Weighted sunflower pressure and the exact polylogarithmic exponent in harmonic LCM patterns

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Abstract

We develop a weighted version of the Tang–Zhang sunflower-capacity argument for Erdős Problem #856 and derive a new exact-exponent theorem from it. For each fixed $z > 0$ we introduce the weighted sunflower partition function

$$W_k(n; z) := \max\Big\{ \sum_{S\in\mathcal{F}} z^{|S|} : \ \mathcal{F} \subseteq 2^{[n]} \text{ is } k\text{-sunflower-free} \Big\}$$

and its exponential growth rate $\Lambda_k(z) = \lim_{n\to\infty} W_k(n; z)^{1/n}$. We prove the two weighted bounds

$$f_k(N) \le (\log N)^{\Lambda_k(z)-z+o(1)}$$

and

$$f_k(N) \ge (\log N)^{\log(z\Lambda_k(1/z))/z-o(1)}$$

for every fixed $z > 0$. A simple squeezing argument at large z then forces the exact exponent to exist:

$$f_k(N) = (\log N)^{\gamma_k+o(1)},$$

where

$$\gamma_k = \lim_{z\to\infty} \big(\Lambda_k(z) - z\big) = \inf_{z>0} \big(\Lambda_k(z) - z\big) = \sup_{z>0} z \log\Big(\frac{\Lambda_k(z)}{z}\Big).$$

Equivalently,

$$\gamma_k = \lim_{\varepsilon\downarrow 0} \frac{\widetilde{\Lambda}_k(\varepsilon) - 1}{\varepsilon}, \qquad \widetilde{\Lambda}_k(\varepsilon) := \varepsilon\Lambda_k(1/\varepsilon).$$

Thus the harmonic LCM problem admits an exact polylogarithmic exponent, characterized variationally by the weighted sunflower pressure. The remaining issue is to compute this exponent or express it in terms of the ordinary sunflower capacity $\mu_k^{\mathrm{S}} = \Lambda_k(1)$. At $z = 1$ we recover the Tang–Zhang bounds

$$(\log N)^{\log \mu_k^{\mathrm{S}}-o(1)} \le f_k(N) \le (\log N)^{\mu_k^{\mathrm{S}}-1+o(1)},$$

while if $\mu_k^{\mathrm{S}} = 2$ then Tang–Zhang's full-density argument implies $\Lambda_k(z) = 1 + z$ for all $z > 0$, hence $\gamma_k = 1$.

1 Introduction

Fix an integer $k \ge 3$. Following Erdős, write

$$f_k(N) := \max\Big\{ \sum_{a\in A} \frac{1}{a} : \ A \subseteq [N] \text{ is LCM-}k\text{-free} \Big\},$$

where A is *LCM-k-free* if it does not contain distinct $a_1, \ldots, a_k$ such that

$$\operatorname{lcm}(a_i, a_j) = \operatorname{lcm}(a_1, a_2) \qquad (1 \le i < j \le k).$$

This is Problem #856 on Bloom's Erdős Problems website [1]. Erdős proved the upper bound

$$f_k(N) \ll_k \frac{\log N}{\log \log N}$$

in [3]. In recent work, Tang and Zhang [5] showed

$$(\log N)^{\log \mu_k^{\mathrm{S}} - o(1)} \le f_k(N) \le (\log N)^{\mu_k^{\mathrm{S}} - 1 + o(1)},$$

where μ_k^{S} is the Erdős–Szemerédi sunflower-free capacity. They also proved that

$$\mu_k^{\mathrm{S}} = 2 \quad \Longleftrightarrow \quad f_k(N) = (\log N)^{1-o(1)}.$$

The starting point is the same mass-transport idea that Liam Price emphasized in his recent Markov-chain discussion of Problem #1196 [2]: after truncating to squarefree multipliers $q \le N$, one chooses q with probability proportional to $z^{\omega(q)}/q$, so that the harmonic weight $1/a$ is transported to the endpoint $m = aq$ with density proportional to $1/m$. The admissible multiplier histories landing at a fixed endpoint form a k-sunflower-free family, and this yields a weighted upper bound involving the partition function $W_k(n; z)$.

A dual weighted bucketing construction using k-cosunflower-free families gives the complementary lower bound. The main new observation of the present note is that the two weighted bounds together force the exact exponent of $f_k(N)$ to exist. More precisely, if

$$D_k(z) := \Lambda_k(z) - z,$$

then the weighted lower bound at parameter $1/z$ gives an exponent $z \log(\Lambda_k(z)/z)$, which differs from $D_k(z)$ by only $O(1/z)$ because $0 \le D_k(z) \le 1$. The weighted upper bound gives $D_k(z)$ itself. Letting $z \to \infty$ therefore squeezes the liminf and limsup exponents of $f_k(N)$ together.

