A Gaussian Construction for an Erdős Problem on Zeros of Successive Derivatives

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Abstract

We give a probabilistic construction of a transcendental entire function f with the following property: for every nonempty open set $U \subset \mathbb{C}$, all sufficiently high derivatives $f^{(n)}$ have a zero in U. This implies that, for every infinite increasing sequence $n_1 < n_2 < \cdots$, the union of zero sets

$$\bigcup_{k\geq 1}\{z \in \mathbb{C} : f^{(n_k)}(z) = 0\}$$

is dense in $\mathbb{C}$. The examples are almost sure samples of the Gaussian entire function

$$f(z) = \sum_{k=0}^{\infty} \xi_k \frac{z^k}{(k!)^{1-\beta}}, \qquad \frac{1}{2} < \beta < 1,$$

where the ξ_k are independent standard complex Gaussian variables. The proof has two main ingredients: a saddle-point estimate for the covariance kernels of the derivatives, and an Offord-type exponential tail estimate for smooth zero statistics. For the latter, we include a short self-contained proof in the form needed here.

1 Statement and cofinite reformulation

The literal version of Erdős Problem #906 asks for a nonzero entire function f such that, for every infinite increasing sequence $n_1 < n_2 < \cdots$, the set

$$\{z \in \mathbb{C} : f^{(n_k)}(z) = 0 \text{ for some } k\}$$

is dense in $\mathbb{C}$. Without an additional hypothesis this is trivial: every nonzero polynomial works, since all sufficiently high derivatives vanish identically. The intended question is therefore the transcendental version. We prove the following stronger cofinite form.

Theorem 1.1 (Main theorem)**.** *There exists a transcendental entire function f such that for every nonempty open set $U \subset \mathbb{C}$ there is an integer $N(U)$ for which*

$$f^{(n)} \text{ has a zero in } U \qquad (n \geq N(U)).$$

Consequently, for every infinite increasing sequence $n_1 < n_2 < \cdots$, the set

$$\bigcup_{k\geq 1}\{z \in \mathbb{C} : f^{(n_k)}(z) = 0\}$$

is dense in $\mathbb{C}$.

We first record the elementary equivalence between the subsequence formulation and the cofinite formulation.

Lemma 1.2 (Cofinite reformulation)**.** *Let f be an entire function. The following assertions are equivalent.*

(i) *For every infinite increasing sequence $n_1 < n_2 < \cdots$ of nonnegative integers, the set*

$$\bigcup_{k\geq 1}\{z \in \mathbb{C} : f^{(n_k)}(z) = 0\}$$

is dense in $\mathbb{C}$.

(ii) *For every nonempty open set $U \subset \mathbb{C}$, the set*

$$A_U := \{n \in \mathbb{N}_0 : f^{(n)} \text{ has a zero in } U\}$$

is cofinite in $\mathbb{N}_0$.

Proof. Assume (ii). Let $n_1 < n_2 < \cdots$ be infinite and let $U \subset \mathbb{C}$ be nonempty and open. Since A_U is cofinite and $n_k \to \infty$, some n_k belongs to A_U. Thus the union of zero sets meets U; since U was arbitrary, the union is dense.

Conversely, suppose that (ii) fails for some nonempty open set U. Then $\mathbb{N}_0 \setminus A_U$ is infinite. Enumerate this complement increasingly as $n_1 < n_2 < \cdots$. No derivative $f^{(n_k)}$ has a zero in U, so the corresponding union of zero sets misses U and is not dense. □

2 The random entire function

Fix

$$\frac{1}{2} < \beta < 1$$

throughout the proof. Let $\xi_0, \xi_1, \xi_2, \ldots$ be independent standard complex Gaussian random variables, normalized by

$$\mathbb{P}(\xi_j \in dw) = \frac{1}{\pi} e^{-|w|^2}\, dA(w).$$

Define

$$f(z) = \sum_{k=0}^{\infty} \xi_k \frac{z^k}{(k!)^{1-\beta}}. \tag{1}$$

Lemma 2.1. *Almost surely, (1) defines a transcendental entire function.*

Proof. For every $\varepsilon > 0$,

$$\sum_{k=1}^{\infty} \mathbb{P}\{|\xi_k| > e^{\varepsilon k}\} = \sum_{k=1}^{\infty} e^{-e^{2\varepsilon k}} < \infty.$$

