An exact frontier theorem and the asymptotic constant for Erdős problem #858

Przemek Chojecki

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Abstract

Let

$$M(N) := \max_{A} \sum_{n\in A} \frac{1}{n}, \qquad \mathcal{M}(N) := \frac{M(N)}{\log N},$$

where the maximum is over all $A \subseteq \{1, \dots, N\}$ such that there is no solution to

$$b = at \qquad (a, b \in A)$$

with $P^-(t) > a$. This is Erdős problem #858 from [4, p. 128]; see also [3].

We attach to the relation a rooted tree with parent map π, and for each cutoff K we consider the frontier antichain

$$A_N(K) := \{n \le N : \ \pi(n) \le K < n\}.$$

Write

$$M_{\mathrm{fr}}(N) := \max_{0\le K\le N} \sum_{n\in A_N(K)} \frac{1}{n}.$$

We prove $M(N) = M_{\mathrm{fr}}(N)$. It is partly computer-assisted in a finite range and completely analytic thereafter. We also record an independent purely analytic Bellman proof of frontier exactness for all sufficiently large N.

The key point is a max-closure reformulation with local coefficients

$$q_N(a) := C_N(a) - \frac{1}{a}, \qquad C_N(a) := \sum_{\pi(n)=a} \frac{1}{n},$$

so that exact frontier optimality follows once the set $\{a : q_N(a) > 0\}$ is an initial segment. We prove this sign theorem by combining three ingredients: (i) a finite exact rational computation of the small thresholds $\nu(a)$ for $1 \le a \le 19$; (ii) a prime-harmonic lower bound showing $q_N(a) > 0$ throughout the low layer $20 \le a \le N^{1/4}$; and (iii) a monotonicity theorem on the upper layer $a > N^{1/4}$ coming from the exact prime–semiprime description of the children.

As a consequence, for every $N \ge 2$ there exists a cutoff $K_*(N)$ such that

$$M(N) = \sum_{n\in A_N(K_*(N))} \frac{1}{n} = M_{\mathrm{fr}}(N).$$

Combining this exact frontier theorem with the prime–semiprime asymptotic analysis of the frontier sweep gives

$$K_*(N) = N^{\alpha_2+o(1)}, \qquad M(N) = \big(c_2 + o(1)\big) \log N,$$

where $\alpha_2 \in (1/4, 1/3)$ is the unique solution of

$$\Phi(u) := \log \frac{1-u}{u} + \int_u^{(1-u)/2} \frac{1}{v} \log \frac{1-u-v}{v}\, dv = 1,$$

and

$$\alpha_2 = 0.2804383098923534\dots, \qquad c_2 = 0.6187712111099834\dots.$$

1 Introduction

We study the finite weighted extremal problem

$$M(N) := \max_{A} \sum_{n\in A} \frac{1}{n},$$

where $A \subseteq \{1,\dots,N\}$ is required to contain no pair a, b with

$$b = at, \qquad P^-(t) > a.$$

This is Erdős problem #858 from [4, p. 128]; see also [3]. The condition is a weakened primitive-set condition, so the problem belongs to the same circle of ideas as the classical work of Behrend [2], Alexander [1], Erdős–Sárközy–Szemerédi [5], and Lichtman [7, 8].

There is an immediate lower bound

$$M(N) \geq \sum_{\sqrt{N}<n\leq N} \frac{1}{n} = \frac{1}{2}\log N + O(1),$$

because every subset of $(\sqrt{N}, N]$ is admissible. The point of this note is that the exact finite optimum is much more rigid than this trivial construction suggests.

For positive integers a, b define

$$a \preceq b \iff b = a \text{ or } b = at \text{ for some } t > 1 \text{ with } P^-(t) > a.$$

The admissibility condition says exactly that A is an antichain for this relation. The relation is a partial order: reflexivity and antisymmetry are immediate, and transitivity follows because if $b = au$ with $P^-(u) > a$ and $c = bv$ with $P^-(v) > b$, then $c = a(uv)$ with $P^-(uv) > a$.

Theorem 1.1 (exact frontier theorem). *For every $N \geq 2$ there exists a cutoff $K_*(N) \in \{0, 1, \dots, N\}$ such that*

$$M(N) = \sum_{n\in A_N(K_*(N))} \frac{1}{n}.$$

Equivalently,

$$M(N) = M_{\mathrm{fr}}(N) := \max_{0\leq K\leq N} \sum_{n\in A_N(K)} \frac{1}{n}.$$

The proof runs through a max-closure reformulation. Let

$$C_N(a) := \sum_{\pi(n)=a} \frac{1}{n}, \qquad q_N(a) := C_N(a) - \frac{1}{a}, \qquad R_N(a) := aC_N(a).$$

The frontier sweep identity reads

$$S_N(K) - S_N(K-1) = q_N(K) = \frac{R_N(K) - 1}{K}, \qquad S_N(K) := \sum_{n\in A_N(K)} \frac{1}{n}.$$

Thus Theorem 1.1 follows once one knows that the positivity set $\{a : R_N(a) > 1\}$ is always an initial segment. That sign theorem is the real new input of the paper.

Define

$$\Phi(u) := \log\frac{1-u}{u} + \int_u^{(1-u)/2} \frac{1}{v}\log\frac{1-u-v}{v}\,dv \qquad \left(\frac{1}{4} < u \leq \frac{1}{2}\right),$$

with the integral interpreted as 0 when $(1-u)/2 \le u$. Let $\alpha_2 \in (1/4, 1/3)$ be the unique solution of $\Phi(\alpha_2) = 1$, and put

$$c_2 := \frac{1}{2} + \int_{\alpha_2}^{1/2} (1 - \Phi(u))\, du.$$

Numerically,

$$\alpha_2 = 0.2804383098923534\ldots, \qquad c_2 = 0.6187712111099834\ldots.$$

Theorem 1.2 (asymptotic law)**.** *Let $K_*(N)$ be any cutoff from Theorem 1.1. Then*

$$K_*(N) = N^{\alpha_2 + o(1)}, \qquad M(N) = (c_2 + o(1)) \log N.$$

Hence

$$\mathcal{M}(N) = c_2 + o(1).$$

For completeness, we also record later a purely analytic Bellman proof that

$$M(N) = M_{\mathrm{fr}}(N)$$

for all sufficiently large N. The finite computer-assisted verification is needed only to promote this to the exact all-N statement of Theorem 1.1.

