# The asymptotic constant in Erdős problem #858

**Abstract**

Let

$$M(N) := \max_A \sum_{n\in A} \frac{1}{n},$$

where the maximum is over all $A \subseteq \{1,\dots,N\}$ containing no pair $a, b$ with

$$b = at, \qquad P^-(t) > a.$$

This is Erdős problem #858 from [1, p. 128]. We prove that

$$M(N) = \bigl(c_2 + o(1)\bigr)\log N,$$

where $c_2$ is defined by

$$\Phi(u) := \log\frac{1-u}{u} + \int_u^{(1-u)/2} \frac{1}{v}\log\frac{1-u-v}{v}\,dv \qquad \left(\frac{1}{4} \le u \le \frac{1}{2}\right),$$

with the integral interpreted as 0 for $u \ge 1/3$, $\alpha_2 \in (1/4, 1/3)$ is the unique solution of $\Phi(\alpha_2) = 1$, and

$$c_2 := \frac{1}{2} + \int_{\alpha_2}^{1/2} (1 - \Phi(u))\,du.$$

Numerically,

$$\alpha_2 = 0.28043830989\dots, \qquad c_2 = 0.6187712111\dots.$$

The proof has two steps. First, a Bellman recursion together with a monotonicity argument above the $N^{1/4}$ layer shows that for all sufficiently large $N$ the exact optimum is attained by a frontier antichain. Second, the frontier sweep is analyzed there, where the children are exactly prime and semiprime extensions.

## 1 Introduction

We study the finite extremal quantity

$$M(N) := \max_A \sum_{n\in A} \frac{1}{n},$$

where $A \subseteq \{1,\dots,N\}$ is required to contain no pair $a, b$ with

$$b = at, \qquad P^-(t) > a.$$

Equivalently, $A$ is an antichain for the relation

$$a \preceq b \iff b = a \text{ or } b = at \text{ for some } t > 1 \text{ with } P^-(t) > a.$$

Our goal is to determine the asymptotic size of $M(N)$.

The main result is the following.

**Theorem 1.1.** *Let* $\alpha_2 \in (1/4, 1/3)$ *be the unique solution of*

$$\Phi(u) = 1, \qquad \Phi(u) := \log \frac{1-u}{u} + \int_u^{(1-u)/2} \frac{1}{v} \log \frac{1-u-v}{v}\, dv,$$

*where the integral is understood as* 0 *for* $u \geq 1/3$. *Then*

$$M(N) = \bigl(c_2 + o(1)\bigr) \log N, \qquad c_2 := \frac{1}{2} + \int_{\alpha_2}^{1/2} \bigl(1 - \Phi(u)\bigr)\, du.$$

*More precisely, for all sufficiently large* $N$ *there exists an integer* $K_N$ *such that*

$$M(N) = \sum_{n \in A_N(K_N)} \frac{1}{n}, \qquad K_N = N^{\alpha_2 + o(1)},$$

*where*

$$A_N(K) := \{n \leq N : \ \pi(n) \leq K < n\}$$

*is the frontier at height* $K$ *in the rooted tree attached to* $\preceq$.

The argument deliberately avoids the finite exact frontier theorem from earlier drafts. Since we only care about the asymptotic law, it is enough to prove frontier exactness for all sufficiently large $N$ and then analyze the frontier sweep.

## 2 Tree recursion and threshold frontiers

Let

$$n = p_1 p_2 \cdots p_r \qquad (p_1 \leq p_2 \leq \cdots \leq p_r)$$

be the ordered prime factorization of $n$, and write

$$P_k := p_1 \cdots p_k \qquad (0 \leq k \leq r),$$

with $P_0 := 1$.

**Lemma 2.1** (ordered-factorization criterion)**.** *The proper ancestors of* $n$ *are exactly the prefix products* $P_k$ *with* $0 \leq k < r$ *and*

$$p_{k+1} > P_k.$$

*Hence the parent* $\pi(n)$ *is the largest such prefix product. In particular, joining each* $n > 1$ *to* $\pi(n)$ *makes* $\{1, \ldots, N\}$ *into a rooted tree with root* 1.

*Proof.* If $a \prec n$, say $n = at$ with $P^-(t) > a$, then every prime factor of $n$ at most $a$ must already lie in $a$ with the same multiplicity as in $n$. So $a = P_k$ for some $k < r$. Since $n/a$ has least prime factor $p_{k+1}$, the condition $a \prec n$ is exactly $p_{k+1} > a = P_k$. The maximal proper ancestor is therefore the largest such prefix product. □

Write $\mathrm{ch}_N(a)$ for the set of children of $a$ in this tree.

