

# Uniquely Represented Products from Two Subsets of an Initial Interval

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## Abstract

For  $A, B \subseteq \{1, \dots, N\}$ , let  $F(A, B)$  denote the number of integers  $m$  having exactly one representation  $m = ab$  with  $a \in A$  and  $b \in B$ . We prove the order-of-magnitude estimate

$$\max_{A, B \subseteq [N]} F(A, B) \asymp \frac{N^2}{(\log N)^\delta (\log \log N)^{3/2}}, \quad \delta = 1 - \frac{1 + \log \log 2}{\log 2}.$$

The upper bound is the immediate comparison with the full multiplication table, using Ford's theorem on the multiplication table problem. The lower bound is obtained by attaching randomly selected large prime labels to the isolated-divisor construction implicit in Ford's estimates for  $H_1(x, y, z)$ , the number of integers with exactly one divisor in a prescribed interval.

## 1 Introduction

Write

$$[N] = \{1, 2, \dots, N\}$$

and, for  $A, B \subseteq [N]$ ,

$$r_{A,B}(m) = \#\{(a, b) \in A \times B : ab = m\}, \quad F(A, B) = \#\{m : r_{A,B}(m) = 1\}.$$

Let

$$U(N) = \max_{A, B \subseteq [N]} F(A, B).$$

All logarithms are natural, and all constants implicit in  $\ll$ ,  $\gg$ , and  $\asymp$  are absolute unless a dependence is explicitly indicated. The problem of estimating  $U(N)$  appears as Erdős Problem #896 in Bloom's compendium [1]. We prove the following theorem.

**Theorem 1.** *Let*

$$\delta = 1 - \frac{1 + \log \log 2}{\log 2}.$$

*Then, as  $N \rightarrow \infty$ ,*

$$U(N) \asymp \frac{N^2}{(\log N)^\delta (\log \log N)^{3/2}}.$$

The upper bound is inherited from the multiplication table problem. Let

$$M(N) = \#\{ab : 1 \leq a, b \leq N\}.$$

Plainly  $F(A, B) \leq M(N)$  for every  $A, B \subseteq [N]$ . Ford's multiplication-table theorem states that

$$M(N) \asymp \frac{N^2}{(\log N)^\delta (\log \log N)^{3/2}};$$

this is Corollary 3 of [2], applied with  $x = N^2$ . Thus only the lower bound in Theorem 1 remains.

The lower bound uses large prime labels to separate many independent divisor-interval problems. For a large prime  $p$ , we put into  $A$  elements of the form  $pr$ , where  $r$  lies in an interval depending on  $p$ . The set  $B$  is sifted to avoid multiples of the selected large labels. This ensures that products coming from distinct labels cannot collide. Inside a fixed label  $p$ , uniqueness is reduced to the condition that an integer has exactly one divisor in a short interval. Ford's estimates for  $H_1(x, y, z)$  provide enough such integers to match the multiplication-table order.

*Remark 2.* The theorem determines the order of magnitude of  $U(N)$ , and in particular the logarithmic exponent. It does not determine a leading constant, nor does it address exact finite- $N$  maximizers.

## 2 Ford's divisor-interval input

For  $1 \leq y < z$  and  $n \in \mathbb{N}$ , define

$$\tau(n; y, z) = \#\{d \mid n : y < d \leq z\}.$$

Also define

$$H(x, y, z) = \#\{n \leq x : \tau(n; y, z) > 0\}, \quad H_1(x, y, z) = \#\{n \leq x : \tau(n; y, z) = 1\}.$$

We shall use the following consequence of Ford's work.

**Proposition 3** (Ford input). *There are absolute constants  $c_0 > 0$  and  $y_0 \geq e^e$  such that, whenever  $N$  is sufficiently large and*

$$y_0 \leq y \leq N^{1/3},$$

one has

$$H_1(Ny, y, 2y) \geq c_0 \frac{Ny}{(\log y)^\delta (\log \log y)^{3/2}}.$$

*Proof.* Ford's estimate for  $H(x, y, cy)$ , with fixed  $c > 1$ , gives

$$H(x, y, cy) \asymp_c \frac{x}{(\log Y)^\delta (\log \log Y)^{3/2}}, \quad Y = \min(y, x/y) + 3,$$