2 The rooted tree and exact dynamic programming

Lemma 2.1. *If $a \preceq n$ and $b \preceq n$ with $a < b < n$, then $a \preceq b$.*

Proof. Write $n = au = bv$ with $P^-(u) > a$ and $P^-(v) > b$. Let $q \le a$ be prime. Since $q < P^-(u)$, we have $q \nmid u$, hence $\nu_q(a) = \nu_q(n)$. Since also $q \le a < b < P^-(v)$, we have $q \nmid v$, hence $\nu_q(b) = \nu_q(n)$. Therefore $\nu_q(a) = \nu_q(b)$ for every prime $q \le a$. It follows that $a \mid b$, and every prime factor of b/a is $> a$. Thus $a \preceq b$. □

Corollary 2.2. *For every $n > 1$, the set of proper ancestors*

$$\{a < n : \ a \preceq n\}$$

is linearly ordered. In particular it has a unique maximal element, which we denote by $\pi(n)$. With the convention $\pi(1) := 0$, joining each $n > 1$ to $\pi(n)$ makes $\{1, \ldots, N\}$ into a rooted tree with root 1.

Lemma 2.3. *Let*

$$n = p_1 p_2 \cdots p_r \qquad (p_1 \le p_2 \le \cdots \le p_r)$$

be the prime factorization of n, listed in nondecreasing order, and put

$$P_k := p_1 \cdots p_k \qquad (0 \le k \le r),$$

with $P_0 := 1$. Then the proper ancestors of n are exactly the prefix products P_k with $0 \le k < r$ and

$$p_{k+1} > P_k.$$

Hence

$$\pi(n) = P_k,$$

where k is the largest index $0 \le k < r$ with $p_{k+1} > P_k$.

Proof. Suppose $a \prec n$. Write $n = at$ with $P^-(t) > a$. Every prime factor of n that is at most a must then already lie in a with the same multiplicity as in n. So a is exactly the product of the first k primes in the ordered factorization of n, namely $a = P_k$ for some $k < r$. Now n/a has least prime factor p_{k+1}, so $a \prec n$ exactly when $p_{k+1} > a = P_k$. The maximal proper ancestor is therefore the stated one. □

For $a \le N$, let

$$\mathcal{T}_N(a) := \{n \le N : \ a \preceq n\}$$

be the subtree rooted at a. Let $F_N(a)$ be the maximum of $\sum_{n \in B} 1/n$ over all antichains $B \subseteq \mathcal{T}_N(a)$. Then $M(N) = F_N(1)$.

Theorem 2.4 (subtree recursion)**.** *For every* $a \le N$*,*

$$F_N(a) = \max\left\{ \frac{1}{a}, \sum_{b \in \mathrm{ch}_N(a)} F_N(b) \right\},$$

where $\mathrm{ch}_N(a)$ *is the set of children of* a *in the rooted tree.*

Proof. Any antichain $B \subseteq \mathcal{T}_N(a)$ either contains a, in which case it contributes exactly $1/a$, or else it avoids a. In the latter case B splits uniquely as the disjoint union of antichains inside the child subtrees $\mathcal{T}_N(b)$, $b \in \mathrm{ch}_N(a)$. Different child subtrees are pairwise incomparable, so the optimal contributions add. This gives the formula. □

Remark 2.5 (Bellman form)*.* If one rescales by

$$V_N(a) := aF_N(a),$$

then Theorem 2.4 becomes

$$V_N(a) = \max\left\{ 1, \sum_{b \in \mathrm{ch}_N(a)} \frac{a}{b} V_N(b) \right\}.$$

Thus the subtree optimization is an optimal-stopping problem on the rooted tree. The max-closure reformulation below linearizes this recursion in terms of continuation sets and frontier increments.

Remark 2.6 (exact finite algorithm)*.* Theorem 2.4 gives an exact dynamic program for $M(N)$. Using a sieve of smallest prime factors, one computes the parents $\pi(n)$ via Lemma 2.3, accumulates child contributions at each parent, and then evaluates $F_N(a)$ for $a = N, N-1, \ldots, 1$. This computes $M(N)$ exactly with essentially sieve complexity.

Lemma 2.7 (prime child lemma)**.** *Let* $a \ge 1$ *and let* $p > a$ *be prime with* $ap \le N$*. Then*

$$\pi(ap) = a.$$

Consequently, for every $1 \le a \le N$*,*

$$C_N(a) \ge \frac{1}{a} \sum_{a < p \le N/a} \frac{1}{p}.$$

Proof. Since $p > a$, we have $a \preceq ap$. Suppose that b is a proper ancestor of ap with $a < b < ap$. By Lemma 2.1, applied to $a \preceq ap$ and $b \preceq ap$, we have $a \preceq b$. Thus $b = au$ for some integer $u > 1$ all of whose prime factors exceed a. In particular $\gcd(a, u) = 1$. Because $b = au$ divides ap, it follows that $u \mid p$. Since p is prime and $u > 1$, we must have $u = p$, hence $b = ap$, contradicting $b < ap$. So no such b exists, and therefore $\pi(ap) = a$. The stated inequality follows by summing $1/(ap)$ over all primes p with $a < p \le N/a$. □

3 Frontiers and a max-closure reformulation

For $0 \le K \le N$ define the frontier

$$A_N(K) := \{n \le N : \ \pi(n) \le K < n\}.$$

This is the set of first vertices strictly above K on the root-to-leaf paths.

Lemma 3.1. *For every K, the set $A_N(K)$ is an antichain.*

Proof. Suppose $x, y \in A_N(K)$ with $x < y$ and $x \preceq y$. Then x is a proper ancestor of y. By Corollary 2.2, all proper ancestors of y lie on a chain whose maximal element is $\pi(y)$. Hence $x \le \pi(y) \le K$. But $x \in A_N(K)$ also gives $x > K$, a contradiction. □

Define

$$C_N(a) := \sum_{\pi(n)=a} \frac{1}{n}, \qquad q_N(a) := C_N(a) - \frac{1}{a}, \qquad R_N(a) := aC_N(a),$$

for $1 \le a \le N$. Also write

$$S_N(K) := \sum_{n \in A_N(K)} \frac{1}{n}.$$

Proposition 3.2 (frontier sweep)**.** *We have $S_N(0) = 1$, and for $1 \le K \le N$,*

$$S_N(K) - S_N(K-1) = q_N(K) = C_N(K) - \frac{1}{K}.$$

Hence

$$S_N(K) = 1 + \sum_{a=1}^{K} q_N(a).$$

Proof. When the cutoff rises from $K-1$ to K, the frontier loses the single vertex K and gains exactly those vertices whose parent is K. Taking reciprocal sums gives the stated identity. □

A *continuation set* is a downward-closed subset $D \subseteq \{1, \dots, N\}$ with respect to the tree order, that is,

$$x \in D \text{ and } y \prec x \implies y \in D.$$

For such a set define its boundary

$$\partial D := \{n \le N : \ n \notin D, \ \pi(n) \in D \cup \{0\}\}.$$

Thus ∂D is the first vertex outside D on each branch.