For $a \leq N$, let

$$\mathcal{T}_N(a) := \{n \leq N : \ a \preceq n\}$$

be the subtree rooted at $a$, and let $F_N(a)$ be the maximum of $\sum_{n \in B} 1/n$ over all antichains $B \subseteq \mathcal{T}_N(a)$. Then $M(N) = F_N(1)$.

**Theorem 2.2** (Bellman recursion)**.** *For every $a \le N$,*

$$F_N(a) = \max\left\{\frac{1}{a}, \sum_{b \in \mathrm{ch}_N(a)} F_N(b)\right\}.$$

*Equivalently, with*

$$B_N(a) := a \sum_{b \in \mathrm{ch}_N(a)} F_N(b),$$

*one has*

$$aF_N(a) = \max\{1, B_N(a)\}.$$

*Thus continuation is optimal at $a$ exactly when $B_N(a) \ge 1$.*

*Proof.* Any antichain $B \subseteq \mathcal{T}_N(a)$ either contains $a$, contributing $1/a$, or else avoids $a$. In the latter case

$$\mathcal{T}_N(a) \setminus \{a\} = \bigsqcup_{b \in \mathrm{ch}_N(a)} \mathcal{T}_N(b),$$

and different child subtrees are pairwise incomparable. So the optimal contribution is the sum of the optimal contributions from the child subtrees. □

For $1 \le K \le N$ define the frontier

$$A_N(K) := \{n \le N : \ \pi(n) \le K < n\},$$

and set $A_N(0) := \{1\}$. Its reciprocal weight is

$$S_N(K) := \sum_{n \in A_N(K)} \frac{1}{n} \qquad (0 \le K \le N).$$

Thus $A_N(K)$ is the set of first vertices strictly above $K$ on the root-to-leaf paths.

**Lemma 2.3** (threshold Bellman policy)**.** *Assume that for some $K$ one has*

$$B_N(a) \ge 1 \quad (a \le K), \qquad B_N(a) < 1 \quad (a > K).$$

*Then*

$$M(N) = S_N(K).$$

*Proof.* Because the tree is finite, Theorem 2.2 can be realized recursively by an optimal stop/continue choice at each vertex. In the tie case $B_N(a) = 1$, choose continuation. Under the present hypothesis, this means: continue at every vertex $a \le K$, and stop at every vertex $a > K$. Consequently the selected vertices are exactly those whose parent is at most $K$ while the vertex itself exceeds $K$, namely the frontier $A_N(K)$. Its weight is therefore $S_N(K)$, and Bellman-optimality at every state gives $M(N) = S_N(K)$. □

## 3 The quarter-power layer and eventual frontier exactness

We shall use the shorthand

$$P_N(a) := \sum_{a<p\le N/a} \frac{1}{p}, \qquad Q_N(a) := \sum_{\substack{a<p\le q\\ apq\le N}} \frac{1}{pq},$$

where all sums are over primes.

**Lemma 3.1** (prime child lemma). *If $p > a$ is prime and $ap \le N$, then $\pi(ap) = a$.*

*Proof.* Certainly $a \prec ap$. If $b$ were a proper ancestor with $a < b < ap$, then $b = au$ with every prime factor of $u$ exceeding $a$. Because $b \mid ap$ and $\gcd(a,u) = 1$, we would have $u \mid p$, hence $u = p$ and $b = ap$, a contradiction. ☐

**Lemma 3.2** (prime–semiprime children above $N^{1/4}$). *If $a > N^{1/4}$, then every child of $a$ is of exactly one of the two forms*

$$ap \qquad or \qquad apq,$$

*with primes $a < p \le q$ and $apq \le N$. Consequently, for $a > N^{1/4}$,*

$$a \sum_{b\in \mathrm{ch}_N(a)} \frac{1}{b} = P_N(a) + Q_N(a).$$

*Proof.* Let $n$ be a child of $a$. Then $n = at$, where every prime factor of $t$ exceeds $a$. Since $a > N^{1/4}$,

$$t \le N/a < a^3,$$

so $t$ has at most two prime factors, each $> a$. Thus $t$ is either a prime $p > a$ or a product $pq$ with $a < p \le q$.