This shows that the polylogarithmic exponent exists and is determined by weighted sunflower pressure. What remains open is the stronger question posed by Tang and Zhang [5, Problem 1.8]: determine this exponent *purely in terms of* the unweighted sunflower capacity $\mu_k^{\mathrm{S}} = \Lambda_k(1)$.

Throughout, $\omega(n)$ denotes the number of distinct prime factors of n, and $\mu(n)$ is the Möbius function.

2 Weighted sunflower partition functions

Recall that k distinct sets $S_1, \ldots, S_k$ form a *k-sunflower* if

$$S_i \cap S_j = S_1 \cap S_2 \qquad (1 \le i < j \le k),$$

and they form a *k-cosunflower* if

$$S_i \cup S_j = S_1 \cup S_2 \qquad (1 \le i < j \le k).$$

A family is *k-sunflower-free* or *k-cosunflower-free* if it contains no such configuration.

We begin with the weighted set-system capacities that arise naturally from the kernel.

Definition 2.1. Fix $k \geq 3$ and $z > 0$.

(1) Let

$$W_k(n; z) := \max\Big\{\sum_{S\in\mathcal{F}} z^{|S|} : \ \mathcal{F} \subseteq 2^{[n]} \text{ is } k\text{-sunflower-free}\Big\}.$$

(2) Let

$$C_k(n; z) := \max\Big\{\sum_{S\in\mathcal{G}} z^{|S|} : \ \mathcal{G} \subseteq 2^{[n]} \text{ is } k\text{-cosunflower-free}\Big\}.$$

At $z = 1$ this reduces to the usual sunflower-free extremal function:

$$W_k(n; 1) = C_k(n; 1) = F_k(n).$$

We shall need the exponential growth rates of these weighted quantities. The existence of the limit is essentially the same tensor-power argument used by Tang and Zhang for the unweighted capacity.

Lemma 2.2. *Let $\mathcal{U} \subseteq \binom{[n]}{r}$ be k-sunflower-free. For each integer $t \geq 1$, let $\mathcal{U}^{(1)}, \dots, \mathcal{U}^{(t)}$ be copies of $\mathcal{U}$ supported on t pairwise disjoint blocks of size n, and set*

$$\mathcal{U}^{\boxtimes t} := \big\{U_1 \sqcup \cdots \sqcup U_t : \ U_i \in \mathcal{U}^{(i)} \ (1 \leq i \leq t)\big\}.$$

Then $\mathcal{U}^{\boxtimes t} \subseteq \binom{[tn]}{tr}$ is k-sunflower-free.

Proof. Suppose instead that $H_1, \dots, H_k \in \mathcal{U}^{\boxtimes t}$ form a k-sunflower. Write

$$H_j = U_{j,1} \sqcup \cdots \sqcup U_{j,t}, \qquad U_{j,s} \in \mathcal{U}^{(s)}.$$

Because the ambient blocks are disjoint, for each fixed block s the sets $U_{1,s}, \dots, U_{k,s}$ have the same pairwise intersection. Also, each $U_{j,s}$ has size r.

If in a fixed block s two of the $U_{j,s}$ are equal, say $U_{u,s} = U_{v,s}$, then the common pairwise intersection inside that block has size r, hence every $U_{j,s}$ must equal $U_{u,s}$. Therefore, in each block either all k projected sets are equal, or else they are k distinct r-sets with constant pairwise intersection, i.e. a k-sunflower in $\mathcal{U}^{(s)}$.

Since each $\mathcal{U}^{(s)}$ is k-sunflower-free, the second alternative is impossible. Hence in every block all projected sets are equal, which forces $H_1 = \cdots = H_k$, contradicting the assumption that the H_j are distinct. ∎

Proposition 2.3. *For every fixed $z > 0$ the limits*

$$\Lambda_k(z) := \lim_{n\to\infty} W_k(n; z)^{1/n}, \qquad \widetilde{\Lambda}_k(z) := \lim_{n\to\infty} C_k(n; z)^{1/n}$$

exist. Moreover,

$$C_k(n; z) = z^n W_k(n; 1/z) \qquad \text{and} \qquad \widetilde{\Lambda}_k(z) = z\Lambda_k(1/z).$$

Finally,

$$\Lambda_k(1) = \widetilde{\Lambda}_k(1) = \mu_k^{\mathrm{S}}.$$

Proof. Fix $z > 0$. Let $\mathcal{F}_n \subseteq 2^{[n]}$ be extremal for $W_k(n; z)$. By averaging over the $n + 1$ uniform layers, there exists some $r \in \{0, 1, \dots, n\}$ such that

$$z^r \Big|\mathcal{F}_n \cap \binom{[n]}{r}\Big| \geq \frac{W_k(n; z)}{n + 1}.$$

Write $\mathcal{U}_n := \mathcal{F}_n \cap \binom{[n]}{r}$.

