A Negative Answer to an Erdős–Nathanson Question on Two Disjoint Bases

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Abstract

Erdős and Nathanson asked whether, if A_1, A_2 are disjoint asymptotic additive bases of order 2, the union $A = A_1 \cup A_2$ must contain a minimal asymptotic basis of order 2. The answer is negative. We give a detailed construction of disjoint bases B, C such that every asymptotic subbasis of $A = B \cup C$ remains a basis after deleting its least element. The proof is a streamlined specialization of Daniel Larsen's "fragile basis" construction, corresponding to the case $(P2)$ = true and $(P3)$ = false in [3]. We also record the randomized block construction which supplies the required two-colour canary template.

1 The question and the result

Throughout, $\mathbb{N} = \{1, 2, 3, \ldots\}$; changing finitely many elements is immaterial. For $A \subseteq \mathbb{N}$, write

$$r_A(n) = \left|\{(a, a') \in A^2 : a \leq a',\ a + a' = n\}\right|.$$

The set A is an *asymptotic additive basis of order* 2, or simply a basis, if $r_A(n) > 0$ for every sufficiently large n. A subbasis $D \subseteq A$ is *minimal* if no proper subset of D is again an asymptotic basis. Equivalently, for every $d \in D$, the set $D \setminus \{d\}$ is not a basis.

Erdős and Nathanson asked in [5, Open Problem 4]:

> If A_1, A_2 are asymptotic bases of order 2 with $A_1 \cap A_2 = \emptyset$, and $A = A_1 \cup A_2$, must A contain a minimal asymptotic basis of order 2?

The problem is also listed as Erdős Problem #869 [1]. Larsen announced on the associated discussion thread that the implication fails, and subsequently proved the stronger independence theorem in [3]. The theorem below is the direct negative answer.

Theorem 1. *There exist disjoint asymptotic bases $B, C \subseteq \mathbb{N}$ of order 2 such that $A = B \cup C$ contains no minimal asymptotic basis of order 2. More strongly, if $D \subseteq A$ is any asymptotic basis and $d_0 = \min D$, then $D \setminus \{d_0\}$ is still an asymptotic basis of order 2.*

The construction has two parts. First we isolate a two-colour canary template. Then we specify a selection rule for the canaries. The template is due to Larsen [3]; our selection rule is a slightly streamlined presentation of the decomposable/no-minimal-subbasis case of his construction.

2 A two-colour canary template

Let

$$N_k = 4^{k+1}, \qquad h(n) = \left\lfloor 10^{-8} n \right\rfloor.$$

For a representation $n = a + a'$ with $a \le a'$, call it *balanced* if $a'/a \in [1, 100]$. Let $\widetilde{r}_A(n)$ denote the number of balanced representations of n by elements of A.

We shall use the following theorem as a template. A proof, following Larsen's random block construction, is given in Section 5.

Theorem 2 (Two-colour canary template)**.** *Suppose that at stage k a selection mechanism, after seeing the earlier data, produces finite sets*

$$G_k \subseteq B(k-1), \qquad H_k \subseteq C(k-1),$$

where

$$B(k-1) = \bigcup_{i<k} (B_i \cup (N_{i+1} - G_i)), \qquad C(k-1) = \bigcup_{i<k} (C_i \cup (N_{i+1} - H_i)),$$

and where $F_k = G_k \cup H_k \subseteq [1, N_k]$. Assume $G_k \cap H_k = \emptyset$ for every possible input.

Then there exist finite blocks $B_k, C_k \subset (N_k, N_{k+1})$ such that, with

$$B = \bigcup_{i \ge 1} (B_i \cup (N_{i+1} - G_i)), \qquad C = \bigcup_{i \ge 1} (C_i \cup (N_{i+1} - H_i)),$$

the following hold.

(i) $B \cap C = \emptyset$.

