfrak{S}(\{L_1(\mathbf{u}), \ldots, L_r(\mathbf{u})\}) \ll_{r,m,A} Y^m,$$

provided the repeated values, if any, are deleted before the singular series is evaluated. Equivalently, after splitting into the finitely many coincidence strata among the L_j, *the same bound holds on each stratum with the resulting distinct forms.*

Proof. When the forms are distinct as functions and we restrict to points where their values are distinct, this is exactly [4, Corollary 2.6]. If two or more values coincide for some choices of $\mathbf{u}$, the coincidence condition cuts out one of finitely many affine sublattices of dimension at most $m-1$; on each such stratum we delete repeated forms and apply the same corollary to the remaining fixed affine-linear family. Since there are only $O_{r,m,A}(1)$ possible coincidence patterns, summing over the strata gives the asserted bound. □

We shall need the following immediate consequences of Lemmas 2 and 3.

Lemma 4. *Let* $L = \log N$ *and let* $2 \leq Y \leq N$. *Then*

$$\sum_{\substack{0<a<b\leq Y,\ 1\leq q\leq Y \\ b\neq a+q}} \#\{m \leq N : m, m+a, m+a+q, m+b, m+b+q \in \mathbb{P}\} \ll Y^3 + \frac{NY^3}{L^5}, \quad (1)$$

$$\sum_{\substack{1\leq b,q\leq Y \\ b\neq q}} \#\{m \leq N : m, m+q, m+b, m+b+q \in \mathbb{P}\} \ll Y^2 + \frac{NY^2}{L^4}, \quad (2)$$

$$\sum_{1\leq a,q\leq Y} \#\{m \leq N : m, m+a, m+a+q, m+a+2q \in \mathbb{P}\} \ll Y^2 + \frac{NY^2}{L^4}, \quad (3)$$

$$\sum_{1\leq q\leq Y} \#\{m \leq N : m, m+q, m+2q \in \mathbb{P}\} \ll Y + \frac{NY}{L^3}. \quad (4)$$

All implied constants are absolute.

Proof. For (1), apply Lemma 2 with the five distinct shifts

$$0,\ a,\ a+q,\ b,\ b+q.$$

The distinctness follows from $a > 0$, $a < b$, $q > 0$, and $b \neq a+q$. Summing the additive 1 in Lemma 2 gives $O(Y^3)$, while Lemma 3, applied to the affine-linear family in the variables (a, b, q), gives

$$\sum_{\substack{0<a<b\leq Y,\ 1\leq q\leq Y \\ b\neq a+q}} \mathfrak{S}(\{0, a, a+q, b, b+q\}) \ll Y^3.$$

This proves (1).

The estimates (2), (3), and (4) are identical applications of Lemmas 2 and 3 to the affine-linear families

$$\{0,q,b,b+q\}, \qquad \{0,a,a+q,a+2q\}, \qquad \{0,q,2q\},$$

respectively. $\square$

3 Proof of the main theorem

We first prove the lower bound along the sequence $M=\pi(N)$.

Proposition 5. *Let $M=\pi(N)$. There is an absolute constant $\eta>0$ such that, for all sufficiently large N,*

$$h(M)\geq \eta(\log N)^{1/3}.$$

Proof. Put

$$L=\log N, \qquad H=\lfloor \eta L^{1/3}\rfloor,$$

where $\eta>0$ will be chosen sufficiently small. Let $A\geq 10$ be a fixed sufficiently large constant, and put

$$Y=AHL.$$

For $1\leq n\leq M-H$, define the total length of the H-gap block starting at n by

$$S_n=p_{n+H}-p_n=\sum_{j=0}^{H-1} d_{n+j}.$$

Then

$$\begin{aligned}\sum_{n\leq M-H} S_n &= \sum_{n\leq M-H}\sum_{j=0}^{H-1} d_{n+j}\\ &\leq H\sum_{m\leq M-1} d_m = H(p_M-p_1)\leq HN.\end{aligned}$$

Hence, by Markov's inequality,

$$\#\{n\leq M-H : S_n>Y\}\leq \frac{HN}{Y}=\frac{N}{AL}.$$

By the prime number theorem, $M\sim N/L$, and $H=o(N/L)$. Thus, if A is fixed sufficiently large, then for all sufficiently large N the set

$$\mathcal{G}=\{1\leq n\leq M-H : S_n\leq Y\}$$

satisfies

$$\#\mathcal{G}\geq c_1\frac{N}{L} \tag{5}$$

for some absolute constant $c_1>0$.

Call a starting position $n \in \mathcal{G}$ bad if the block

$$d_n, d_{n+1}, \ldots, d_{n+H-1}$$

contains a repeated value. We shall show that, if η is sufficiently small, the number of bad positions in $\mathcal{G}$ is smaller than $\#\mathcal{G}$.