Lemma 3.3. *Every maximum-weight antichain may be chosen in the form ∂D for some continuation set D.*

Proof. Let B be any antichain. If a root-to-leaf path contains no element of B, then the leaf ℓ of that path is incomparable with every element of B; otherwise some element of B would lie on that path. So $B \cup \{\ell\}$ is still an antichain and has larger weight. Iterating this procedure, we reach a stopping antichain meeting every root-to-leaf path exactly once.

Now let B be such a stopping antichain, and let D be the set of vertices that lie strictly below B on their root-to-leaf paths, that is,

$$D := \{a \le N : \text{ no ancestor of } a \text{ lies in } B\}.$$

Then D is downward-closed, and the first vertex outside D on each branch is exactly the unique element of B on that branch. Hence $\partial D = B$. □

Proposition 3.4 (max-closure identity)**.** *For every continuation set D,*

$$\sum_{n\in\partial D}\frac{1}{n}=1+\sum_{a\in D}q_N(a).$$

Consequently,

$$M(N)=1+\max_D\sum_{a\in D}q_N(a),$$

where the maximum is over all continuation sets D.

Proof. If $D=\varnothing$, then $\partial D=\{1\}$ and the displayed identity is immediate. So assume $D\neq\varnothing$. Since D is downward-closed, this forces $1\in D$.

Now every child of a vertex in D lies either in D or in ∂D. Equivalently, for $n>1$ one has $\pi(n)\in D$ if and only if $n\in D\cup\partial D$. Hence

$$\sum_{a\in D}C_N(a)=\sum_{n\in D\setminus\{1\}}\frac{1}{n}+\sum_{n\in\partial D}\frac{1}{n}.$$

Therefore

$$1+\sum_{a\in D}q_N(a)=1+\sum_{a\in D}C_N(a)-\sum_{a\in D}\frac{1}{a}=\sum_{n\in\partial D}\frac{1}{n},$$

because $1\in D$ and every term from D cancels.

For the consequence, first note that ∂D is always an antichain. Indeed, if $x,y\in\partial D$ with $x\prec y$, then $\pi(y)\in D$ and $x\preceq\pi(y)$, so downward closure would force $x\in D$, contradicting $x\in\partial D$. Hence the displayed identity gives

$$1+\sum_{a\in D}q_N(a)=\sum_{n\in\partial D}\frac{1}{n}\le M(N)$$

for every continuation set D.

Conversely, by Lemma 3.3 there is a maximum-weight antichain B of the form $B=\partial D$ for some continuation set D. Applying the displayed identity to this D yields

$$M(N)=\sum_{n\in B}\frac{1}{n}=1+\sum_{a\in D}q_N(a)\le 1+\max_D\sum_{a\in D}q_N(a).$$

Combining the two inequalities proves the formula for $M(N)$. □

Corollary 3.5. *Assume there is an integer K such that*

$$q_N(a)>0\quad(1\le a\le K),\qquad q_N(a)\le 0\quad(a>K).$$

Then

$$M(N)=S_N(K)=M_{\mathrm{fr}}(N).$$

Proof. By Proposition 3.4, it suffices to maximize $\sum_{a\in D}q_N(a)$ over continuation sets D.

Let D be any continuation set and put

$$D':=D\cap\{1,\dots,K\}.$$

Since every descendant of a vertex $>K$ is also $>K$, the set D' is still downward-closed. Moreover,

$$\sum_{a\in D'}q_N(a)\ge\sum_{a\in D}q_N(a),$$

because only terms with $a > K$ are removed and all of them are nonpositive.

Now $[1, K]$ is itself a continuation set, because every ancestor of a vertex $a \le K$ is $< a$. Since every $q_N(a)$ is positive on $[1, K]$, enlarging D' inside $[1, K]$ can only increase the objective. Hence

$$\sum_{a \in D} q_N(a) \le \sum_{a=1}^{K} q_N(a),$$

with equality for $D = [1, K]$. Therefore $[1, K]$ is optimal. Its boundary is $\partial[1, K] = A_N(K)$. Applying Proposition 3.4 and Proposition 3.2 gives

$$M(N) = 1 + \sum_{a=1}^{K} q_N(a) = S_N(K) = M_{\mathrm{fr}}(N).$$

□

Thus Theorem 1.1 reduces to showing that the positivity set of $q_N(a)$, or equivalently of $R_N(a) - 1$, is always an initial segment.

4 The sign of the frontier increment

4.1 Small fixed values of the root

For fixed a, the function $N \mapsto R_N(a)$ is nondecreasing because increasing N can only add new children of a. Define the threshold

$$\nu(a) := \min\{N \ge 2 : \ R_N(a) > 1\}.$$

Proposition 4.1 (computer-assisted exact small thresholds)**.** *The values of $\nu(a)$ for $1 \le a \le 19$ are*

$$\begin{aligned}
&\nu(1) = 4,\ \nu(2) = 30,\ \nu(3) = 195,\ \nu(4) = 260,\ \nu(5) = 985,\ \nu(6) = 1182,\\
&\nu(7) = 3451,\ \nu(8) = 3944,\ \nu(9) = 4437,\ \nu(10) = 4930,\ \nu(11) = 10351,\\
&\nu(12) = 11292,\ \nu(13) = 21853,\ \nu(14) = 23534,\ \nu(15) = 25215,\\
&\nu(16) = 26896,\ \nu(17) = 46019,\ \nu(18) = 48726,\ \nu(19) = 80807.
\end{aligned}$$

In particular, $\nu(a)$ is strictly increasing on $\{1, \ldots, 19\}$.

Proof. This is a finite exact computation with rational arithmetic. For each fixed $a \le 19$, one enumerates the integers n with $\pi(n) = a$ in increasing order and accumulates the exact rational sum

$$\sum_{\substack{n \le N \\ \pi(n) = a}} \frac{1}{n}$$

until it first exceeds $1/a$. The values above are the first such crossing points. A short verification script accompanying the note carries out this computation exactly. □

Remark 4.2. Proposition 4.1 is the only computer-assisted ingredient in the proof of Theorem 1.1. Everything beyond the finite range $a \le 19$ is proved analytically.