(ii) *For all sufficiently large $n \notin \{N_i : i \ge 1\}$,*

$$\widetilde{r}_B(n) \ge h(n), \qquad \widetilde{r}_C(n) \ge h(n).$$

(iii) *For all sufficiently large k, every representation of N_{k+1} as a sum of two elements of $B \cup C$ has at least one summand in F_k.*

The set F_k is the canary for the exceptional integer N_{k+1}. The construction makes all representations of N_{k+1} pass through F_k, while keeping many balanced representations of all non-exceptional integers in each colour separately.

3 The selection rule

We now define the canaries. Put

$$\phi(k) = k, \qquad \psi(k) = k.$$

After the previous stages have been constructed, a finite set S is called *k-eligible* if

$$S = \{s_1 < \cdots < s_{\phi(k)}\} \subseteq B(k-1) \cup C(k-1),$$

$$\psi(k) \le s_1 < \cdots < s_{\phi(k)} \le N_k,$$

and S meets both colours:

$$S \cap B(k-1) \neq \emptyset, \qquad S \cap C(k-1) \neq \emptyset.$$

A set is *cumulatively k-eligible* if it is i-eligible for some $i \leq k$.

Fix once and for all a total ordering of finite subsets of $\mathbb{N}$ which first compares the largest element and then breaks ties lexicographically. At stage k, let $\mathcal{E}_k$ be the family of cumulatively k-eligible sets which have not previously been used as one of $F_1, \dots, F_{k-1}$. If $\mathcal{E}_k = \emptyset$, set $F_k = \emptyset$. Otherwise let F_k be the first element of $\mathcal{E}_k$ in the fixed ordering. Finally set

$$G_k = F_k \cap B(k-1), \qquad H_k = F_k \cap C(k-1).$$

This is a legitimate input for Theorem 2: when $F_k \neq \emptyset$ the two pieces are separated by colour, and $F_k \subseteq [1, N_k]$.

Let B, C be the sets supplied by Theorem 2 for this selection mechanism, and set $A = B \cup C$.

Lemma 3. *For all sufficiently large k, there are at least k distinct k-eligible sets. Consequently $F_k \neq \emptyset$ for all sufficiently large k.*

Proof. Suppose first that $n = N_k - 1$. For large k, this is not one of the exceptional integers N_i, so Theorem 2 gives

$$\widetilde{r}_B(N_k - 1) \geq h(N_k - 1), \qquad \widetilde{r}_C(N_k - 1) \geq h(N_k - 1).$$

Every balanced representation of $N_k - 1$ uses only summands $< N_k$, hence summands belonging to $B(k-1)$ or to $C(k-1)$ respectively. Therefore

$$|B(k-1)| \geq h(N_k - 1)^{1/2}, \qquad |C(k-1)| \geq h(N_k - 1)^{1/2}.$$

After discarding the finitely many elements below k, both colours still contain more than k elements in $[k, N_k]$, for large k. Hence one may choose many k-element subsets of $B(k-1) \cup C(k-1)$ lying in $[k, N_k]$ and meeting both colours. In particular there are at least k such sets.

At stage k fewer than k sets have previously been used. Thus, once there are at least k k-eligible sets, at least one cumulatively k-eligible set remains unused, so $F_k \neq \emptyset$. □

Lemma 4. *Every eligible set is eventually used as some F_j.*

Proof. Let S be i-eligible, and let $m = \max S$. Then S is cumulatively k-eligible for every $k \geq i$ unless it has already been used. There are only finitely many finite subsets of $\mathbb{N}$ whose maximum is at most m. The selection rule always chooses the least unused cumulatively eligible set in the fixed order. By Lemma 3, the process chooses a non-empty F_k at every sufficiently large stage. Therefore all unused cumulatively eligible sets preceding or equal to S in the fixed order are exhausted after finitely many stages, and then S itself is chosen. □

Lemma 5. *Both B and C are asymptotic bases of order* 2.

Proof. Let n be sufficiently large. If $n \notin \{N_i\}$, then Theorem 2 gives a balanced representation of n in $B + B$ and also one in $C + C$.