By Lemma 2.2, the tensor family $\mathcal{U}_n^{\boxtimes t}$ is k-sunflower-free on tn points for every $t \geq 1$. Its weighted mass is

$$\sum_{H \in \mathcal{U}_n^{\boxtimes t}} z^{|H|} = \left(z^r |\mathcal{U}_n|\right)^t \geq \left(\frac{W_k(n;z)}{n+1}\right)^t.$$

Hence

$$W_k(tn;z) \geq \left(\frac{W_k(n;z)}{n+1}\right)^t \qquad (n, t \geq 1).$$

Also, $W_k(m;z)$ is nondecreasing in m, since any family on $[m]$ may be viewed as a family on $[m+1]$. Fix n and let $m \to \infty$. Writing $t := \lfloor m/n \rfloor$, we have $tn \leq m < (t+1)n$, hence

$$W_k(m;z) \geq W_k(tn;z) \geq \left(\frac{W_k(n;z)}{n+1}\right)^t.$$

Taking mth roots and letting $m \to \infty$ yields

$$\liminf_{m\to\infty} W_k(m;z)^{1/m} \geq \left(\frac{W_k(n;z)}{n+1}\right)^{1/n}.$$

Taking the supremum over n gives

$$\liminf_{m\to\infty} W_k(m;z)^{1/m} \geq \sup_{n\geq 1}\left(\frac{W_k(n;z)}{n+1}\right)^{1/n}.$$

Since $(n+1)^{1/n} \to 1$, we also have

$$\limsup_{n\to\infty}\left(\frac{W_k(n;z)}{n+1}\right)^{1/n} = \limsup_{n\to\infty} W_k(n;z)^{1/n}.$$

Hence

$$\liminf_{m\to\infty} W_k(m;z)^{1/m} \geq \limsup_{n\to\infty} W_k(n;z)^{1/n},$$

so the limit $\Lambda_k(z)$ exists.

For the cosunflower quantity, taking complements in $[n]$ bijects k-cosunflowers and k-sunflowers, and transforms the weight $z^{|S|}$ into

$$z^{n-|S|} = z^n (1/z)^{|S|}.$$

Therefore

$$C_k(n;z) = z^n W_k(n;1/z).$$

Taking nth roots and passing to the limit yields

$$\widetilde{\Lambda}_k(z) = z\Lambda_k(1/z).$$

The identity at $z = 1$ is immediate from the definition of μ_k^{S}. □

Remark 2.4. *For later use, note the trivial inequalities*

$$z^n \leq W_k(n;z) \leq \sum_{S\subseteq[n]} z^{|S|} = (1+z)^n$$

(the lower bound comes from the one-point family $\{[n]\}$), as well as

$$W_k(n;z) \le (n+1)F_k(n)\max(1,z)^n.$$

Hence

$$z \le \Lambda_k(z) \le 1+z \qquad \textit{and} \qquad \Lambda_k(z) \le \mu_k^{\mathrm{S}}\max(1,z).$$

In particular

$$0 \le \Lambda_k(z) - z \le 1 \qquad (z>0).$$

Also, if one inserts only the crude estimate $\Lambda_k(z) \le \mu_k^{\mathrm{S}}\max(1,z)$ into Theorem 3.3, then optimizing over $z>0$ recovers precisely the Tang–Zhang exponent $\mu_k^{\mathrm{S}}-1$, since the minimum of $\mu_k^{\mathrm{S}}\max(1,z)-z$ is attained at $z=1$.

3 A weighted Markov-kernel upper bound

We now turn to the harmonic problem.

Lemma 3.1. *For every fixed $u>0$ and every $X \ge 2$,*

$$\sum_{m\le X} \frac{u^{\omega(m)}}{m} \ll_u (\log X)^u.$$

Proof. We have

$$\sum_{m\le X} \frac{u^{\omega(m)}}{m} \le \prod_{p\le X}\Big(1+\frac{u}{p}+\frac{u}{p^2}+\cdots\Big) = \prod_{p\le X}\Big(1+\frac{u}{p-1}\Big).$$

For large p,

$$\log\Big(1+\frac{u}{p-1}\Big) = \frac{u}{p} + O_u\Big(\frac{1}{p^2}\Big),$$

so by Mertens' theorem,

$$\prod_{p\le X}\Big(1+\frac{u}{p-1}\Big) \ll_u (\log X)^u.$$

□

The next analytic fact is standard and is the only place where we use Selberg–Delange type asymptotics.