Suppose $n \in \mathcal{G}$ is bad. Choose indices $0 \leq i < j \leq H-1$ such that

$$d_{n+i} = d_{n+j} = q.$$

Set

$$a = p_{n+i} - p_n, \qquad b = p_{n+j} - p_n.$$

Since $n \in \mathcal{G}$, all offsets inside the block are at most Y; hence

$$0 \leq a < b \leq Y, \qquad 1 \leq q \leq Y.$$

Moreover the five integers

$$p_n, \quad p_n + a, \quad p_n + a + q, \quad p_n + b, \quad p_n + b + q \tag{6}$$

are prime. Thus every bad short block gives at least one prime-pattern witness of the form (6).

There are two degenerate possibilities: either $a = 0$, or $b = a + q$. If neither occurs, then the five offsets in (6) are distinct, and (1) gives at most

$$\ll Y^3 + \frac{NY^3}{L^5}$$

such witnesses. If $a = 0$ and $b \neq q$, the witness is bounded by the four-prime pattern

$$m, \quad m + q, \quad m + b, \quad m + b + q,$$

and (2) applies. If $a = 0$ and $b = q$, the witness is bounded by the three-prime pattern

$$m, \quad m + q, \quad m + 2q,$$

and (4) applies. Finally, if $b = a + q$, the witness is bounded by the four-prime pattern

$$m, \quad m + a, \quad m + a + q, \quad m + a + 2q,$$

and (3) applies.

Consequently the number of bad positions in $\mathcal{G}$ is

$$\ll Y^3 + \frac{NY^3}{L^5} + Y^2 + \frac{NY^2}{L^4} + Y + \frac{NY}{L^3}. \tag{7}$$

Since $Y = AHL$ and $H \leq \eta L^{1/3}$, we have

$$\begin{aligned} \frac{NY^3}{L^5} &\leq A^3 \eta^3 \frac{N}{L}, \\ \frac{NY^2}{L^4} &\leq A^2 \eta^2 \frac{N}{L^{4/3}}, \\ \frac{NY}{L^3} &\leq A\eta \frac{N}{L^{5/3}}. \end{aligned}$$

The purely polynomial terms $Y^3 + Y^2 + Y$ are $O_A(L^4)$ and hence are $o(N/L)$. Therefore (7) gives

$$\#\{n \in \mathcal{G} : n \text{ bad}\} \le CA^3\eta^3 \frac{N}{L} + o\left(\frac{N}{L}\right)$$

for an absolute constant C. Choosing $\eta > 0$ so small that $CA^3\eta^3 < c_1/2$, and then taking N sufficiently large, we obtain

$$\#\{n \in \mathcal{G} : n \text{ bad}\} < \#\mathcal{G}.$$

Thus some $n \in \mathcal{G}$ is not bad. For this n, the H gaps

$$d_n, d_{n+1}, \dots, d_{n+H-1}$$

are pairwise distinct. Since $n \le M - H < M$, this proves

$$h(M) \ge H \gg L^{1/3},$$

as claimed. □

We now pass from the prime-value cutoff N to the index variable in the definition of $h(x)$.

Proof of Theorem 1. Let x be large and put $M = \lfloor x \rfloor$. Apply Proposition 5 with $N = p_M$, so that $\pi(N) = M$. Then

$$h(M) \gg (\log p_M)^{1/3}.$$

By the prime number theorem, $p_M \sim M \log M$, and hence $\log p_M \sim \log M \sim \log x$. Since $h(x) \ge h(M)$ for $M \le x$, we obtain

$$h(x) \gg (\log x)^{1/3}.$$

The final assertion follows immediately: if $c < 1/3$, then $(\log x)^{1/3} \gg (\log x)^c$ for all sufficiently large x. □

Remark 6. *The argument gives a block of* $H \asymp (\log N)^{1/3}$ *distinct consecutive gaps contained in an interval of length* $O(H \log N) = O((\log N)^{4/3})$*. The exponent* $1/3$ *comes from comparing the number* $\asymp N/\log N$ *of available short blocks with the sieve upper bound for the three-parameter family of five-prime witnesses. Any improvement using this strategy would have to exploit additional information not used here, such as the absence of intermediate primes inside the equal gaps.*

References

[1] T. F. Bloom, *Erdős Problem #852*, `https://www.erdosproblems.com/852`, accessed 24 April 2026.

[2] P. Erdős, *On some of my problems in number theory I would most like to see solved*, in *Number Theory, Ootacamund, 1984*, Lecture Notes in Mathematics, vol. 1122, Springer, Berlin, 1985, pp. 74–84.

[3] J. Friedlander and H. Iwaniec, *Opera de Cribro*, American Mathematical Society Colloquium Publications, vol. 57, American Mathematical Society, Providence, RI, 2010.

[4] J. D. Lichtman and J. Teräväinen, *On the Hardy–Littlewood–Chowla conjecture on average*, Forum of Mathematics, Sigma **10** (2022), Paper No. e57.