It remains to consider the exceptional integers. By Lemma 3, $F_k \neq \emptyset$ for all sufficiently large k; by eligibility, such an F_k meets both colours. Thus G_k and H_k are non-empty. If $g \in G_k$, then

$$N_{k+1} = g + (N_{k+1} - g)$$

is a representation in $B+B$. If $h \in H_k$, then

$$N_{k+1} = h + (N_{k+1} - h)$$

is a representation in $C+C$. Hence each of B and C represents every sufficiently large exceptional integer as well. □

4 No subbasis is minimal

Let $D \subseteq A$ be an arbitrary asymptotic basis. We prove that deleting the least element of D leaves a basis.

Lemma 6. *There exists K such that, for every $k \geq K$, the set D meets every k-eligible set.*

Proof. Since D is a basis, $D+D$ contains N_j for all sufficiently large j. By Theorem 2, every representation of $N_j = N_{(j-1)+1}$ by elements of A meets F_{j-1}, for all sufficiently large j. Hence $D \cap F_{j-1} \neq \emptyset$ for all sufficiently large j for which F_{j-1} is relevant.

Now let S be k-eligible. By Lemma 4, $S = F_j$ for some $j \geq k$. If k is sufficiently large, then j is sufficiently large for the preceding paragraph to apply. Thus $D \cap S \neq \emptyset$. □

Lemma 7. *For all sufficiently large M, if k is the least integer with $N_k > M$, then*

$$|(B \setminus D) \cap [M, 100M]| < k+4 \quad \textit{or} \quad |(C \setminus D) \cap [M, 100M]| < k+4.$$

Proof. Assume the contrary for some large M. Choose k minimal with $N_k > M$. If both displayed cardinalities are at least $k+4$, then we can choose a $(k+4)$-element subset of $[M, 100M]$ avoiding D and meeting both colours. For large M we have

$$k+4 \leq M, \qquad 100M < N_{k+4},$$

because $M < N_k$ and $N_{k+4} = 4^4 N_k = 256 N_k$. Hence the chosen set is $(k+4)$-eligible and is disjoint from D, contradicting Lemma 6. □

Lemma 8. *For each fixed $a \in A$, only finitely many of the canaries F_k contain a.*

Proof. If an eligible set S contains a and is i-eligible, then $i = \psi(i) \leq a$ and $S \subseteq [1, N_i]$. There are only finitely many choices for such an S, since $i \leq a$. By the selection rule, each eligible set is used at most once. Hence a belongs to only finitely many F_k. □

Proof of Theorem 1. By Lemma 5, B and C are disjoint bases, so $A = B \cup C$ has the required decomposition. Let $D \subseteq A$ be any asymptotic basis and put $d_0 = \min D$, $D' = D \setminus \{d_0\}$.

First consider a sufficiently large non-exceptional integer $n \notin \{N_i\}$. Write

$$n = 101M - t, \qquad 0 \leq t \leq 100,$$

with $M \to \infty$ as $n \to \infty$. Let k be minimal with $N_k > M$. By Lemma 7, at least one colour is missed by D fewer than $k+4$ times on $[M, 100M]$. Suppose it is B; the case of C is identical.

Every balanced representation $n = b + b'$ with $b \leq b'$ and $b, b' \in B$ has both summands in $[M, 100M]$: indeed $b \geq n/101 = M - t/101$, so $b \geq M$, and then $b' \leq n - M \leq 100M$. Theorem 2 gives at least $h(n)$ such balanced B-representations. Each element of $(B \setminus D) \cap [M, 100M]$ can destroy at most one of them. Therefore

$$r_{D\cap[M,100M]}(n) \geq h(n) - (k+4).$$

Since $k = O(\log M)$ and $h(n) \gg M$, the right side is > 1 for all sufficiently large M. Deleting d_0 destroys at most one further representation, so $r_{D'}(n) > 0$ for all sufficiently large non-exceptional n.