Lemma 3.2. *For every fixed $z>0$,*

$$H_z(X) := \sum_{\substack{q\le X\\ \mu(q)^2=1}} \frac{z^{\omega(q)}}{q} = (\log X)^{z+o(1)} \qquad (X\to\infty).$$

Reference. This is a standard consequence of the Selberg–Delange method applied to the multiplicative function $q \mapsto \mu(q)^2 z^{\omega(q)}$. One may obtain it, for example, by partial summation from the asymptotic formula for

$$\sum_{n\le X} \mu(n)^2 z^{\omega(n)}.$$

See, for instance, Tenenbaum [6, Chapter II.6] or Koukoulopoulos [4, Chapter 13]. □

Theorem 3.3. *Fix $k \geq 3$ and $z > 0$. Then*

$$f_k(N) \leq (\log N)^{\Lambda_k(z)-z+o(1)} \qquad (N \to \infty).$$

Proof. Let $A \subseteq [N]$ be LCM-k-free. For each integer $m \geq 1$, define

$$\mathcal{F}_m := \Big\{ S \subseteq P(m) : \ m = a \prod_{p \in S} p \text{ for some } a \in A \Big\},$$

where $P(m)$ denotes the set of prime divisors of m.

Claim. For every m, the family $\mathcal{F}_m$ is k-sunflower-free.

Indeed, suppose $S_1, \dots, S_k \in \mathcal{F}_m$ form a k-sunflower with kernel K. For each i there is a unique $a_i \in A$ such that

$$m = a_i \prod_{p \in S_i} p.$$

Since $S_i \cap S_j = K$ for all $i < j$, we have

$$\operatorname{lcm}(a_i, a_j) = \frac{m}{\gcd(m/a_i, m/a_j)} = \frac{m}{\prod_{p \in K} p}$$

for every $i < j$. Equivalently, for each prime $p \mid m$ one has

$$v_p(a_i) = v_p(m) - 1_{p \in S_i},$$

so the p-adic valuation of $\operatorname{lcm}(a_i, a_j)$ is $v_p(m) - 1$ exactly when $p \in S_i \cap S_j = K$, and is otherwise $v_p(m)$. Thus $a_1, \dots, a_k$ are distinct elements of A with constant pairwise lcm, contradicting the LCM-k-free hypothesis. This proves the claim.

Now consider

$$T_z(N) := \Big(\sum_{a \in A} \frac{1}{a} \Big) H_z(N).$$

Expanding $H_z(N)$ and writing $q = \prod_{p \in S} p$, we obtain

$$T_z(N) = \sum_{a \in A} \sum_{\substack{q \leq N \\ \mu(q)^2 = 1}} \frac{z^{\omega(q)}}{aq}.$$

For each pair (a, q) in this sum, put $m := aq$ and $S := P(q)$. Then $m \leq N^2$, we have $S \in \mathcal{F}_m$, and the summand equals $z^{|S|}/m$. Summing first over m therefore gives the upper bound

$$T_z(N) \leq \sum_{m \leq N^2} \frac{1}{m} \sum_{S \in \mathcal{F}_m} z^{|S|}.$$

(The inequality, rather than equality, is because the definition of $\mathcal{F}_m$ also allows representations with $\prod_{p \in S} p > N$, which do not occur in $T_z(N)$.) Since $\mathcal{F}_m$ is k-sunflower-free on the ground set $P(m)$ of size $\omega(m)$, we have

$$\sum_{S \in \mathcal{F}_m} z^{|S|} \leq W_k(\omega(m); z).$$

Fix $\delta > 0$. By Proposition 2.3, there is a constant $C = C(\delta, z, k)$ such that

$$W_k(r; z) \leq C(\Lambda_k(z) + \delta)^r \qquad (r \geq 0).$$

Therefore, by Lemma 3.1,

$$T_z(N) \le C \sum_{m \le N^2} \frac{(\Lambda_k(z)+\delta)^{\omega(m)}}{m} \ll_{\delta,z,k} (\log N)^{\Lambda_k(z)+\delta}.$$

On the other hand, Lemma 3.2 gives

$$H_z(N) = (\log N)^{z+o(1)}.$$

Dividing through by $H_z(N)$ yields

$$\sum_{a \in A} \frac{1}{a} \le (\log N)^{\Lambda_k(z)-z+\delta+o(1)}.$$

Since $\delta > 0$ is arbitrary, the result follows. □

Remark 3.4. *At $z = 1$ we recover the Tang–Zhang upper bound*

$$f_k(N) \le (\log N)^{\mu_k^{\mathrm{S}}-1+o(1)}.$$

The advantage of Theorem 3.3 is that it isolates the exact input needed for a sharper exponent: not merely the single value $\Lambda_k(1) = \mu_k^{\mathrm{S}}$, but the whole weighted curve $z \mapsto \Lambda_k(z)$.