whenever  $1/(c-1) \leq y \leq x/c$ ; see [2, Corollary 2]. Apply this with  $c = 2$  and  $x = Ny$ . Since  $y \leq N^{1/3} < N$  for large  $N$ , we have  $Y = y + 3$ , whence

$$H(Ny, y, 2y) \asymp \frac{Ny}{(\log y)^\delta (\log \log y)^{3/2}}.$$

It remains to compare  $H_1$  with  $H$ . Ford's theorem on integers with exactly one divisor in an interval states that, for fixed  $c > 0$ , if  $y$  is sufficiently large and

$$y + 1 \leq z \leq x^{5/8}, \quad yz \leq x^{1-c},$$

then

$$\frac{H_1(x, y, z)}{H(x, y, z)} \asymp_c \frac{\log \log(z/y + 10)}{\log(z/y + 10)};$$

see [2, Theorem 4]. Set  $x = Ny$  and  $z = 2y$ . For  $y \leq N^{1/3}$ , the hypotheses hold, for example with  $c = 1/4$ , once  $N$  is large. Indeed,

$$2y \leq (Ny)^{5/8}$$

and

$$2y^2 \leq (Ny)^{3/4}$$

follow uniformly from  $y \leq N^{1/3}$  for all sufficiently large  $N$ . Since  $z/y = 2$ , the displayed ratio is bounded below by a positive absolute constant. Combining the two estimates proves the proposition.  $\square$

For later reference, set

$$\Phi(t) = (\log t)^\delta (\log \log t)^{3/2} \quad (t > e^e).$$

### 3 A prime-sum estimate

The lower-bound construction will require the following elementary consequence of the prime number theorem.

**Lemma 4.** *Let  $Y = N^{1/3}$ , and let  $y_0 > e^e$  be fixed. Then, as  $N \rightarrow \infty$ ,*

$$\sum_{N/(2Y) < p \leq N/(2y_0)} \frac{1}{p \Phi(N/(2p))} \gg \frac{1}{(\log N)^\delta (\log \log N)^{3/2}},$$

where the sum is over primes.

*Proof.* Let  $L = \log N$  and  $J = \lfloor L/3 \rfloor$ . Choose an integer  $j_0$  sufficiently large in terms of  $y_0$ . For  $j_0 \leq j \leq J - 2$ , consider primes in

$$\frac{N}{2e^{j+1}} < p \leq \frac{N}{2e^j}.$$

These intervals are contained in the prime range of the lemma. For such primes,

$$\frac{N}{2p} \asymp e^j,$$

and hence

$$\Phi\left(\frac{N}{2p}\right) \asymp j^\delta (\log j)^{3/2}.$$

By the prime number theorem, or Mertens' theorem for primes, uniformly for  $j \leq L/3$ ,

$$\sum_{N/(2e^{j+1}) < p \leq N/(2e^j)} \frac{1}{p} \asymp \frac{1}{L-j} \asymp \frac{1}{L}.$$

Therefore the full prime sum is bounded below by

$$\gg \frac{1}{L} \sum_{j_0 \leq j \leq J-2} \frac{1}{j^\delta (\log j)^{3/2}}.$$

Since  $0 < \delta < 1$ ,

$$\sum_{j_0 \leq j \leq J-2} \frac{1}{j^\delta (\log j)^{3/2}} \gg \frac{J^{1-\delta}}{(\log J)^{3/2}}.$$

As  $J \asymp L$ , this gives

$$\gg \frac{L^{-\delta}}{(\log L)^{3/2}},$$

which is the desired estimate.  $\square$

## 4 The lower-bound construction

We now prove the lower bound in Theorem 1. Let  $y_0$  be sufficiently large for Proposition 3 and also larger than  $e^e$ , and put

$$Y = N^{1/3}.$$

Let

$$\mathcal{Q} = \left\{ p \text{ prime} : \frac{N}{2Y} < p \leq \frac{N}{2y_0} \right\}.$$