Conversely, $ap$ is a child by Lemma 3.1. Now let $n = apq$ with $a < p \le q$ prime and $apq \le N$. By Lemma 2.1, the only possible proper ancestor of $n$ strictly above $a$ is $ap$. But

$$ap > a^2 > N^{1/2}, \qquad q \le \frac{N}{ap} < N^{1/2} < ap,$$

so $q \not> ap$ and therefore $ap$ is not an ancestor. Hence $\pi(apq) = a$. The displayed identity follows by summing $1/b$ over these children. ☐

**Proposition 3.3** (eventual continuation interval). *For all sufficiently large $N$, there exists an integer $K_N \ge \lfloor N^{1/4} \rfloor$ such that*

$$B_N(a) \ge 1 \iff a \le K_N.$$

*Consequently,*

$$M(N) = S_N(K_N)$$

*for all sufficiently large $N$.*

*Proof.* Let $L := \lfloor N^{1/4} \rfloor$.

First take $a \le L$. By Lemma 3.1, every prime $p$ with $a < p \le N/a$ contributes the child $ap$, and certainly $F_N(ap) \ge 1/(ap)$. Hence

$$B_N(a) \ge a \sum_{a<p\le N/a} \frac{1}{ap} = P_N(a) \ge P_N(L).$$

By Mertens' theorem in the form

$$\sum_{p \le x} \frac{1}{p} = \log\log x + B_1 + o(1),$$

we obtain

$$P_N(L) = \log \frac{\log(N/L)}{\log L} + o(1) = \log 3 + o(1) > 1.$$

So $B_N(a) > 1$ for every $a \le L$ once $N$ is large enough.

Now take $a > L$. By Lemma 3.2, each child $b$ of $a$ satisfies $b > a^2 > N^{1/2}$. Any further child of $b$ would have the form $bt$ with $t > b$, hence would exceed $b^2 > N$. So every child of $a$ is a leaf and

$$F_N(b) = \frac{1}{b} \qquad (b \in \mathrm{ch}_N(a)).$$

Therefore

$$B_N(a) = a \sum_{b \in \mathrm{ch}_N(a)} \frac{1}{b} = P_N(a) + Q_N(a) \qquad (a > L).$$

The right-hand side is nonincreasing in $a$, because both summation domains shrink with $a$. It is also 0 for $a > \sqrt{N}$. Hence there is an integer $K_N \ge L$ such that

$$B_N(a) \ge 1 \iff a \le K_N.$$

Lemma 2.3 now gives $M(N) = S_N(K_N)$. □

# 4 Asymptotics of the frontier sweep

For $a \le N$ define

$$C_N(a) := \sum_{\pi(n) = a} \frac{1}{n}.$$

The frontier weight changes by deleting $K$ and inserting all children of $K$.

**Lemma 4.1** (frontier sweep)**.** *We have $S_N(0) = 1$, and for $1 \le K \le N$,*

$$S_N(K) - S_N(K-1) = C_N(K) - \frac{1}{K}.$$

*Proof.* Passing from $A_N(K-1)$ to $A_N(K)$ removes the vertex $K$ and adds precisely the children $n$ with $\pi(n) = K$. □

**Proposition 4.2** (exact frontier identity above $N^{1/4}$)**.** *If $\lfloor N^{1/4} \rfloor \le K \le \lfloor \sqrt{N} \rfloor$, then*

$$S_N(K) = H_N - H_{\lfloor \sqrt{N} \rfloor} + \sum_{K < a \le \lfloor \sqrt{N} \rfloor} \frac{1 - P_N(a) - Q_N(a)}{a},$$

*where $H_m := \sum_{n \le m} 1/n$. In particular, for integers $a > N^{1/4}$,*

$$S_N(a) - S_N(a-1) = \frac{P_N(a) + Q_N(a) - 1}{a}.$$

*Proof.* Since every $n \in \{2, \ldots, N\}$ has exactly one parent,

$$\sum_{a=1}^{N} C_N(a) = \sum_{n=2}^{N} \frac{1}{n} = H_N - 1.$$

Also $C_N(a) = 0$ for $a > \sqrt{N}$. Summing Lemma 4.1 from 1 to $K$ gives

$$S_N(K) = 1 + \sum_{a=1}^{K} \left( C_N(a) - \frac{1}{a} \right) = H_N - H_K - \sum_{K < a \leq \lfloor \sqrt{N} \rfloor} C_N(a).$$

If $K \geq \lfloor N^{1/4} \rfloor$, then every $a > K$ satisfies $a > N^{1/4}$, so Lemma 3.2 yields

$$C_N(a) = \frac{P_N(a) + Q_N(a)}{a} \qquad (K < a \leq \lfloor \sqrt{N} \rfloor).$$

Substituting and using

$$H_{\lfloor \sqrt{N} \rfloor} - H_K = \sum_{K < a \leq \lfloor \sqrt{N} \rfloor} \frac{1}{a}$$

gives the identity. The difference formula follows by subtraction. ☐

We now record the analytic input.