By Borel–Cantelli, almost surely $|\xi_k| \leq e^{\varepsilon k}$ for all sufficiently large k. Since

$$\left((k!)^{1-\beta}\right)^{1/k} \sim (k/e)^{1-\beta} \to \infty,$$

the Taylor series has infinite radius of convergence. Also $\mathbb{P}(\xi_k = 0) = 0$ for each k, hence, by countable intersection, almost surely all coefficients are nonzero. Thus f is not a polynomial. □

For $n \geq 0$, termwise differentiation gives

$$F_n(z) := f^{(n)}(z) = \sum_{m=0}^{\infty} \xi_{n+m} \frac{((n+m)!)^\beta}{m!} z^m. \tag{2}$$

For each fixed n, F_n is a Gaussian analytic function. Its covariance kernel on the diagonal is

$$K_n(z,z) := \mathbb{E}|F_n(z)|^2 = \sum_{m=0}^{\infty} \frac{((n+m)!)^{2\beta}}{(m!)^2} |z|^{2m}. \tag{3}$$

Let Z_n denote the zero counting measure of F_n, counting multiplicity, and set $\mu_n := \mathbb{E}Z_n$.

We shall use the following standard form of the Edelman–Kostlan formula. It is included to fix normalizations.

Lemma 2.2 (Edelman–Kostlan formula)**.** *Let X be a Gaussian analytic function on a plane domain G, and suppose that its covariance $K_X(z,z) = \mathbb{E}|X(z)|^2$ is positive on G. If Z_X is the zero counting measure of X, then*

$$\mathbb{E}Z_X = \frac{1}{4\pi} \Delta \log K_X(z,z),$$

in the sense of distributions, where $\Delta = \partial_x^2 + \partial_y^2$.

Proof. The Poincare–Lelong formula in this normalization is

$$Z_X = \frac{1}{2\pi} \Delta \log |X|.$$

For each fixed z, the random variable $X(z)$ has the same distribution as $K_X(z,z)^{1/2}\zeta$, where ζ is a standard complex Gaussian. Therefore

$$\mathbb{E} \log |X(z)| = \frac{1}{2} \log K_X(z,z) + \mathbb{E} \log |\zeta|.$$

The last term is a finite constant, whose Laplacian is zero. Taking expectations in the distributional identity gives the result. ☐

Consequently,

$$\mu_n = \frac{1}{4\pi} \Delta \log K_n(z,z). \tag{4}$$

3 Expected zero measures

The deterministic core of the proof is the following asymptotic for the expected zero measures of the derivatives.

Proposition 3.1. *As $n \to \infty$,*

$$n^{-\beta} \mu_n \longrightarrow \nu := \frac{1}{2\pi|z|} \, dA(z) \tag{5}$$

vaguely on $\mathbb{C}$. Equivalently, for every $\varphi \in C_c^\infty(\mathbb{C})$,

$$\frac{1}{n^\beta} \int_{\mathbb{C}} \varphi \, d\mu_n \longrightarrow \int_{\mathbb{C}} \varphi(z) \frac{dA(z)}{2\pi|z|}. \tag{6}$$

The density $1/|z|$ is locally integrable at the origin.

Write $\lambda_n := n^\beta$. Factoring out the constant $(n!)^{2\beta}$ from (3), we have

$$K_n(z,z) = (n!)^{2\beta} A_n(|z|), \tag{7}$$

where

$$A_n(r) = \sum_{m=0}^{\infty} a_{n,m}(r), \qquad a_{n,m}(r) = \frac{r^{2m}}{(m!)^2} \prod_{\ell=1}^{m} (n+\ell)^{2\beta}. \tag{8}$$

The empty product is 1.