For  $p \in \mathcal{Q}$ , define

$$y_p = \frac{N}{2p}, \quad R_p = \{r \in \mathbb{N} : y_p < r \leq 2y_p\}.$$

Thus  $y_0 \leq y_p < Y$ . We randomly choose a subset  $\mathcal{P} \subseteq \mathcal{Q}$  by keeping each prime  $p \in \mathcal{Q}$  independently with probability  $1/2$ . Given  $\mathcal{P}$ , define

$$A = A(\mathcal{P}) = \{pr : p \in \mathcal{P}, r \in R_p\}$$

and

$$B = B(\mathcal{P}) = \{b \leq N : b \text{ is divisible by no prime } p \in \mathcal{P}\}.$$

For  $N$  large,

$$2Y < \frac{N}{2Y} < p \quad (p \in \mathcal{Q}).$$

Consequently every  $r \in R_p$  satisfies  $r \leq 2Y < p$ , and

$$pr \leq p \cdot 2y_p = N.$$

Thus  $A \subseteq [N]$  and  $B \subseteq [N]$ . The same inequality also implies that the label representation of an element of  $A$  is unique: if  $pr = qs$  with distinct primes  $p, q \in \mathcal{Q}$  and  $r \in R_p, s \in R_q$ , then, say,  $p \mid qs$ ; since  $p \neq q$  and  $s < p$ , this is impossible.

For  $p \in \mathcal{Q}$ , let

$$\mathcal{G}_p = \{n \leq Ny_p : \tau(n; y_p, 2y_p) = 1\}.$$

For  $n \in \mathcal{G}_p$ , denote by  $r_p(n)$  the unique divisor of  $n$  lying in  $R_p$ , and put

$$b_p(n) = \frac{n}{r_p(n)}.$$

Since  $r_p(n) > y_p$  and  $n \leq Ny_p$ , we have  $b_p(n) \leq N$ .

We discard the exceptional set for which  $p \mid b_p(n)$ , and define

$$\mathcal{G}_p^* = \{n \in \mathcal{G}_p : p \nmid b_p(n)\}.$$

The discarded set has size  $O(y_p^2)$ . Indeed, if  $p \mid b_p(n)$ , then  $b_p(n) = pt$  with

$$1 \leq t \leq \frac{N}{p} = 2y_p,$$

while  $r_p(n) \in R_p$ , giving at most  $O(y_p^2)$  possible pairs  $(r_p(n), t)$ , and hence at most  $O(y_p^2)$  possible  $n$ . Proposition 3 gives

$$|\mathcal{G}_p| \gg \frac{Ny_p}{\Phi(y_p)}.$$

Since  $y_p \leq N^{1/3}$ ,

$$y_p^2 = o\left(\frac{Ny_p}{\Phi(y_p)}\right)$$

uniformly for  $p \in \mathcal{Q}$ . Hence

$$|\mathcal{G}_p^*| \gg \frac{Ny_p}{\Phi(y_p)} \tag{1}$$

uniformly for  $p \in \mathcal{Q}$ .

For a fixed pair  $(p, n)$  with  $p \in \mathcal{Q}$  and  $n \in \mathcal{G}_p^*$ , consider the event

$$E_{p,n} = \{p \in \mathcal{P} \text{ and } b_p(n) \in B(\mathcal{P})\}.$$

Every prime in  $\mathcal{Q}$  is larger than  $N^{1/2}$  for large  $N$ , and  $b_p(n) \leq N$ ; hence  $b_p(n)$  is divisible by at most one prime from  $\mathcal{Q}$ . Also  $p \nmid b_p(n)$  by definition of  $\mathcal{G}_p^*$ . Therefore

$$\mathbb{P}(E_{p,n}) \geq \frac{1}{4}.$$

Indeed, if no prime of  $\mathcal{Q}$  divides  $b_p(n)$ , then the probability is  $1/2$ ; if a prime  $q \in \mathcal{Q}$ ,  $q \neq p$ , divides  $b_p(n)$ , then the probability is

$$\mathbb{P}(p \in \mathcal{P}, q \notin \mathcal{P}) = \frac{1}{4}.$$

Whenever  $E_{p,n}$  occurs, the product

$$m = pn = (pr_p(n))b_p(n)$$

lies in  $A(\mathcal{P})B(\mathcal{P})$ . We claim that this representation is unique. Suppose

$$pr_p(n)b_p(n) = qsc, \quad q \in \mathcal{P}, \quad s \in R_q, \quad c \in B(\mathcal{P}).$$