4.2 The low layer

Write

$$T(x) := \sum_{p \le x} \frac{1}{p}.$$

Lemma 4.3. *For every integer $a \ge 20$,*

$$\sum_{a < p \le a^3} \frac{1}{p} > 1.$$

Proof. For $20 \le a \le 26$, the claim follows by direct exact computation:

$$\sum_{20<p\le 20^3} \frac{1}{p} = 1.0028139859\ldots, \qquad \sum_{21<p\le 21^3} \frac{1}{p} = 1.0190536998\ldots,$$
$$\sum_{22<p\le 22^3} \frac{1}{p} = 1.0342730599\ldots, \qquad \sum_{23<p\le 23^3} \frac{1}{p} = 1.0047377040\ldots,$$
$$\sum_{24<p\le 24^3} \frac{1}{p} = 1.0183048829\ldots, \qquad \sum_{25<p\le 25^3} \frac{1}{p} = 1.0310735407\ldots,$$
$$\sum_{26<p\le 26^3} \frac{1}{p} = 1.0430797534\ldots.$$

Now let $a \ge 27$. By the explicit Mertens bounds recorded in [6, Lemma 2],

$$T(x) = \log\log x + B + E(x)$$

with

$$E(x) > -\frac{1}{2\log^2 x}, \qquad E(x) < \frac{1}{\log^2 x} \qquad (x > 1).$$

Therefore

$$T(a^3) - T(a) > \log 3 - \frac{1}{18\log^2 a} - \frac{1}{\log^2 a} = \log 3 - \frac{19}{18\log^2 a}.$$

At $a = 27$ the right-hand side is already larger than 1, and it increases with a. So $T(a^3) - T(a) > 1$ for every $a \ge 27$. □

Corollary 4.4. *Let $N \ge 2$ and let $a \le N^{1/4}$. If either $a \le 19$ and $N \ge \nu(a)$, or $a \ge 20$, then*

$$R_N(a) > 1.$$

In particular, if $20 \le a \le N^{1/4}$, then $R_N(a) > 1$.

Proof. If $a \le 19$ and $N \ge \nu(a)$, the conclusion is the definition of $\nu(a)$. Now assume $a \ge 20$. Since $a \le N^{1/4}$, we have $N/a \ge a^3$. By Lemma 2.7, every prime p with $a < p \le N/a$ contributes the child ap, and each such child contributes $1/p$ to $R_N(a)$. Therefore

$$R_N(a) \ge \sum_{a<p\le N/a} \frac{1}{p} \ge \sum_{a<p\le a^3} \frac{1}{p}.$$

Apply Lemma 4.3. □

4.3 The upper layer

Define

$$P_N(a) := \sum_{a<p\le N/a} \frac{1}{p}, \qquad Q_N(a) := \sum_{\substack{a<p\le q\\ apq\le N}} \frac{1}{pq},$$

where both sums run over primes.

Lemma 4.5 (prime–semiprime description)**.** *If $a > N^{1/4}$, then every child of a is exactly of one of the two forms*

$$ap \qquad \text{or} \qquad apq,$$

with $a < p \le q$ prime and $apq \le N$. Consequently,

$$R_N(a) = P_N(a) + Q_N(a) \qquad (a > N^{1/4}).$$

Proof. Let n be a child of a. Then $n = at$ with every prime factor of t exceeding a. Since $a > N^{1/4}$, we have

$$t \le N/a < a^3,$$

so t has at most two prime factors, each $> a$. Thus t is either a prime $p > a$, giving $n = ap$, or a product pq of primes with $a < p \le q$, giving $n = apq$.

Conversely, if $n = ap$ with $a < p \le N/a$, then $a \prec ap$ and no proper ancestor can lie strictly between a and ap, so $\pi(ap) = a$.

Now let $n = apq$ with $a < p \le q$ prime and $apq \le N$. By Lemma 2.3, the only possible proper ancestor between a and n is ap. But

$$ap > a^2 > N^{1/2}, \qquad q \le \frac{N}{ap} < N^{1/2} < ap,$$

so $q \not> ap$. Hence ap is not an ancestor, and therefore $\pi(apq) = a$. This proves the description of the children and the displayed identity. □

Proposition 4.6 (upper-layer monotonicity)**.** *For fixed N, the function $a \mapsto R_N(a)$ is nonincreasing on the range $a > N^{1/4}$. In particular, for every fixed N, the set*

$$\{a > N^{1/4} : R_N(a) > 1\}$$

is an initial segment of the upper layer.

Proof. By Lemma 4.5, for $a > N^{1/4}$ we have

$$R_N(a) = \sum_{a<p\le N/a} \frac{1}{p} + \sum_{\substack{a<p\le q\\ apq\le N}} \frac{1}{pq}.$$

If $a < b$, then every prime counted for $R_N(b)$ is also counted for $R_N(a)$, and the same is true for every prime pair (p, q) counted for $R_N(b)$. Thus both summation domains shrink as a grows, so $R_N(a)$ is nonincreasing on that range. The final assertion is immediate. □

4.4 The sign theorem

Theorem 4.7 (sign of the frontier increment)**.** *For every $N \geq 2$ there exists an integer $K(N)$ such that*

$$R_N(a) > 1 \quad (1 \leq a \leq K(N)), \qquad R_N(a) \leq 1 \quad (a > K(N)).$$

Equivalently,

$$q_N(a) > 0 \quad (1 \leq a \leq K(N)), \qquad q_N(a) \leq 0 \quad (a > K(N)).$$

Proof. Let

$$k := \lfloor N^{1/4} \rfloor.$$

First consider the range $a \leq 19$. By Proposition 4.1, the thresholds $\nu(a)$ are strictly increasing. Therefore the set

$$\{1 \leq a \leq 19 : R_N(a) > 1\} = \{1 \leq a \leq 19 : N \geq \nu(a)\}$$

is always an initial segment of $\{1, \dots, 19\}$.

If $k \geq 20$, then $N \geq 20^4 > \nu(19)$, so $R_N(a) > 1$ for all $a \leq 19$. By Corollary 4.4, we also have $R_N(a) > 1$ for every $20 \leq a \leq k$. Finally Proposition 4.6 shows that the positivity set in the upper layer $a > k$ is an initial segment. Hence the full positivity set $\{a : R_N(a) > 1\}$ is an initial segment of $\{1, \dots, N\}$.

Now assume $k < 20$. Then the first upper-layer index is $k+1 \leq 19$. On $\{1, \dots, 19\}$ the positivity set is already an initial segment by the threshold table. Starting at $a = k + 1$, Proposition 4.6 shows that $R_N(a)$ is nonincreasing in a. So once the sign turns nonpositive at some upper-layer index, it stays nonpositive for all larger a. Again the full positivity set is an initial segment. This proves the theorem. □

Corollary 4.8 (exact frontier theorem, again)**.** *For every $N \geq 2$,*

$$M(N) = M_{\text{fr}}(N) = \max_{0 \leq K \leq N} S_N(K).$$

More precisely, if $K(N)$ is the cutoff from Theorem 4.7, then

$$M(N) = S_N(K(N)) = \sum_{n \in A_N(K(N))} \frac{1}{n}.$$

Proof. Apply Corollary 3.5 with $K = K(N)$. □

This proves Theorem 1.1.