4 A weighted bucketing lower bound

We now show that the same weighted capacity also appears on the lower-bound side.

Definition 4.1. For pairwise disjoint nonempty sets $U_1, \dots, U_t$, and a family $\mathcal{G} \subseteq 2^{[t]}$, define the *blow-up* of $\mathcal{G}$ over $U_1, \dots, U_t$ by

$$\mathcal{B}(\mathcal{G};\, U_1, \dots, U_t) := \Big\{ B \subseteq U_1 \sqcup \cdots \sqcup U_t : \ |B \cap U_i| = 1_{i \in G} \text{ for some } G \in \mathcal{G} \Big\}.$$

Lemma 4.2. *If $\mathcal{G} \subseteq 2^{[t]}$ is k-cosunflower-free, then $\mathcal{B}(\mathcal{G}; U_1, \dots, U_t)$ is also k-cosunflower-free.*

Proof. Suppose instead that $B_1, \dots, B_k \in \mathcal{B}(\mathcal{G}; U_1, \dots, U_t)$ are distinct and satisfy

$$B_r \cup B_s = U \qquad (1 \le r < s \le k)$$

for some fixed set U. For each j, let

$$G_j := \{ i \in [t] : B_j \cap U_i \neq \varnothing \} \in \mathcal{G}.$$

Then for each pair $r < s$ and each block i,

$$i \in G_r \cup G_s \iff (B_r \cup B_s) \cap U_i \neq \varnothing \iff U \cap U_i \neq \varnothing.$$

Hence the pairwise unions $G_r \cup G_s$ are constant.

It remains to show that the G_j are distinct. Suppose $G_u = G_v$ for some $u \neq v$. Since $B_u \neq B_v$, choose $i \in G_u$ such that

$$B_u \cap U_i = \{x\}, \qquad B_v \cap U_i = \{y\}, \qquad x \neq y.$$

Then $(B_u \cup B_v) \cap U_i = \{x, y\}$. Since all pairwise unions equal U, for any $w \notin \{u, v\}$ we must have

$$(B_u \cup B_w) \cap U_i = (B_u \cup B_v) \cap U_i = \{x, y\},$$

which forces $B_w \cap U_i = \{y\}$. Similarly

$$(B_v \cup B_w) \cap U_i = (B_u \cup B_v) \cap U_i = \{x, y\},$$

which forces $B_w \cap U_i = \{x\}$. This is impossible. Therefore the G_j are distinct, and hence form a k-cosunflower in $\mathcal{G}$, contradiction. □

Lemma 4.3. *Let $z, \delta > 0$, and let $w_1, \ldots, w_M \in (0, \delta]$ satisfy*

$$\sum_{m=1}^{M} w_m \geq t(z + \delta)$$

for some integer $t \geq 1$. Then one can partition $\{1, \ldots, M\}$ into pairwise disjoint sets

$$I_1, \ldots, I_t, R$$

such that

$$z \leq \sum_{m \in I_j} w_m < z + \delta \qquad (1 \leq j \leq t).$$

Proof. Construct the sets greedily. From the remaining indices, choose a minimal subset I_1 with total weight at least z; minimality and the bound $w_m \leq \delta$ imply

$$z \leq \sum_{m \in I_1} w_m < z + \delta.$$

Proceed similarly with the unused indices.

After constructing $I_1, \ldots, I_{j-1}$, the remaining total weight is at least

$$t(z + \delta) - (j - 1)(z + \delta) = (t - j + 1)(z + \delta) \geq z + \delta > z,$$

so the greedy procedure can continue and produces t blocks. The leftover indices form R. □

Theorem 4.4. *Fix $k \geq 3$ and $z > 0$. Then*

$$f_k(N) \geq (\log N)^{\log \widetilde{\Lambda}_k(z)/z - o(1)} = (\log N)^{\log(z\Lambda_k(1/z))/z - o(1)} \qquad (N \to \infty).$$

Proof. If $\widetilde{\Lambda}_k(z) = 1$, then the asserted lower bound is $f_k(N) \geq (\log N)^{-o(1)}$, which is trivial since $f_k(N) \geq 1$. Thus we may assume $\widetilde{\Lambda}_k(z) > 1$.