Lemma 3.2 (Laplace estimate)**.** *For every $R > 0$,*

$$\sup_{0\le r\le R} \left| \frac{1}{\lambda_n} \log A_n(r) - 2r \right| \longrightarrow 0. \tag{9}$$

Consequently,

$$\frac{1}{n^\beta} \left(\log K_n(z,z) - 2\beta \log(n!)\right) \longrightarrow 2|z| \tag{10}$$

locally uniformly on $\mathbb{C}$.

Proof. We prove (9). For the lower bound, fix $0 < \varepsilon < R$ and let $r \in [\varepsilon, R]$. Put $m_n = \lfloor r\lambda_n \rfloor$. Since $\beta < 1$, we have $m_n/n \to 0$ uniformly for $r \in [\varepsilon, R]$. Also, uniformly in this range,

$$\sum_{\ell=1}^{m_n} \log(n+\ell) = m_n \log n + O\left(\frac{m_n^2}{n}\right) = m_n \log n + o(\lambda_n),$$

and Stirling's formula gives

$$\log(m_n!) = m_n \log m_n - m_n + o(\lambda_n).$$

Using $\lambda_n = n^\beta$, we obtain

$$\begin{aligned} \log a_{n,m_n}(r) &= 2m_n \log r + 2\beta \sum_{\ell=1}^{m_n} \log(n+\ell) - 2\log(m_n!) \\ &= 2m_n \log r + 2m_n \log \lambda_n - 2m_n \log m_n + 2m_n + o(\lambda_n) \\ &= 2r\lambda_n + o(\lambda_n), \end{aligned}$$

uniformly for $r \in [\varepsilon, R]$. Hence

$$\liminf_{n\to\infty} \inf_{\varepsilon\le r\le R} \left(\frac{1}{\lambda_n} \log A_n(r) - 2r \right) \ge 0.$$

For $0 \le r < \varepsilon$, the single term $m = 0$ gives $A_n(r) \ge 1$, hence

$$\frac{1}{\lambda_n} \log A_n(r) \ge 0 \ge 2r - 2\varepsilon.$$

Letting $\varepsilon \downarrow 0$ gives the required uniform lower bound on $[0, R]$.

For the upper bound, observe that the ratio of consecutive terms is

$$\frac{a_{n,m+1}(r)}{a_{n,m}(r)} = r^2 \frac{(n+m+1)^{2\beta}}{(m+1)^2}. \tag{11}$$

Choose $M > 0$ so large that $R^2 2^{2\beta}/M^2 < 1/4$. If $M\lambda_n \le m \le n$, then, for all sufficiently large n and all $r \le R$, the ratio in (11) is at most $1/4$. If $m > n$, then the same ratio is bounded by $C_R m^{-2(1-\beta)}$, hence is also at most $1/4$ for all sufficiently large n. Thus, uniformly for $0 \le r \le R$, the tail $m \ge M\lambda_n$ is geometrically dominated by its first term. It follows that, after replacing M by $M+1$ if necessary,

$$A_n(r) \le C_R(1+\lambda_n) \max_{0 \le m \le M\lambda_n} a_{n,m}(r) \tag{12}$$

for all large n.

It remains to estimate the maximum in (12). Let $x = m/\lambda_n$. If $r = 0$, then $a_{n,m}(0) = 0$ for $m > 0$ and $a_{n,0}(0) = 1$, so the desired upper bound is immediate. Assume therefore that $r > 0$. For $1 \le m \le M\lambda_n$, the estimate

$$\sum_{\ell=1}^{m} \log(n+\ell) = m \log n + O\left(\frac{m^2}{n}\right) = m \log n + o(\lambda_n)$$

is uniform, because $m \le M\lambda_n$ and $\lambda_n/n \to 0$. The elementary lower bound $\log(m!) \ge m \log m - m$ then gives, uniformly for $0 < r \le R$ and $1 \le m \le M\lambda_n$,

$$\frac{1}{\lambda_n} \log a_{n,m}(r) \le 2x \log \frac{er}{x} + o(1), \tag{13}$$