5 The frontier problem above the quarter-power layer

We next reuse the prime–semiprime analysis of the frontier sweep. The exact frontier theorem now upgrades the frontier asymptotics to asymptotics for the original Erdős maximum.

Proposition 5.1 (exact frontier identity above $N^{1/4}$)**.** *If $\lfloor N^{1/4} \rfloor \leq K \leq \lfloor \sqrt{N} \rfloor$, then*

$$S_N(K) = H_N - H_{\lfloor \sqrt{N} \rfloor} + \sum_{K < a \leq \lfloor \sqrt{N} \rfloor} \frac{1 - P_N(a) - Q_N(a)}{a},$$

where

$$H_m := \sum_{n \leq m} \frac{1}{n}$$

is the mth harmonic number. In particular, for integers a with $N^{1/4} < a \le \sqrt{N}$*,*

$$S_N(a) - S_N(a-1) = \frac{P_N(a) + Q_N(a) - 1}{a}.$$

Proof. Since every integer $n \in \{2, \dots, N\}$ has exactly one parent,

$$\sum_{a=1}^{N} C_N(a) = \sum_{n=2}^{N} \frac{1}{n} = H_N - 1.$$

Also $C_N(a) = 0$ for $a > \sqrt{N}$. Hence Proposition 3.2 gives

$$S_N(K) = 1 + \sum_{a=1}^{K} \left(C_N(a) - \frac{1}{a} \right) = H_N - H_K - \sum_{K < a \le \lfloor \sqrt{N} \rfloor} C_N(a).$$

If $K \ge \lfloor N^{1/4} \rfloor$, then Lemma 4.5 yields

$$C_N(a) = \frac{P_N(a) + Q_N(a)}{a} \qquad (K < a \le \lfloor \sqrt{N} \rfloor).$$

Substituting this identity and using

$$H_{\lfloor \sqrt{N} \rfloor} - H_K = \sum_{K < a \le \lfloor \sqrt{N} \rfloor} \frac{1}{a}$$

gives the formula for $S_N(K)$. The difference formula follows by subtracting the cases $K = a$ and $K = a - 1$. □

We now use three standard analytic tools.

Lemma 5.2 (Mertens on polynomial intervals)**.** *Fix* $\beta \in (0, 1)$*. Uniformly for real numbers* x, y *with*

$$N^\beta \le x \le y \le N,$$

one has

$$\sum_{x < p \le y} \frac{1}{p} = \log \frac{\log y}{\log x} + o(1) \qquad (N \to \infty).$$

Proof. Write

$$B(x) := \sum_{p \le x} \frac{1}{p}.$$

By Mertens' theorem (see, for instance, [9, Chap. I.1]),

$$B(x) = \log \log x + B_1 + r(x), \qquad r(x) \to 0.$$

Hence

$$\sum_{x < p \le y} \frac{1}{p} = B(y) - B(x) = \log \frac{\log y}{\log x} + r(y) - r(x).$$

If $N^\beta \le x \le y \le N$, then both x and y tend to infinity uniformly with N, so $r(x)$ and $r(y)$ are uniformly $o(1)$ on that region. □

Lemma 5.3 (prime-harmonic Riemann sums)**.** *Fix α with $1/4 < \alpha < 1/3$, and let*

$$D_\alpha := \{(u,v):\ \alpha \le u \le 1/3,\ u \le v \le (1-u)/2\}.$$

If $G : D_\alpha \to \mathbb{R}$ is continuous, then uniformly for $u \in [\alpha, 1/3]$,

$$\sum_{N^u < p \le N^{(1-u)/2}} \frac{G\Big(u, \frac{\log p}{\log N}\Big)}{p} = \int_u^{(1-u)/2} \frac{G(u,v)}{v}\, dv + o(1).$$

Proof. Since D_α is compact, G is bounded and uniformly continuous. Fix $\eta > 0$, and choose a partition

$$\alpha = t_0 < t_1 < \cdots < t_J = \frac{1}{2}$$

with mesh at most δ, where $\delta = \delta(\eta) > 0$ is small enough that

$$|G(u,v) - G(u,v')| < \eta$$

whenever $(u,v), (u,v') \in D_\alpha$ and v, v' lie in the same interval $[t_{j-1}, t_j]$.

Fix $u \in [\alpha, 1/3]$. The interval $[u, (1-u)/2]$ is the disjoint union of some full partition intervals together with at most two boundary fragments, whose total dv/v-mass is $O_\alpha(\delta)$. By Lemma 5.2, those boundary fragments contribute $O_\alpha(\delta) + o(1)$ to the prime-harmonic sum, uniformly in u.

On each full interval $I_j = [t_{j-1}, t_j] \subseteq [u, (1-u)/2]$, choose a sample point $\xi_j \in I_j$. Uniform continuity gives

$$\sum_{\substack{N^u < p \le N^{(1-u)/2} \\ \log p/\log N \in I_j}} \frac{G\Big(u, \frac{\log p}{\log N}\Big)}{p} = G(u,\xi_j) \sum_{\substack{N^u < p \le N^{(1-u)/2} \\ \log p/\log N \in I_j}} \frac{1}{p} + O\left(\eta \sum_{\substack{N^u < p \le N^{(1-u)/2} \\ \log p/\log N \in I_j}} \frac{1}{p}\right).$$

Summing over all full intervals and invoking Lemma 5.2 on each of them yields

$$\sum_{N^u < p \le N^{(1-u)/2}} \frac{G\Big(u, \frac{\log p}{\log N}\Big)}{p} = \sum_{I_j \subseteq [u,(1-u)/2]} G(u,\xi_j) \int_{I_j} \frac{dv}{v} + O_\alpha(\eta) + O_\alpha(\delta) + o(1),$$

uniformly in u. The displayed sum is a Riemann sum for $\int_u^{(1-u)/2} G(u,v)\, dv/v$, and because the family $v \mapsto G(u,v)$ is uniformly continuous and bounded on D_α, the Riemann sums converge uniformly in u as $\delta \to 0$. Now let $\eta \to 0$. $\square$

Lemma 5.4 (harmonic Riemann sums)**.** *Let $0 \le \alpha < \beta \le 1$, and let $f : [\alpha, \beta] \to \mathbb{R}$ be continuous. Then*

$$\frac{1}{\log N} \sum_{N^\alpha < a \le N^\beta} \frac{f\Big(\frac{\log a}{\log N}\Big)}{a} \to \int_\alpha^\beta f(u)\, du \qquad (N \to \infty).$$

More generally, if λ_N, μ_N satisfy

$$\alpha \le \lambda_N \le \mu_N \le \beta,$$

then

$$\frac{1}{\log N} \sum_{N^{\lambda_N} < a \le N^{\mu_N}} \frac{f\Big(\frac{\log a}{\log N}\Big)}{a} = \int_{\lambda_N}^{\mu_N} f(u)\, du + o(1),$$

uniformly in the choice of λ_N, μ_N. If $\alpha = 0$, the same conclusion remains true with the sum taken over $1 \le a \le N^\beta$.