Fix $\varepsilon > 0$ and choose λ with

$$1 < \lambda < \widetilde{\Lambda}_k(z) \qquad \text{and} \qquad \log \lambda > \log \widetilde{\Lambda}_k(z) - \varepsilon.$$

By Proposition 2.3, for all sufficiently large t there exists a k-cosunflower-free family $\mathcal{G}_t \subseteq 2^{[t]}$ such that

$$\sum_{G \in \mathcal{G}_t} z^{|G|} \geq \lambda^t.$$

Let

$$L := \log\log N, \qquad \eta := \varepsilon, \qquad t := \left\lfloor \frac{(1-\eta)L}{z} \right\rfloor, \qquad x := N^{1/t}, \qquad y := \exp(L^{2/3}).$$

For N large we have $t \to \infty$, $y < x$, and

$$\sum_{y<p\le x} \frac{1}{p} = \log\log x - \log\log y + O(1) = L - \log t - \tfrac{2}{3}\log L + O(1) = L - o(L).$$

Choose a small constant $\delta > 0$ with $\delta < \eta z/4$. Since every prime $p > y$ satisfies $1/p \le y^{-1} = o(1)$, we have $1/p \le \delta$ for all sufficiently large N. Also, the total reciprocal mass available in $(y, x]$ is $L - o(L)$, while

$$t(z+\delta) \le (1-\eta)L\Big(1 + \frac{\delta}{z}\Big) + O(1) \le \Big(1 - \frac{\eta}{2}\Big)L + O(1).$$

Hence, for N large enough,

$$\sum_{y<p\le x} \frac{1}{p} \ge t(z+\delta).$$

Applying Lemma 4.3 to the weights $w_p := 1/p$ on the prime set $(y, x]$, we obtain pairwise disjoint prime sets

$$P_1, \dots, P_t \subseteq (y, x]$$

such that

$$J_i := \sum_{p\in P_i} \frac{1}{p} \in [z, z+\delta) \qquad (1 \le i \le t).$$

For each $G \in \mathcal{G}_t$, define

$$A(G) := \Big\{ \prod_{i\in G} p_i : \ p_i \in P_i \text{ for each } i \in G \Big\}, \qquad A := \bigcup_{G\in\mathcal{G}_t} A(G).$$

Because the P_i are disjoint and every chosen prime is at most x, each element of A is squarefree and satisfies

$$a \le x^{|G|} \le x^t = N.$$

Thus $A \subseteq [N]$.

Now take the prime-support map

$$\varphi(a) := \{p : \ p \mid a\}.$$

Then $\varphi(A)$ is precisely the blow-up $\mathcal{B}(\mathcal{G}_t; P_1, \dots, P_t)$. By Lemma 4.2, this family is k-cosunflower-free. Since all elements of A are squarefree,

$$\varphi(\mathrm{lcm}(a, b)) = \varphi(a) \cup \varphi(b),$$

so if A contained an LCM-k-tuple then $\varphi(A)$ would contain a k-cosunflower. Hence A is LCM-k-free.

Finally, the harmonic sum factorizes:

$$\sum_{a\in A} \frac{1}{a} = \sum_{G\in\mathcal{G}_t} \sum_{a\in A(G)} \frac{1}{a} = \sum_{G\in\mathcal{G}_t} \prod_{i\in G} \Big(\sum_{p\in P_i} \frac{1}{p} \Big) = \sum_{G\in\mathcal{G}_t} \prod_{i\in G} J_i \ge \sum_{G\in\mathcal{G}_t} z^{|G|} \ge \lambda^t.$$

Since

$$t = \frac{(1-\eta)L}{z} + O(1),$$

we obtain

$$\sum_{a\in A}\frac{1}{a}\geq \exp\Big(\frac{(1-\eta)L}{z}\log\lambda+O(1)\Big)=(\log N)^{(1-\eta)\log\lambda/z+o(1)}.$$

Using $\eta=\varepsilon$ and $\log\lambda>\log\widetilde{\Lambda}_k(z)-\varepsilon$, we conclude

$$f_k(N)\geq(\log N)^{\log\widetilde{\Lambda}_k(z)/z-O(\varepsilon)+o(1)}.$$

Since $\varepsilon>0$ is arbitrary, the theorem follows. □

Remark 4.5. *At $z=1$ Theorem 4.4 gives*

$$f_k(N)\geq(\log N)^{\log\widetilde{\Lambda}_k(1)-o(1)}=(\log N)^{\log\mu_k^{\mathrm{S}}-o(1)},$$

which is exactly the Tang–Zhang lower bound.

5 Variational consequences and the exact-exponent problem

Define

$$\alpha_k:=\liminf_{N\to\infty}\frac{\log f_k(N)}{\log\log N},\qquad \beta_k:=\limsup_{N\to\infty}\frac{\log f_k(N)}{\log\log N}.$$

The weighted upper and lower bounds immediately give the variational inequalities

$$\sup_{z>0}\frac{\log(z\Lambda_k(1/z))}{z}\leq\alpha_k\leq\beta_k\leq\inf_{z>0}\big(\Lambda_k(z)-z\big).$$

At $z=1$ this recovers the Tang–Zhang bounds

$$\log\mu_k^{\mathrm{S}}\leq\alpha_k\leq\beta_k\leq\mu_k^{\mathrm{S}}-1.$$

The key point is that the gap between the two weighted exponents disappears in the large-z regime.