Now let $n = N_i$ be exceptional and sufficiently large. Since D is a basis, $r_D(N_i) > 0$. By Lemma 8, $d_0 \notin F_{i-1}$ for all sufficiently large i. Also $F_{i-1} \subseteq [1, N_{i-1}]$, while $N_i - d_0 > N_{i-1}$ for large i. Hence a representation of N_i using d_0 would meet neither summand in F_{i-1}, contradicting the canary property in Theorem 2. Thus no A-representation of N_i uses d_0, and consequently

$$r_{D'}(N_i) = r_D(N_i) > 0$$

for all sufficiently large exceptional N_i.

Combining the non-exceptional and exceptional cases, D' is an asymptotic basis of order 2. Since this holds for every asymptotic subbasis $D \subseteq A$, no such D is minimal. □

5 Larsen's randomized block construction

For completeness, and to make the dependence on Larsen's construction explicit, we recall the proof of Theorem 2. This is the construction in [3, Section 4], specialized to the two-colour decomposable setting.

Let X_1, X_2, X_3 be a random partition of $\mathbb{N}$: independently for each integer, choose one of the three colours with probability 1/3. For $X \subseteq \mathbb{N}$ and an interval I, write $X_i(I) = X_i \cap I$. If $I = (a, b)$ is an interval, put

$$I_{\text{core}} = \left(a + \frac{b-a}{100},\ b - \frac{b-a}{100}\right).$$

The same notation applies to the interval $I + J$.

We use two elementary large-deviation lemmas. The constants are deliberately wasteful.

Lemma 9 (Random interval sums)**.** *There is $\varepsilon > 0$ such that the following holds. Let I, J be intervals of positive integers of length at least m. With probability at least $1 - (1-\varepsilon)^m$, every element of $(I+J)_{\text{core}}$ has at least $m/2000$ representations as $x + y$ with $x \in X_i(I)$ and $y \in X_i(J)$.*

Proof. Fix $z \in (I+J)_{\text{core}}$. The definition of the core implies that $z = x + y$ has at least $m/100$ solutions with $x \in I$ and $y \in J$. From these choose at least $m/200$ pairs whose entries are all distinct. Each chosen pair lands in $X_i(I) \times X_i(J)$ with probability 1/9, independently over the chosen pairs. A Chernoff bound gives an exponentially small probability that fewer than $m/2000$ pairs survive. The number of possible z is $O(m)$, and the union bound is absorbed by changing ε. □

Lemma 10 (Random reflected sums)**.** *There is $\varepsilon > 0$ such that the following holds for all sufficiently large N. Let I, J be intervals of positive integers of length $> N/100$, with every element of J less than $4N$. Then, for any $i, j \in \{1, 2, 3\}$, with probability at least $1 - (1-\varepsilon)^N$, every*

$$z \in (I + 4N - J)_{\text{core}} \setminus \{4N\}$$

has more than $10^{-7}N$ *representations*

$$z = x + (4N - y), \qquad x \in X_i(I), \quad y \in X_j(J), \quad 4N - y \in X_i.$$

Proof. Write $z = 4N + \Delta$. We need many $y \in J$ for which $y + \Delta \in I$, $y \in X_j$, and both $y + \Delta$ and $4N - y$ lie in X_i. Let

$$S = (I - \Delta) \cap J.$$

The assumption that z lies in the core implies $|S| \gg N$, say $|S| > N/5000$. Choose a maximal subset $Y \subseteq S$ such that the triples

$$\{y,\ y + \Delta,\ 4N - y\} \qquad (y \in Y)$$

are pairwise disjoint; then $|Y| \geq |S|/9$. For each $y \in Y$, the desired colouring event has probability at least $1/27$, independently over Y. Another Chernoff bound, followed by a union bound over z, proves the claim after adjusting ε. $\square$

We now construct the blocks. Suppose that stages $< k$ have been completed and that the selection mechanism has produced G_k, H_k. Set

$$N = N_k, \qquad A(k-1) = B(k-1) \cup C(k-1).$$

Define

$$B_k = \Big((4N/3, 2N) \cup (8N/3, 3N) \cup \big(4N - ((0, N) \setminus A(k-1))\big)\Big) \cap X_1, \tag{1}$$

$$C_k = \Big((4N/3, 2N) \cup (8N/3, 3N) \cup \big(4N - ((0, N) \setminus A(k-1))\big)\Big) \cap X_2. \tag{2}$$