Proof. Put

$$u_a := \frac{\log a}{\log N}.$$

Then

$$u_{a+1} - u_a = \frac{\log(1+1/a)}{\log N} = \frac{1}{a\log N} + O\Big(\frac{1}{a^2\log N}\Big)$$

uniformly in a. Since f is bounded on $[\alpha,\beta]$,

$$\frac{1}{\log N}\sum_{N^{\lambda_N}<a\le N^{\mu_N}} \frac{f(u_a)}{a} = \sum_{N^{\lambda_N}<a\le N^{\mu_N}} f(u_a)(u_{a+1}-u_a) + O\Bigg(\frac{1}{\log N}\sum_{a>N^\alpha}\frac{1}{a^2}\Bigg).$$

The error term is $o(1)$, uniformly in λ_N, μ_N. The main sum is the left-endpoint Riemann sum for $\int_{\lambda_N}^{\mu_N} f(u)\,du$, and its mesh tends to 0 uniformly on subintervals of $[\alpha,\beta]$. This proves the variable-endpoint form, from which the fixed-endpoint statement is the special case $\lambda_N \equiv \alpha$, $\mu_N \equiv \beta$. If $\alpha = 0$, the extra term $a = 1$ contributes only $f(0)/\log N = o(1)$. □

For

$$\frac{1}{4} < u \le \frac{1}{2}$$

define

$$\Phi(u) := \log\frac{1-u}{u} + \int_u^{(1-u)/2} \frac{1}{v}\log\frac{1-u-v}{v}\,dv,$$

with the understanding that the integral is 0 when $(1-u)/2 \le u$, i.e. when $u \ge 1/3$.

Lemma 5.5 (uniform prime–semiprime asymptotics)**.** *Fix $1/4 < \alpha < 1/2$. Then, uniformly for $N^\alpha \le a \le \sqrt{N}$,*

$$P_N(a) + Q_N(a) = \Phi(u) + o(1), \qquad u := \frac{\log a}{\log N}.$$

Proof. Write

$$u := \frac{\log a}{\log N}.$$

By Lemma 5.2,

$$P_N(a) = \sum_{a<p\le N/a}\frac{1}{p} = \log\frac{1-u}{u} + o(1)$$

uniformly for $N^\alpha \le a \le \sqrt{N}$.

If $\alpha \ge 1/3$, then $u \ge 1/3$ throughout the range, so the outer prime range in $Q_N(a)$ is empty. Thus $Q_N(a) = 0$ and the claim follows from the identity

$$\Phi(u) = \log\frac{1-u}{u} \qquad \Big(\frac{1}{3} \le u \le \frac{1}{2}\Big).$$

If $\alpha < 1/3$, it remains only to treat the subrange $u \in [\alpha, 1/3]$, since for $u \ge 1/3$ we again have $Q_N(a) = 0$. Define

$$G(u,v) := \log\frac{1-u-v}{v}$$

on the compact triangle

$$D_\alpha = \{(u,v):\ \alpha \le u \le 1/3,\ u \le v \le (1-u)/2\}.$$

This function is continuous. For each outer prime p, write

$$v := \frac{\log p}{\log N}.$$

Then Lemma 5.2 gives

$$\sum_{p \le q \le N/(ap)} \frac{1}{q} = G(u, v) + o(1)$$

uniformly for $(u, v) \in D_\alpha$. Since

$$\sum_{a < p \le N^{(1-u)/2}} \frac{1}{p} \ll_\alpha 1,$$

the total contribution of the uniform $o(1)$ term is still $o(1)$. Therefore

$$Q_N(a) = \sum_{a < p \le N^{(1-u)/2}} \frac{G\Big(u, \frac{\log p}{\log N}\Big)}{p} + o(1)$$

uniformly for $u \in [\alpha, 1/3]$. Applying Lemma 5.3 yields

$$Q_N(a) = \int_u^{(1-u)/2} \frac{1}{v} \log \frac{1-u-v}{v}\, dv + o(1)$$

uniformly in u. Combining this with the prime term proves the claim. □

Proposition 5.6. *The function* Φ *is continuous and strictly decreasing on* $[1/4, 1/2]$. *Moreover,*

$$\Phi\left(\frac{1}{4}\right) = 1.2458329656\ldots > 1, \qquad \Phi\left(\frac{1}{3}\right) = \log 2 < 1.$$

Hence there is a unique

$$\alpha_2 \in \left(\frac{1}{4}, \frac{1}{3}\right)$$

with

$$\Phi(\alpha_2) = 1.$$

Numerically,

$$\alpha_2 = 0.2804383098923534\ldots.$$

Proof. Continuity is clear. For $u \in [1/3, 1/2]$ we have

$$\Phi(u) = \log \frac{1-u}{u},$$

which is strictly decreasing.

Now take $u \in (1/4, 1/3)$ and write

$$I(u) := \int_u^{(1-u)/2} \frac{1}{v} \log \frac{1-u-v}{v}\, dv.$$

By Leibniz' rule, since the integrand vanishes at the upper endpoint $v = (1-u)/2$,

$$I'(u) = -\frac{1}{u} \log \frac{1-2u}{u} - \int_u^{(1-u)/2} \frac{dv}{v(1-u-v)}.$$

Also

$$\int_u^{(1-u)/2} \frac{dv}{v(1-u-v)} = \frac{1}{1-u}\log\frac{1-2u}{u}.$$

Therefore

$$I'(u) = -\left(\frac{1}{u} + \frac{1}{1-u}\right)\log\frac{1-2u}{u} < 0,$$

because $u < 1/3$ implies $(1-2u)/u > 1$. Since

$$\frac{d}{du}\log\frac{1-u}{u} = -\frac{1}{1-u} - \frac{1}{u} < 0,$$

we conclude that $\Phi'(u) < 0$ on $(1/4, 1/3)$ as well. So Φ is strictly decreasing on the whole interval $[1/4, 1/2]$. The numerical values follow from direct evaluation. Strict monotonicity then gives the unique solution α_2 of $\Phi(u) = 1$. □

Set

$$\alpha := \frac{1}{e+1} = 0.2689414213\ldots, \qquad c_2 := \frac{1}{2} + \int_{\alpha_2}^{1/2} (1 - \Phi(u))\, du.$$

Lemma 5.7 (prime-only lower ramp)**.** *For every $\varepsilon > 0$ and all sufficiently large N, the function $K \mapsto S_N(K)$ is strictly increasing for*

$$1 \le K \le N^{\alpha-\varepsilon}.$$

In particular, every maximizing cutoff satisfies

$$K_*(N) \ge N^{\alpha-\varepsilon}.$$

Proof. It is enough to treat the case $0 < \varepsilon < \alpha$, since for $\varepsilon \ge \alpha$ the range

$$1 \le K \le N^{\alpha-\varepsilon}$$

is eventually empty or consists only of $K = 1$, and the stated lower bound for $K_*(N)$ is then trivial.