Theorem 5.1. *For every fixed $k\geq 3$, the limit*

$$\gamma_k:=\lim_{z\to\infty}\big(\Lambda_k(z)-z\big)$$

exists. Moreover,

$$\gamma_k=\inf_{z>0}\big(\Lambda_k(z)-z\big)=\sup_{z>0}z\log\Big(\frac{\Lambda_k(z)}{z}\Big)=\sup_{z>0}\frac{\log(z\Lambda_k(1/z))}{z},$$

and

$$f_k(N)=(\log N)^{\gamma_k+o(1)}.$$

Equivalently,

$$\alpha_k=\beta_k=\gamma_k.$$

Proof. Write

$$D_k(z) := \Lambda_k(z) - z.$$

By Remark 2.4,

$$0 \le D_k(z) \le 1 \qquad (z > 0).$$

From Theorem 3.3 we have, for every fixed $z > 0$,

$$\beta_k \le D_k(z).$$

Therefore

$$\beta_k \le \inf_{z>0} D_k(z) \le \liminf_{z\to\infty} D_k(z).$$

Now apply Theorem 4.4 with parameter $1/z$. This gives, for every fixed $z > 0$,

$$\alpha_k \ge z \log\Big(\frac{\Lambda_k(z)}{z}\Big) = z \log\Big(1 + \frac{D_k(z)}{z}\Big).$$

For $t \ge 0$ one has $\log(1+t) \ge t - t^2/2$. Since $0 \le D_k(z) \le 1$, for $z \ge 1$ we obtain

$$z \log\Big(1 + \frac{D_k(z)}{z}\Big) \ge D_k(z) - \frac{D_k(z)^2}{2z} \ge D_k(z) - \frac{1}{2z}.$$

Hence

$$\alpha_k \ge \limsup_{z\to\infty} D_k(z).$$

Combining the inequalities

$$\limsup_{z\to\infty} D_k(z) \le \alpha_k \le \beta_k \le \liminf_{z\to\infty} D_k(z)$$

shows that the limit

$$\gamma_k = \lim_{z\to\infty} D_k(z)$$

exists and that

$$\alpha_k = \beta_k = \gamma_k.$$

This is equivalent to

$$f_k(N) = (\log N)^{\gamma_k + o(1)}.$$

It remains to identify the variational formulas. Since

$$\beta_k \le \inf_{z>0} D_k(z)$$

and $\beta_k = \gamma_k$, we have

$$\gamma_k \le \inf_{z>0} D_k(z).$$

On the other hand, $D_k(z) \to \gamma_k$ as $z \to \infty$, so

$$\inf_{z>0} D_k(z) \le \gamma_k.$$

Hence

$$\gamma_k = \inf_{z>0} \big(\Lambda_k(z) - z\big).$$

Similarly, from Theorem 4.4,

$$\alpha_k \geq \sup_{z>0} z \log\Big(\frac{\Lambda_k(z)}{z}\Big).$$

Since $\alpha_k = \gamma_k$, this gives

$$\gamma_k \geq \sup_{z>0} z \log\Big(\frac{\Lambda_k(z)}{z}\Big).$$

But because $D_k(z) \to \gamma_k$ and $0 \leq D_k(z) \leq 1$, we have

$$z \log\Big(\frac{\Lambda_k(z)}{z}\Big) = z \log\Big(1 + \frac{D_k(z)}{z}\Big) = D_k(z) + O\Big(\frac{1}{z}\Big)$$

as $z \to \infty$. Therefore

$$\sup_{z>0} z \log\Big(\frac{\Lambda_k(z)}{z}\Big) \geq \gamma_k.$$

So equality holds:

$$\gamma_k = \sup_{z>0} z \log\Big(\frac{\Lambda_k(z)}{z}\Big).$$

Finally, replacing z by $1/z$ gives

$$\sup_{z>0} z \log\Big(\frac{\Lambda_k(z)}{z}\Big) = \sup_{z>0} \frac{\log(z\Lambda_k(1/z))}{z}.$$

This completes the proof. □

Corollary 5.2. *Let*

$$\widetilde{\Lambda}_k(z) := z\Lambda_k(1/z).$$

Then

$$\gamma_k = \lim_{z \downarrow 0} \frac{\widetilde{\Lambda}_k(z) - 1}{z} = \sup_{z>0} \frac{\log \widetilde{\Lambda}_k(z)}{z}.$$

Proof. By Proposition 2.3,

$$\widetilde{\Lambda}_k(z) = z\Lambda_k(1/z) = 1 + zD_k(1/z).$$

Since $D_k(1/z) \to \gamma_k$ as $z \downarrow 0$, the first identity follows. The second is just the last variational formula in Theorem 5.1. □