with the usual convention that the right hand side is 0 when $x = 0$. The case $m = 0$ is also covered, since $a_{n,0}(r) = 1$. For fixed $r > 0$, the function

$$x \mapsto 2x \log \frac{er}{x} \qquad (x > 0)$$

has maximum $2r$, attained at $x = r$. Hence, for every $\delta > 0$ and all sufficiently large n,

$$a_{n,m}(r) \le \exp\{(2r+\delta)\lambda_n\}$$

for all $0 \le r \le R$ and $0 \le m \le M\lambda_n$. Combining this with (12) and using $\log \lambda_n = o(\lambda_n)$ gives

$$\limsup_{n\to\infty} \sup_{0 \le r \le R} \left(\frac{1}{\lambda_n} \log A_n(r) - 2r \right) \le 0.$$

Together with the lower bound, this proves (9). Formula (10) follows from (7). □

Proof of Proposition 3.1. Define

$$u_n(z) = \frac{1}{n^\beta} \left(\log K_n(z,z) - 2\beta \log(n!) \right).$$

By Lemma 3.2, $u_n \to u := 2|z|$ locally uniformly, hence in L^1_{loc}. Taking distributional Laplacians and using (4), we get

$$n^{-\beta}\mu_n = \frac{1}{4\pi}\Delta u_n \longrightarrow \frac{1}{4\pi}\Delta(2|z|).$$

Since $\Delta|z| = 1/|z|$ on $\mathbb{C} \setminus \{0\}$ and this identity has no extra atom at the origin in the distributional sense, the limit is

$$\frac{1}{2\pi|z|}\, dA(z).$$

This proves the proposition. □

4 An Offord estimate for smooth zero statistics

The next estimate is the only probabilistic input beyond elementary Gaussian facts. It is a standard Offord-type estimate; the proof below is included because the constants and exact normalization are unimportant for the application.

Lemma 4.1 (Logarithmic tail bound)**.** *Let ζ be a standard complex Gaussian random variable. There is a universal constant C such that, for every event E with probability $p \in [0,1]$,*

$$\mathbb{E}\big(|\log|\zeta||\,\mathbf{1}_E\big) \le Cp\log\frac{e}{p},$$

where the right hand side is interpreted as 0 when $p = 0$.

Proof. Set $Y = |\log|\zeta||$. For $s \ge 0$,

$$\mathbb{P}\{Y > s\} = \mathbb{P}\{|\zeta| > e^s\} + \mathbb{P}\{|\zeta| < e^{-s}\} = e^{-e^{2s}} + 1 - e^{-e^{-2s}} \le 2e^{-s}.$$

For any $a \ge 0$ and any event E of probability p,

$$\mathbb{E}(Y\mathbf{1}_E) \le ap + \mathbb{E}(Y\mathbf{1}_{Y>a}) = ap + a\mathbb{P}\{Y > a\} + \int_a^\infty \mathbb{P}\{Y > s\}\,ds \le ap + 2(a+1)e^{-a}.$$

Choosing $a = \log(e/p)$ for $0 < p \le 1$ gives $a \ge 1$ and $e^{-a} = p/e$, so the last expression is at most a universal constant times $p\log(e/p)$. □

Lemma 4.2 (Offord estimate)**.** *Let X be a Gaussian analytic function on a plane domain G, and suppose $K_X(z,z) = \mathbb{E}|X(z)|^2$ is positive on G. Let Z_X be the zero counting measure of X and $\mu_X = \mathbb{E}Z_X$. Then there are universal constants $C_1, C_2 > 0$ such that, for every $\phi \in C_c^\infty(G)$ and every $t > 0$,*

$$\mathbb{P}\left\{\left|\int_G \phi\, d(Z_X - \mu_X)\right| \ge t\right\} \le C_1 \exp\left(-C_2\frac{t}{\|\Delta\phi\|_{L^1(G)}}\right), \tag{14}$$

provided $\Delta\phi \not\equiv 0$. If $\Delta\phi \equiv 0$, the left hand side is zero.