The function $a \mapsto P_N(a)$ is nonincreasing. By Mertens' theorem,

$$P_N(\lfloor N^{\alpha-\varepsilon} \rfloor) = \log\frac{1-(\alpha-\varepsilon)}{\alpha-\varepsilon} + o(1).$$

Because $\alpha = 1/(e+1)$, the limiting quantity is strictly larger than 1. So for some $\eta = \eta(\varepsilon) > 0$ and all sufficiently large N we have

$$P_N(a) \ge 1 + \eta \qquad (1 \le a \le N^{\alpha-\varepsilon}).$$

By Lemma 2.7, every prime p with $a < p \le N/a$ contributes the child ap to $C_N(a)$, so

$$C_N(a) \ge \frac{P_N(a)}{a}.$$

Hence Proposition 3.2 gives

$$S_N(a) - S_N(a-1) = C_N(a) - \frac{1}{a} \ge \frac{P_N(a) - 1}{a} > 0$$

throughout that range. This proves the claim. □

Theorem 5.8 (frontier asymptotic law)**.** *Let $K_*(N)$ be any cutoff from Corollary 4.8. Then*

$$K_*(N) = N^{\alpha_2+o(1)}.$$

Moreover,

$$M(N) = \big(c_2 + o(1)\big)\log N.$$

Numerically,

$$\alpha_2 = 0.2804383098923534\ldots, \qquad c_2 = 0.6187712111099834\ldots.$$

Proof. By Corollary 4.8, it is enough to analyze the frontier maximizer. The argument is the same as in the frontier-only setting, but now yields the true maximum.

Fix $\varepsilon > 0$.

For the upper bound on $K_*(N)$, we may assume

$$0 < \varepsilon < \frac{1}{2} - \alpha_2,$$

since $K_*(N) \le \sqrt{N}$ for every N by Proposition 3.2 and the fact that $C_N(a) = 0$ for $a > \sqrt{N}$. Then Proposition 5.6 gives

$$\Phi(u) \le 1 - \eta \qquad (\alpha_2 + \varepsilon \le u \le 1/2)$$

for some $\eta = \eta(\varepsilon) > 0$. By Lemma 5.5, uniformly for integers a with

$$N^{\alpha_2+\varepsilon} \le a \le \sqrt{N},$$

we have

$$P_N(a) + Q_N(a) \le 1 - \eta/2$$

for all sufficiently large N. Proposition 5.1 then yields

$$S_N(a) - S_N(a-1) = \frac{P_N(a) + Q_N(a) - 1}{a} \le -\frac{\eta}{2a} < 0$$

throughout that interval. For $a > \sqrt{N}$ we have $C_N(a) = 0$, so

$$S_N(a) - S_N(a-1) = -\frac{1}{a} < 0.$$

Therefore every maximizing cutoff satisfies

$$K_*(N) \le N^{\alpha_2+\varepsilon}$$

for all sufficiently large N.

For the lower bound, first note that if

$$\varepsilon > \alpha_2 - \alpha,$$

then with

$$\delta := \varepsilon - (\alpha_2 - \alpha) > 0$$

Lemma 5.7 gives

$$K_*(N) \ge N^{\alpha-\delta} = N^{\alpha_2-\varepsilon}$$

for all sufficiently large N. The boundary value $\varepsilon = \alpha_2 - \alpha$ then follows from any smaller value of ε, which gives a stronger lower bound. So only the case

$$0 < \varepsilon < \alpha_2 - \alpha$$

needs further work.

Choose any fixed β with

$$\frac{1}{4} < \beta < \alpha < \alpha_2 - \varepsilon.$$

By Lemma 5.7, every maximizing cutoff satisfies

$$K_*(N) \geq N^\beta$$

once N is large. On the interval $[\beta, \alpha_2 - \varepsilon]$, Proposition 5.6 gives

$$\Phi(u) \geq 1 + \eta'$$

for some $\eta' = \eta'(\beta, \varepsilon) > 0$. By Lemma 5.5, uniformly for integers a with

$$N^\beta \leq a \leq N^{\alpha_2 - \varepsilon},$$

we have

$$P_N(a) + Q_N(a) \geq 1 + \eta'/2$$

for all sufficiently large N. Using Proposition 5.1 again,

$$S_N(a) - S_N(a-1) = \frac{P_N(a) + Q_N(a) - 1}{a} \geq \frac{\eta'}{2a} > 0$$

throughout that interval. Since a maximizing cutoff already lies at or beyond N^β, it must in fact lie at or beyond $N^{\alpha_2 - \varepsilon}$. Thus

$$N^{\alpha_2 - \varepsilon} \leq K_*(N) \leq N^{\alpha_2 + \varepsilon}$$

for all large N. Because $\varepsilon > 0$ is arbitrary, this proves

$$K_*(N) = N^{\alpha_2 + o(1)}.$$

Set

$$\kappa_N := \frac{\log K_*(N)}{\log N}.$$

Then $\kappa_N \to \alpha_2$. Since $\alpha_2 > 1/4$, we have $K_*(N) > N^{1/4}$ for all sufficiently large N, so Proposition 5.1 applies:

$$M(N) = S_N(K_*(N)) = H_N - H_{\lfloor \sqrt{N} \rfloor} + \sum_{K_*(N) < a \leq \lfloor \sqrt{N} \rfloor} \frac{1 - P_N(a) - Q_N(a)}{a}.$$

Also

$$H_N - H_{\lfloor \sqrt{N} \rfloor} = \frac{1}{2} \log N + O(1).$$

Fix $\delta > 0$ so small that $\alpha_2 - \delta > 1/4$. For large N, the lower limit $K_*(N)$ lies in $[N^{\alpha_2 - \delta}, N^{\alpha_2 + \delta}]$, and Lemma 5.5 applies uniformly on $[\alpha_2 - \delta, 1/2]$. Hence the sum is a harmonic Riemann sum:

$$\frac{1}{\log N} \sum_{K_*(N) < a \leq \lfloor \sqrt{N} \rfloor} \frac{1 - P_N(a) - Q_N(a)}{a} = \int_{\kappa_N}^{1/2} (1 - \Phi(u))\, du + o(1).$$

Since $\kappa_N \to \alpha_2$ and $1-\Phi(u)$ is continuous on $[\alpha_2-\delta, 1/2]$,

$$\int_{\kappa_N}^{1/2} (1-\Phi(u))\,du \to \int_{\alpha_2}^{1/2} (1-\Phi(u))\,du.$$

Therefore

$$\frac{M(N)}{\log N} \to \frac{1}{2} + \int_{\alpha_2}^{1/2} (1-\Phi(u))\,du = c_2.$$

□

6 A purely analytic eventual frontier theorem

The finite verification in Proposition 4.1 is only needed to obtain the exact frontier theorem for every N. The Bellman viewpoint already gives eventual frontier exactness without any computer assistance.