**Lemma 4.3** (Mertens on polynomial intervals)**.** *Fix $\beta \in (0, 1)$. Uniformly for real numbers $x, y$ with*

$$N^{\beta} \leq x \leq y \leq N,$$

*one has*

$$\sum_{x < p \leq y} \frac{1}{p} = \log \frac{\log y}{\log x} + o(1) \qquad (N \to \infty).$$

*Proof.* Let $\mathcal{B}(x) := \sum_{p \leq x} 1/p$. By Mertens' theorem [2, Chap. I.1],

$$\mathcal{B}(x) = \log \log x + B_1 + r(x), \qquad r(x) \to 0.$$

Hence

$$\sum_{x < p \leq y} \frac{1}{p} = \mathcal{B}(y) - \mathcal{B}(x) = \log \frac{\log y}{\log x} + r(y) - r(x).$$

If $N^{\beta} \leq x \leq y \leq N$, then $x$ and $y$ tend to infinity uniformly with $N$, so $r(x)$ and $r(y)$ are uniformly $o(1)$. ☐

**Lemma 4.4** (prime-harmonic Riemann sums)**.** *Fix $\alpha$ with $1/4 \leq \alpha < 1/3$, and let*

$$D_{\alpha} := \{(u, v) : \ \alpha \leq u \leq 1/3, \ u \leq v \leq (1-u)/2\}.$$

*If $G : D_{\alpha} \to \mathbb{R}$ is continuous, then uniformly for $u \in [\alpha, 1/3]$,*

$$\sum_{N^u < p \leq N^{(1-u)/2}} \frac{G\left(u, \frac{\log p}{\log N}\right)}{p} = \int_u^{(1-u)/2} \frac{G(u, v)}{v} \, dv + o(1).$$

*Proof.* This is the usual Riemann-sum argument, using Lemma 4.3 on short multiplicative slabs. Since $D_\alpha$ is compact, $G$ is uniformly continuous and bounded there; partitioning the $v$-interval into pieces of mesh $\delta$ and summing the prime-harmonic masses over each piece gives a Riemann sum for $\int G(u,v)\,dv/v$, uniformly in $u$. ☐

**Lemma 4.5** (harmonic Riemann sums)**.** *Let* $0<\alpha<\beta\le 1$*, let* $f:[\alpha,\beta]\to\mathbb{R}$ *be continuous, and suppose that*

$$\alpha \le \lambda_N \le \mu_N \le \beta.$$

*Then*

$$\frac{1}{\log N}\sum_{N^{\lambda_N}<a\le N^{\mu_N}} \frac{f\Big(\frac{\log a}{\log N}\Big)}{a} = \int_{\lambda_N}^{\mu_N} f(u)\,du + o(1),$$

*uniformly in the choice of* $\lambda_N, \mu_N$*.*

*Proof.* Put $u_a := \log a/\log N$. Then

$$u_{a+1}-u_a = \frac{\log(1+1/a)}{\log N} = \frac{1}{a\log N} + O\Big(\frac{1}{a^2\log N}\Big),$$

uniformly in $a$. Thus the displayed sum is a left-endpoint Riemann sum for $\int_{\lambda_N}^{\mu_N} f(u)\,du$, with an $o(1)$ total error because $\sum_{a\ge N^\alpha} a^{-2} = O(N^{-\alpha})$. ☐

For $1/4\le u\le 1/2$ define

$$\Phi(u) := \log\frac{1-u}{u} + \int_u^{(1-u)/2} \frac{1}{v}\log\frac{1-u-v}{v}\,dv,$$

where the integral is taken to be 0 when $u\ge 1/3$.

**Lemma 4.6** (uniform prime–semiprime asymptotics)**.** *Fix* $\alpha$ *with* $1/4\le\alpha<1/2$*. Then uniformly for* $N^\alpha\le a\le\sqrt{N}$*,*

$$P_N(a)+Q_N(a) = \Phi(u)+o(1), \qquad u := \frac{\log a}{\log N}.$$

*Proof.* Write $u := \log a/\log N$. By Lemma 4.3,

$$P_N(a) = \sum_{a<p\le N/a}\frac{1}{p} = \log\frac{1-u}{u} + o(1)$$

uniformly for $N^\alpha \le a\le\sqrt{N}$.