Proof. Put

$$L_X(z) := \log|X(z)| - \frac{1}{2}\log K_X(z,z).$$

This is a random locally integrable function on G; we use it only as a real-valued normalized logarithm, not as the logarithm of an analytic normalization. For each fixed $z \in G$, $L_X(z)$ has the same distribution as $\log|\zeta|$, where ζ is a standard complex Gaussian. By Poincare–Lelong and Lemma 2.2,

$$\int_G \phi\, d(Z_X - \mu_X) = \frac{1}{2\pi}\int_G \Delta\phi(z)L_X(z)\,dA(z) \tag{15}$$

in the distributional sense.

Let I denote the left hand side of (15), let $E_t = \{|I| \ge t\}$, and put $p = \mathbb{P}(E_t)$. If $p = 0$ there is nothing to prove. Otherwise, using (15), Fubini's theorem, and Lemma 4.1, we get

$$\begin{aligned} tp &\le \mathbb{E}(|I|\mathbf{1}_{E_t}) \\ &\le \frac{1}{2\pi}\int_G |\Delta\phi(z)|\,\mathbb{E}\big(|L_X(z)|\,\mathbf{1}_{E_t}\big)\,dA(z) \\ &\le C\|\Delta\phi\|_{L^1(G)}\,p\log\frac{e}{p}. \end{aligned}$$

After cancelling p, this implies

$$p \le e \exp\left(-\frac{t}{C\|\Delta\phi\|_{L^1(G)}}\right),$$

which is (14) after adjusting the constants. $\square$

5 Summable hole probabilities

Proposition 5.1 (Hole probabilities)**.** *Let $U \subset \mathbb{C}$ be nonempty and open. Then there are constants $c_U, C_U > 0$ such that, for all sufficiently large n,*

$$\mathbb{P}\{F_n \textit{ has no zero in } U\} \le C_U e^{-c_U n^\beta}. \tag{16}$$

In particular,

$$\sum_{n=1}^{\infty} \mathbb{P}\{F_n \textit{ has no zero in } U\} < \infty. \tag{17}$$

Proof. Choose a nonzero, nonnegative test function $\phi \in C_c^\infty(U)$. Since a compactly supported harmonic function on a plane domain is zero, this choice has $\Delta\phi \not\equiv 0$. Proposition 3.1 gives

$$\frac{1}{n^\beta}\int \phi \, d\mu_n \longrightarrow c_\phi := \int_{\mathbb{C}} \phi(z)\frac{dA(z)}{2\pi|z|} > 0.$$

Therefore, for all sufficiently large n,

$$M_n := \int \phi \, d\mu_n \ge \frac{c_\phi}{2} n^\beta. \tag{18}$$

On the event that F_n has no zero in U, we have $\int \phi \, dZ_n = 0$, because $\operatorname{supp}\phi \subset U$. Hence

$$\left|\int \phi \, d(Z_n - \mu_n)\right| = M_n.$$

Applying Lemma 4.2 to $X = F_n$ and $t = M_n$, and then using (18), gives

$$\mathbb{P}\{F_n \text{ has no zero in } U\} \le C_1 \exp\left(-C_2 \frac{M_n}{\|\Delta\phi\|_1}\right) \le C_U e^{-c_U n^\beta}.$$

The series in (17) converges because $\beta > 0$. $\square$

6 Almost sure proof of the main theorem

Proof of Theorem 1.1. Let $\mathcal{B}$ be a countable base for the topology of $\mathbb{C}$, for instance all open disks with centers in $\mathbb{Q} + i\mathbb{Q}$ and positive rational radii.

Fix $B \in \mathcal{B}$. By Proposition 5.1,

$$\sum_{n=1}^{\infty} \mathbb{P}\{F_n \text{ has no zero in } B\} < \infty.$$

The first Borel–Cantelli lemma, which does not require independence, implies that almost surely only finitely many of the events

$$\{F_n \text{ has no zero in } B\}$$

occur. Equivalently, almost surely there exists $N(B)$ such that $F_n = f^{(n)}$ has a zero in B for every $n \geq N(B)$.