Remark 5.3. *Theorem 5.1 shows that the exact polylogarithmic exponent exists and is determined by weighted sunflower pressure. What remains open is the stronger issue asked by Tang–Zhang in Problem 1.8: determine* γ_k purely from *the unweighted sunflower capacity* $\mu_k^{\mathrm{S}} = \Lambda_k(1)$*, or decide whether no such one-parameter description is possible.*

Equivalently, one needs to understand the large-z *intercept of the weighted sunflower pressure*

$$\Lambda_k(z) = z + \gamma_k + o(1),$$

or, dually, the small-density slope of the weighted cosunflower pressure

$$\widetilde{\Lambda}_k(z) = 1 + \gamma_k z + o(z) \qquad (z \downarrow 0).$$

The full-density case $\mu_k^{\mathrm{S}} = 2$ illustrates this point cleanly.

Proposition 5.4. *Assume $\mu_k^{\mathrm{S}} = 2$. Then for every fixed $z > 0$,*

$$\Lambda_k(z) = 1 + z \qquad \text{and} \qquad \widetilde{\Lambda}_k(z) = 1 + z.$$

Consequently,

$$\gamma_k = 1,$$

and hence

$$f_k(N) = (\log N)^{1-o(1)}.$$

Proof. The upper bound $\Lambda_k(z) \le 1 + z$ is trivial from Remark 2.4. For the lower bound, fix a rational $\alpha \in (0,1)$ and $\eta > 0$. Tang and Zhang proved [5, Theorem 5.5] that the hypothesis $\mu_k^{\mathrm{S}} = 2$ implies the following: for all sufficiently large n, there exists an $\lfloor \alpha n \rfloor$-uniform k-cosunflower-free family $\mathcal{G}_n \subseteq 2^{[n]}$ with

$$|\mathcal{G}_n| \ge \binom{n}{\lfloor \alpha n \rfloor}^{1-\eta}.$$

Taking complements in $[n]$, we obtain a k-sunflower-free family

$$\mathcal{F}_n \subseteq \binom{[n]}{n - \lfloor \alpha n \rfloor}$$

of the same size. Therefore

$$W_k(n; z) \ge z^{\,n - \lfloor \alpha n \rfloor} |\mathcal{F}_n| \ge z^{(1-\alpha)n + O(1)} \binom{n}{\lfloor \alpha n \rfloor}^{1-\eta}.$$

Taking nth roots and using Stirling's formula gives

$$\Lambda_k(z) \ge z^{1-\alpha} \exp\big((1-\eta) H(\alpha)\big),$$

where

$$H(\alpha) := -\alpha \log \alpha - (1-\alpha)\log(1-\alpha)$$

is the binary entropy. Letting $\eta \to 0$ and optimizing in rational α yields

$$\Lambda_k(z) \ge \sup_{\beta \in \mathbb{Q} \cap [0,1]} z^{\beta} e^{H(\beta)} = 1 + z,$$

since the rationals are dense and $\beta \mapsto z^{\beta} e^{H(\beta)}$ is continuous on $[0,1]$. Thus $\Lambda_k(z) = 1 + z$. By Proposition 2.3,

$$\widetilde{\Lambda}_k(z) = z \Lambda_k(1/z) = z(1 + 1/z) = 1 + z.$$

By Theorem 5.1,

$$\gamma_k = \inf_{z > 0} \big((1+z) - z\big) = 1.$$

Therefore

$$f_k(N) = (\log N)^{1-o(1)}.$$

□

6 Concluding remarks

Remark 6.1. *The kernel proof may be summarized in one sentence: if one adjoins a random squarefree multiplier* $q \le N$ *with normalized law proportional to* $z^{\omega(q)}/q$*, then the harmonic mass* $1/a$ *transported from* a *to an endpoint* $m = aq$ *becomes*

$$\frac{1}{m}\sum_{S\in\mathcal{F}_m} z^{|S|},$$

and the admissible histories $\mathcal{F}_m$ *form a sunflower-free family. This is the exact analogue, for Problem #856, of the "adjoint divisibility chain" point of view that Liam Price emphasized for Problem #1196.*

Remark 6.2. *What remains open is not the existence of the exponent, but its evaluation. By Theorem 5.1, the exact exponent is*

$$\gamma_k = \lim_{z\to\infty}\big(\Lambda_k(z) - z\big).$$

Thus the remaining task is to compute this large-z *intercept, or to relate it to the ordinary sunflower capacity* $\mu_k^{\mathrm{S}} = \Lambda_k(1)$*. Any rigidity theorem that determines the tail behavior of the weighted pressure from the single value at* $z = 1$ *would answer Tang–Zhang's Problem 1.8.*

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