Then extend

$$B(k) = B(k-1) \cup B_k \cup (4N - G_k), \qquad C(k) = C(k-1) \cup C_k \cup (4N - H_k).$$

Equations (1)–(2) are Larsen's block construction. The middle intervals provide ordinary random sums; the reflected part $4N - ((0, N) \setminus A(k-1))$ prevents unwanted old-new representations of $4N$; and the mirrors $4N - G_k$, $4N - H_k$ create precisely the canary representations through F_k.

Proposition 11. *With probability* 1*, for all sufficiently large* k *and every*

$$n \in [3N_k/2, 6N_k) \setminus \{4N_k\},$$

one has

$$\widetilde{r}_{B(k)}(n) \geq h(n), \qquad \widetilde{r}_{C(k)}(n) \geq h(n).$$

Proof. We prove the assertion for $B(k)$; the proof for $C(k)$ is the same with X_2 in place of X_1. The construction ensures that $B(k)$ contains the following random pieces:

$$X_1\Big(\frac{N}{24}, \frac{3N}{64}\Big), \quad X_1\Big(\frac{2N}{3}, \frac{3N}{4}\Big), \quad X_1\Big(\frac{4N}{3}, 2N\Big), \quad X_1\Big(\frac{8N}{3}, 3N\Big), \quad X_1(4N - X_3(N/4, 3N/4)).$$

The first piece comes from an earlier block and the second from a nearer earlier block; the last inclusion follows because, for $i < k$, the mirrors $N_{i+1} - G_i$ lie outside the central interval $(N/4, 3N/4)$.

Apply Lemma 9 with $i = 1$ to the fixed intervals above, and apply Lemma 10 with $i = 1$, $j = 3$, and $J = (N/4, 3N/4)$. The resulting sums cover the full interval $[3N/2, 6N)$ except for $4N$. More explicitly:

$$\Big(\frac{N}{24}, \frac{3N}{64}\Big) + \Big(\frac{4N}{3}, 2N\Big), \quad \Big(\frac{2N}{3}, \frac{3N}{4}\Big) + \Big(\frac{4N}{3}, 2N\Big), \quad \Big(\frac{4N}{3}, 2N\Big) + \Big(\frac{4N}{3}, 2N\Big),$$
$$\Big(\frac{8N}{3}, 3N\Big) + \Big(\frac{4N}{3}, 2N\Big), \quad \Big(\frac{8N}{3}, 3N\Big) + \Big(\frac{8N}{3}, 3N\Big),$$

handle the ordinary ranges, and the reflected sums

$$I + 4N - (N/4, 3N/4), \qquad I \in \Big\{\Big(\frac{2N}{3}, \frac{3N}{4}\Big), \Big(\frac{4N}{3}, 2N\Big), \Big(\frac{8N}{3}, 3N\Big)\Big\},$$

fill the remaining gaps around $4N$ and above $29N/6$, excluding $4N$ itself. The lower bounds from Lemmas 9 and 10 are all $\gg N$, in fact larger than $10^{-7}N$ after decreasing constants, while $h(n) \le 6 \cdot 10^{-8}N$ for $n < 6N$. All summands used above have ratio at most 100.

The failure probability at stage k is $O((1-\varepsilon)^{N_k})$, and $\sum_k (1-\varepsilon)^{N_k} < \infty$. Borel–Cantelli gives the asserted eventual validity with probability 1. □

Proof of Theorem 2. Choose a random partition X_1, X_2, X_3 for which Proposition 11 holds eventually. Such a partition exists with probability 1, so fix one.