Set

$$B_N(a) := a \sum_{b\in \mathrm{ch}_N(a)} F_N(b).$$

By Theorem 2.4, continuation is optimal at the state a exactly when

$$B_N(a) \geq 1.$$

Lemma 6.1 (threshold Bellman policy)**.** *Fix $K \in \{0,\dots,N\}$. For each $a \leq N$, let $G_{N,K}(a)$ be the reciprocal weight obtained inside the subtree $\mathcal{T}_N(a)$ by the deterministic policy that continues from a vertex x when $x \leq K$ and stops when $x > K$. Then*

$$G_{N,K}(a) = \begin{cases} \dfrac{1}{a}, & a > K, \\ \sum_{b\in \mathrm{ch}_N(a)} G_{N,K}(b), & a \leq K. \end{cases}$$

At the root one has

$$G_{N,K}(1) = S_N(K).$$

Consequently, if for some K we have

$$F_N(a) = \frac{1}{a} \qquad (a > K),$$

and

$$F_N(a) = \sum_{b\in \mathrm{ch}_N(a)} F_N(b) \qquad (a \leq K),$$

then

$$M(N) = F_N(1) = S_N(K) = M_{\mathrm{fr}}(N).$$

Proof. If $a > K$, the policy stops immediately at a, so $G_{N,K}(a) = 1/a$. If $a \leq K$, the policy continues from a and then acts independently on each child subtree, giving

$$G_{N,K}(a) = \sum_{b\in \mathrm{ch}_N(a)} G_{N,K}(b).$$

At the root, this policy selects exactly the first vertices strictly above K on the root-to-leaf paths, namely the frontier $A_N(K)$, so $G_{N,K}(1) = S_N(K)$.

Now argue by descending induction on a. For $a > K$ we have $F_N(a) = 1/a = G_{N,K}(a)$ by hypothesis. If $a \le K$, then the induction hypothesis gives $F_N(b) = G_{N,K}(b)$ for all children b of a, and therefore

$$F_N(a) = \sum_{b \in \mathrm{ch}_N(a)} F_N(b) = \sum_{b \in \mathrm{ch}_N(a)} G_{N,K}(b) = G_{N,K}(a).$$

Hence $F_N(a) = G_{N,K}(a)$ for all a, and in particular

$$M(N) = F_N(1) = G_{N,K}(1) = S_N(K) \le M_{\mathrm{fr}}(N) \le M(N).$$

So equality holds throughout. □

Proposition 6.2 (eventual continuation interval)**.** *For all sufficiently large N, the set*

$$\mathcal{C}_N := \{a \le N : \ B_N(a) \ge 1\}$$

is an initial interval

$$\mathcal{C}_N = \{1, 2, \dots, K_\infty(N)\}$$

for some integer $K_\infty(N) \ge \lfloor N^{1/4} \rfloor$.

Proof. Let

$$L := \lfloor N^{1/4} \rfloor.$$

First take $a \le L$. For every prime p with $a < p \le N/a$, the prime child lemma gives $\pi(ap) = a$, and certainly

$$F_N(ap) \ge \frac{1}{ap}.$$

Hence

$$B_N(a) = a \sum_{b \in \mathrm{ch}_N(a)} F_N(b) \ge a \sum_{a < p \le N/a} \frac{1}{ap} = P_N(a) \ge P_N(L).$$

By Lemma 5.2,

$$P_N(L) = \log 3 + o(1) > 1.$$

Thus $B_N(a) > 1$ for every $a \le L$ once N is large enough.

Now take $a > L$. By Lemma 4.5, every child b of a is of the form ap or apq with primes $a < p \le q$. In particular,

$$b > a^2 > N^{1/2}.$$

So b has no child below N, hence b is a leaf and

$$F_N(b) = \frac{1}{b} \qquad (b \in \mathrm{ch}_N(a)).$$

Therefore

$$B_N(a) = a \sum_{b \in \mathrm{ch}_N(a)} \frac{1}{b} = P_N(a) + Q_N(a).$$

By Proposition 4.6, the right-hand side is nonincreasing in a throughout the range $a > L$. Also $B_N(a) = 0 < 1$ for $a > \sqrt{N}$. So there is some $K_\infty(N) \ge L$ such that

$$B_N(a) \ge 1 \iff a \le K_\infty(N).$$

This is exactly the stated interval property. □

Theorem 6.3 (purely analytic eventual frontier exactness)**.** *For all sufficiently large N, there exists an integer $K_\infty(N)$ such that*

$$M(N) = S_N(K_\infty(N)) = M_{\rm fr}(N).$$

Proof. By Proposition 6.2, for all sufficiently large N the Bellman continuation set is the initial interval $\{1, \dots, K_\infty(N)\}$. At states with $B_N(a) = 1$, both actions are optimal, and we choose continuation. Thus

$$F_N(a) = \sum_{b \in {\rm ch}_N(a)} F_N(b) \quad (a \le K_\infty(N)), \qquad F_N(a) = \frac{1}{a} \quad (a > K_\infty(N)).$$

Lemma 6.1 then yields

$$M(N) = S_N(K_\infty(N)) = M_{\rm fr}(N).$$

□

7 Sample exact values

By Corollary 4.8, the following are exact values of the original extremal problem.

N	$K_*(N)$	$M(N)$	$M(N)/\log N$
10^2	2	3.071866240609930	0.667047278720890
10^3	5	4.423451396033411	0.640360177421516
10^4	10	5.797395213934269	0.629444187705994
10^5	22	7.190091161965688	0.624523383204608
10^6	42	8.593119797840970	0.621990751755987
10^7	84	10.003706292017160	0.620650634457699

These exact values decrease steadily and are already close to the asymptotic constant $c_2 = 0.6187712111\dots$.

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