Intersect this probability-one event over the countable family $\mathcal{B}$, and also intersect with the probability-one event from Lemma 2.1. For every outcome in the resulting event, f is transcendental entire and satisfies

$$\forall B \in \mathcal{B} \quad \exists N(B) \quad \forall n \geq N(B), \quad f^{(n)} \text{ has a zero in } B.$$

Fix such an outcome.

Let $U \subset \mathbb{C}$ be nonempty and open. Choose $B \in \mathcal{B}$ with $B \subset U$. Then $f^{(n)}$ has a zero in $B \subset U$ for every $n \geq N(B)$. This proves the cofinite property.

Finally, if $n_1 < n_2 < \cdots$ is any infinite increasing sequence and $U \subset \mathbb{C}$ is nonempty and open, choose $B \in \mathcal{B}$ with $B \subset U$. Since $n_k \to \infty$, some n_k is at least $N(B)$, and then $f^{(n_k)}$ has a zero in U. Hence the union of zero sets along the subsequence meets every nonempty open set, and is dense in $\mathbb{C}$. □

7 Order of the examples

The construction gives examples of every prescribed order strictly larger than 2.

Proposition 7.1. *For the random function* (1)*, almost surely*

$$\operatorname{ord}(f) = \frac{1}{1-\beta} > 2.$$

Consequently, for every $p > 2$, choosing $\beta = 1 - 1/p$ gives examples of order p.

Proof. Let

$$c_k = \frac{\xi_k}{(k!)^{1-\beta}}$$

be the Taylor coefficients of f. Fix $\varepsilon > 0$. Since

$$\sum_{k=1}^{\infty} \mathbb{P}\{|\xi_k| > e^{\varepsilon k \log k}\} < \infty$$

and

$$\sum_{k=1}^{\infty} \mathbb{P}\{|\xi_k| < e^{-\varepsilon k \log k}\} < \infty,$$

Borel–Cantelli gives, almost surely,

$$e^{-\varepsilon k \log k} \leq |\xi_k| \leq e^{\varepsilon k \log k}$$

for all sufficiently large k. Applying this to the countable sequence $\varepsilon = 1/j$ and intersecting the resulting probability-one events gives

$$\log |\xi_k| = o(k \log k)$$

almost surely. Therefore

$$-\log|c_k| = (1-\beta)\log(k!) + o(k\log k) = (1-\beta)k\log k + o(k\log k).$$

The standard coefficient formula for the order of an entire function,

$$\mathrm{ord}(f) = \left(\liminf_{k\to\infty} \frac{-\log|c_k|}{k\log k}\right)^{-1},$$

therefore gives $\mathrm{ord}(f) = 1/(1-\beta)$. Because $\beta > 1/2$, this order is greater than 2. □

8 Referee-style audit of the proof

We close by spelling out the main points at which hidden gaps could otherwise enter.

First, no independence between the events for different n is used. The proof uses only the first Borel–Cantelli lemma, so the strong dependence among the derivatives $f^{(n)}$ is harmless.

Second, the Offord estimate is used only for smooth linear statistics. The passage from a hole event to a linear-statistic deviation is made by choosing $\phi \geq 0$ with compact support inside the open set U. If F_n has no zero in U, then $\int \phi\, dZ_n = 0$ exactly. In the proof of the Offord estimate, the normalized logarithm $L_X(z) = \log|X(z)| - \frac{1}{2}\log K_X(z,z)$ is used pointwise; no analyticity of a normalized process is asserted or needed.

Third, the singular density $1/|z|$ in Proposition 3.1 is locally integrable and has no atom at the origin. Therefore every nonzero nonnegative test function $\phi \in C_c^\infty(U)$ has a positive limiting expected statistic, even when U is a small neighborhood of the origin.

Finally, the constant factor $(n!)^{2\beta}$ in the covariance kernel is removed only inside a logarithm and contributes a spatial constant. Its Laplacian is zero, so it has no effect on the expected zero measure.

References

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