The blocks B_k, C_k lie in (N_k, N_{k+1}). They are disjoint from each other because $X_1 \cap X_2 = \emptyset$, and they are disjoint from the older blocks by size. The mirrors $N_{k+1} - G_k$ and $N_{k+1} - H_k$ lie in $(3N_k, N_{k+1})$ and are disjoint from each other because $G_k \cap H_k = \emptyset$. They are also disjoint from $B_k \cup C_k$, since $B_k \cup C_k$ contains the reflected point $N_{k+1} - x$ only when $x \notin A(k-1)$, whereas $G_k \cup H_k \subseteq A(k-1)$. Thus $B \cap C = \emptyset$.

Let n be sufficiently large and non-exceptional. There is a unique k such that

$$n \in [3N_k/2, 6N_k),$$

because $N_{k+1} = 4N_k$ and these intervals overlap enough to cover all large integers. Since $n \ne 4N_k = N_{k+1}$, Proposition 11 gives the lower bounds for $\widetilde{r}_B(n)$ and $\widetilde{r}_C(n)$.

Finally consider $N_{k+1} = 4N_k$. There are no old-old representations, since old elements are at most N_k. There are no representations using the ordinary intervals $(4N/3, 2N)$ and $(8N/3, 3N)$, by the interval placement. If an old element $x \in A(k-1)$ is paired with a reflected point $4N - y$ from (1) or (2), then equality $x + (4N - y) = 4N$ forces $x = y$, impossible because those reflected points are inserted only for $y \notin A(k-1)$. The only remaining way to represent $4N$ is to use one of the deliberately inserted mirrors $4N - g$ or $4N - h$, with $g \in G_k$ or $h \in H_k$. Hence every representation of N_{k+1} meets $F_k = G_k \cup H_k$. □

6 Comparison with Larsen's statement

Larsen considers three properties of a basis A of order 2: divergent representation function $(P1)$, decomposability as a union of two disjoint bases $(P2)$, and the existence of a minimal subbasis $(P3)$ [3]. His Theorem 1 states that all eight truth-value patterns occur. The present note extracts the row

$$(P1, P2, P3) = (\text{true}, \text{true}, \text{false}), \qquad A = B \cup C, \qquad \phi(n) = n, \qquad \psi(n) = n.$$

In particular, it gives exactly the negation of the Erdős–Nathanson implication $(P2) \Rightarrow (P3)$.

Our selection rule differs only notationally from Larsen's "largest element minimal among not yet occurred cumulatively eligible sets" rule. The role of $\phi(n) = n$ is to make the omitted part of any subbasis sparse on each interval $[M, 100M]$ in at least one colour. The role of $\psi(n) = n$ is to ensure that each fixed element lies in only finitely many canaries, so deleting the least element of a subbasis does not affect sufficiently late exceptional integers. The random blocks are Larsen's construction verbatim, with the constants retained to keep the proof uniform and insensitive to endpoints.

References

[1] T. F. Bloom, *Erdős Problem #869*, Erdős Problems, https://www.erdosproblems.com/869, accessed 24 April 2026.

[2] D. Larsen, comment on *Erdős Problem #869*, 25 January 2026, https://www.erdosproblems.com/forum/thread/869, accessed 24 April 2026.

[3] D. Larsen, *Three Questions of Erdős–Nathanson on Asymptotic Bases of Order 2*, arXiv:2603.03472, 2026. DOI: 10.48550/arXiv.2603.03472.

[4] P. Erdős and M. B. Nathanson, *Systems of distinct representatives and minimal bases in additive number theory*, in: *Number Theory, Carbondale 1979*, Lecture Notes in Math. **751**, Springer, Berlin, 1979, 89–107.

[5] P. Erdős and M. B. Nathanson, *Partitions of bases into disjoint unions of bases*, J. Number Theory **29** (1988), 1–9.

[6] E. Härtter, *Ein Beitrag zur Theorie der Minimalbasen*, J. Reine Angew. Math. **196** (1956), 170–204.

[7] M. B. Nathanson, *Minimal bases and maximal nonbases in additive number theory*, J. Number Theory **6** (1974), 324–333.