If  $q \neq p$ , then the prime  $p$  divides  $qsc$ . Since  $p \neq q$ ,  $p \nmid q$ . Also  $s \leq 2Y < p$ , so  $p \nmid s$ . Hence  $p \mid c$ , contradicting  $c \in B(\mathcal{P})$  and  $p \in \mathcal{P}$ . Therefore  $q = p$ . After cancelling  $p$ , we get

$$n = sc.$$

But  $n \in \mathcal{G}_p$ , so  $n$  has exactly one divisor in  $R_p$ ; hence  $s = r_p(n)$  and  $c = b_p(n)$ . This proves uniqueness.

The same argument also proves distinctness of the products obtained from distinct counted pairs. If two pairs  $(p, n)$  and  $(q, n')$  for which the corresponding events occur produced the same product, the representation arising from  $(q, n')$  would be an alternative representation of the product arising from  $(p, n)$ . By the uniqueness just proved, this forces  $p = q$ ,  $r_p(n) = r_q(n')$ , and  $b_p(n) = b_q(n')$ , and hence  $n = n'$ .

Let  $X(\mathcal{P})$  denote the number of pairs  $(p, n)$ , with  $p \in \mathcal{Q}$  and  $n \in \mathcal{G}_p^*$ , for which  $E_{p,n}$  occurs. The preceding paragraphs show that every such pair produces a distinct product counted by  $F(A(\mathcal{P}), B(\mathcal{P}))$ . Thus

$$F(A(\mathcal{P}), B(\mathcal{P})) \geq X(\mathcal{P}).$$

Taking expectations and using (1),

$$\mathbb{E}X(\mathcal{P}) \geq \frac{1}{4} \sum_{p \in \mathcal{Q}} |\mathcal{G}_p^*| \gg N \sum_{p \in \mathcal{Q}} \frac{y_p}{\Phi(y_p)}.$$

Since  $y_p = N/(2p)$ ,

$$\mathbb{E}X(\mathcal{P}) \gg N^2 \sum_{N/(2Y) < p \leq N/(2y_0)} \frac{1}{p \Phi(N/(2p))}.$$

By Lemma 4,

$$\mathbb{E}X(\mathcal{P}) \gg \frac{N^2}{(\log N)^\delta (\log \log N)^{3/2}}.$$

Consequently there exists at least one choice of  $\mathcal{P}$  for which

$$F(A(\mathcal{P}), B(\mathcal{P})) \gg \frac{N^2}{(\log N)^\delta (\log \log N)^{3/2}}.$$

This proves the desired lower bound for  $U(N)$ .

## 5 Completion of the proof of the main theorem

For every  $A, B \subseteq [N]$ , every uniquely represented product  $m$  is certainly an element of the ordinary  $N \times N$  multiplication table. Hence

$$F(A, B) \leq M(N) = \#\{ab : 1 \leq a, b \leq N\}.$$

Equivalently,  $M(N) = A(N^2)$  in Ford's notation, where  $A(x)$  counts integers  $n \leq x$  representable as  $n = m_1 m_2$  with  $m_1, m_2 \leq \sqrt{x}$ . Ford's Corollary 3 gives

$$M(N) \asymp \frac{N^2}{(\log N)^\delta (\log \log N)^{3/2}},$$

the replacement of  $\log(N^2)$  by  $\log N$  changing only the implicit constants. Together with the lower bound from the preceding section, this proves Theorem 1.

## 6 Concluding remarks

The proof gives an order-of-magnitude determination. It does not optimize constants. In particular, it does not assert the existence of a limiting leading constant for  $U(N)$  after normalization by

$$\frac{N^2}{(\log N)^\delta (\log \log N)^{3/2}}.$$

The essential point is the use of random large prime labels. The labels isolate many copies of Ford's one-divisor interval problem: collisions between different labels are eliminated by the sieve defining  $B$ , while collisions inside a fixed label are precisely controlled by  $H_1$ .

## References

- [1] T. F. Bloom, *Erdős Problem #896*, Erdős Problems, <https://www.erdosproblems.com/896>.
- [2] K. Ford, *The distribution of integers with a divisor in a given interval*, *Annals of Mathematics* **168** (2008), no. 2, 367–433.