If $u\ge 1/3$, then $Q_N(a)=0$, and by definition the integral term in $\Phi(u)$ is also 0. Thus only the range $u\in[\alpha,1/3]$ needs analysis; this range is empty when $\alpha\ge 1/3$. Assume therefore that $\alpha<1/3$, and let $u\in[\alpha,1/3]$. Define

$$G(u,v) := \log\frac{1-u-v}{v}$$

on the compact triangle $D_\alpha$. This function is continuous. For each outer prime $p$, with $v := \log p/\log N$, another application of Lemma 4.3 gives

$$\sum_{p\le q\le N/(ap)}\frac{1}{q} = G(u,v)+o(1)$$

uniformly for $(u, v) \in D_\alpha$. Since $\sum_{a<p\le N^{(1-u)/2}} 1/p \ll 1$, the total contribution of the uniform $o(1)$ term is still $o(1)$. Therefore

$$Q_N(a) = \sum_{a<p\le N^{(1-u)/2}} \frac{G\Big(u, \frac{\log p}{\log N}\Big)}{p} + o(1),$$

uniformly for $u \in [\alpha, 1/3]$. Applying Lemma 4.4 yields

$$Q_N(a) = \int_u^{(1-u)/2} \frac{1}{v} \log \frac{1-u-v}{v}\, dv + o(1),$$

uniformly for $u \in [\alpha, 1/3]$. Combining this with the prime term proves the lemma. $\square$

**Proposition 4.7.** *The function* $\Phi$ *is continuous and strictly decreasing on* $[1/4, 1/2]$. *Moreover,*

$$\Phi\left(\frac{1}{4}\right) = 1.2458329656\ldots > 1, \qquad \Phi\left(\frac{1}{3}\right) = \log 2 < 1.$$

*Hence there is a unique*

$$\alpha_2 \in \left(\frac{1}{4}, \frac{1}{3}\right)$$

*with* $\Phi(\alpha_2) = 1$*, namely*

$$\alpha_2 = 0.28043830989\ldots.$$

*Proof.* Continuity at $u = 1/3$ is immediate, since the interval of integration collapses there and the integrand vanishes at $v = (1-u)/2$. For $u \in [1/3, 1/2]$ one has $\Phi(u) = \log((1-u)/u)$, so $\Phi$ is strictly decreasing there. For $u \in (1/4, 1/3)$, write

$$I(u) := \int_u^{(1-u)/2} \frac{1}{v} \log \frac{1-u-v}{v}\, dv.$$

Since the integrand vanishes at $v = (1-u)/2$, Leibniz' rule gives

$$I'(u) = -\frac{1}{u} \log \frac{1-2u}{u} - \int_u^{(1-u)/2} \frac{dv}{v(1-u-v)}.$$

Also

$$\int_u^{(1-u)/2} \frac{dv}{v(1-u-v)} = \frac{1}{1-u} \log \frac{1-2u}{u}.$$

Therefore

$$I'(u) = -\left(\frac{1}{u} + \frac{1}{1-u}\right) \log \frac{1-2u}{u} < 0,$$

because $u < 1/3$ implies $(1-2u)/u > 1$. Adding the derivative of $\log((1-u)/u)$ shows that $\Phi'(u) < 0$ on $(1/4, 1/3)$ as well. The stated endpoint values and strict monotonicity yield the unique root. $\square$

*Proof of Theorem 1.1.* Let $K_N$ be given by Proposition 3.3. Then

$$M(N) = S_N(K_N)$$

for all sufficiently large $N$. Since each $A_N(K)$ is an admissible antichain, one has $S_N(K) \le M(N)$ for every $K$, so $K_N$ is a maximizing cutoff for the frontier sweep. Also $K_N \ge \lfloor N^{1/4} \rfloor$, hence Proposition 4.2 applies to $K_N$.

We first locate $K_N$.

For the upper bound, fix $\varepsilon$ with $0 < \varepsilon < 1/2 - \alpha_2$. By Proposition 4.7, there exists $\eta > 0$ such that

$$\Phi(u) \le 1 - \eta \qquad (\alpha_2 + \varepsilon \le u \le 1/2).$$

By Lemma 4.6, uniformly for integers $a$ with

$$N^{\alpha_2+\varepsilon} \le a \le \sqrt{N},$$

one has

$$P_N(a) + Q_N(a) \le 1 - \eta/2$$

for all sufficiently large $N$. Proposition 4.2 then yields

$$S_N(a) - S_N(a-1) = \frac{P_N(a) + Q_N(a) - 1}{a} \le -\frac{\eta}{2a} < 0$$

throughout that range. Hence every maximizing cutoff satisfies

$$K_N \le N^{\alpha_2+\varepsilon}$$

for all large $N$. Since larger values of $\varepsilon$ are weaker, this proves the upper bound for every $\varepsilon > 0$.

For the lower bound, first assume

$$0 < \varepsilon < \alpha_2 - \frac{1}{4}.$$

Again by Proposition 4.7, there exists $\eta' > 0$ such that

$$\Phi(u) \ge 1 + \eta' \qquad \left(\frac{1}{4} \le u \le \alpha_2 - \varepsilon\right).$$

Applying Lemma 4.6 uniformly on that interval, we get

$$P_N(a) + Q_N(a) \ge 1 + \eta'/2$$

for all integers $a$ with

$$\lfloor N^{1/4} \rfloor < a \le N^{\alpha_2-\varepsilon}$$

and all sufficiently large $N$. By Proposition 4.2,

$$S_N(a) - S_N(a-1) \ge \frac{\eta'}{2a} > 0$$

throughout that range. Since $K_N \ge \lfloor N^{1/4} \rfloor$ and $S_N(K_N)$ is maximal, it follows that

$$K_N \ge N^{\alpha_2-\varepsilon}$$

for all sufficiently large $N$. This proves the lower bound when $0 < \varepsilon < \alpha_2 - 1/4$. For larger $\varepsilon$, the same conclusion follows from any smaller positive value of $\varepsilon$, which yields a stronger lower bound.

Therefore

$$K_N = N^{\alpha_2+o(1)}.$$

Set

$$\kappa_N := \frac{\log K_N}{\log N}.$$

Then $\kappa_N \to \alpha_2$. By Proposition 4.2,

$$M(N) = H_N - H_{\lfloor\sqrt{N}\rfloor} + \sum_{K_N < a \le \lfloor\sqrt{N}\rfloor} \frac{1 - P_N(a) - Q_N(a)}{a}.$$

Also

$$H_N - H_{\lfloor\sqrt{N}\rfloor} = \frac{1}{2}\log N + O(1).$$

Choose $\delta > 0$ so small that $\alpha_2 - \delta > 1/4$. For large $N$, the lower limit $K_N$ lies in $[N^{\alpha_2-\delta}, N^{\alpha_2+\delta}]$, and Lemma 4.6 applies uniformly on $[\alpha_2 - \delta, 1/2]$. Thus

$$1 - P_N(a) - Q_N(a) = 1 - \Phi\left(\frac{\log a}{\log N}\right) + o(1)$$

uniformly for $K_N < a \le \lfloor\sqrt{N}\rfloor$. Since

$$\sum_{K_N < a \le \lfloor\sqrt{N}\rfloor} \frac{1}{a} = O(\log N),$$

the total contribution of the uniform $o(1)$ term is $o(\log N)$. Lemma 4.5 therefore gives

$$\frac{1}{\log N} \sum_{K_N < a \le \lfloor\sqrt{N}\rfloor} \frac{1 - P_N(a) - Q_N(a)}{a} = \int_{\kappa_N}^{1/2} (1 - \Phi(u))\, du + o(1).$$

Since $\kappa_N \to \alpha_2$ and $1 - \Phi(u)$ is continuous,

$$\int_{\kappa_N}^{1/2} (1 - \Phi(u))\, du \to \int_{\alpha_2}^{1/2} (1 - \Phi(u))\, du.$$

Hence

$$\frac{M(N)}{\log N} \to \frac{1}{2} + \int_{\alpha_2}^{1/2} (1 - \Phi(u))\, du = c_2.$$

This completes the proof. $\square$

# References

[1] P. Erdős, *Some extremal problems in combinatorial number theory*, in *Mathematical Essays Dedicated to A. J. Macintyre*, Ohio Univ. Press, Athens, OH, 1970, pp. 123–133.

[2] G. Tenenbaum, *Introduction to Analytic and Probabilistic Number Theory*, third edition, Graduate Studies in Mathematics 163, American Mathematical Society, Providence, RI, 2015.