THE ORDER OF GROWTH OF PLANAR SETS AVOIDING INTEGER DISTANCES

Abstract. Let $M(R)$ be the supremum of the measures of measurable sets $A \subset B_R(0) \subset \mathbb{R}^2$ such that $|a-b|$ is not a positive integer for any distinct $a, b \in A$. We prove

$$M(R) \ll R^{1/2} \qquad (R \geq 1).$$

Combined with Sárközy's lower bound for point sets whose distances stay away from the integers, this gives $M(R) = R^{1/2+o(1)}$. Thus the order-of-growth form of Erdős Problem #953 is settled. The proof reduces the measurable problem to the uniform robust estimate $N(X, \delta) \ll \delta^{-2} X^{1/2}$ and proves that estimate by a positive Poisson–Bessel kernel whose Poisson expansion has explicitly signed summands away from the integer radii.

1. Statement and reduction

Let

$$M(R) = \sup\{|A| : \ A \subset B_R(0) \subset \mathbb{R}^2 \text{ measurable and } |a-b| \notin \mathbb{Z}_{>0} \text{ for } a \neq b \in A\}.$$

Erdős and Sárközy asked for the order of this quantity; the problem is recorded as Erdős Problem #953, and the related discussion thread is acknowledged in [1, 2]. We prove the following.

Theorem 1. *For $R \geq 1$,*

$$M(R) \ll R^{1/2}.$$

Consequently $M(R) = R^{1/2+o(1)}$ as $R \to \infty$.

For $0 < \delta < 1/10$, let $N(X, \delta)$ be the maximum size of a set $P \subset B_X(0)$ such that

$$\big\| |p - p'| \big\|_{\mathbb{Z}} \geq \delta \qquad (p \neq p', \ p, p' \in P),$$

where $\|x\|_{\mathbb{Z}} = \operatorname{dist}(x, \mathbb{Z})$. The measurable problem is equivalent, up to absolute constants, to bounding $\delta^2 N(X, \delta)$. All implicit constants below are absolute.

Lemma 2. *For $R \geq 1$,*

$$M(R) \ll \sup_{0<\delta<1/10} \delta^2 N(R, \delta).$$

Conversely,

$$M(R) \gg \sup_{0<\delta<1/10} \delta^2 N(R-1, \delta).$$

Proof. For the upper bound, take an admissible A and compact $K \subset A$ with $|K| > |A| - \varepsilon$. The compact distance set $\{|x-y| : x, y \in K\}$ avoids the finite set $\{1, \dots, \lfloor 2R \rfloor\}$, so its positive distances are separated from the positive integers by some $\eta > 0$. Choose $0 < \delta < \min(\eta, 1/10)$ and let $P \subset K$ be maximal δ-separated. Then P is counted by $N(R, \delta)$, and the disks $B_\delta(p)$, $p \in P$, cover K. Hence $|K| \leq \pi \delta^2 N(R, \delta)$, and then $\varepsilon \downarrow 0$.

For the lower bound, replace each point of a set counted by $N(R-1, \delta)$ by a disk of radius $\delta/4$. The disks are disjoint, they lie in $B_R(0)$, and the change in any inter-centre distance is at most $\delta/2$; hence no positive integer distance is created. The resulting measurable set has area $\gg \delta^2 N(R-1, \delta)$. □

It remains to prove the uniform robust estimate. For each fixed δ without uniform dependence, this type of bound is due to Konyagin [4].

Theorem 3. *For $X \geq 1$ and $0 < \delta < 1/10$,*

$$N(X, \delta) \ll \delta^{-2} X^{1/2}.$$

2. The kernel

For $0 < s < 1$ define

$$K_s(t) = \sum_{k=1}^{\infty} (k + 2sk^2) e^{-sk} J_0(2\pi k t), \qquad t \geq 0, \tag{2.1}$$

where J_0 is the Bessel function. Since

$$J_0(2\pi k|x|) = \int_{S^1} e^{2\pi i k\omega\cdot x}\, d\sigma(\omega)$$

with σ normalized arclength measure, $x \mapsto K_s(|x|)$ is positive definite on $\mathbb{R}^2$. Also

$$K_s(0) = \sum_{k\geq 1} (k + 2sk^2) e^{-sk} \ll s^{-2}. \tag{2.2}$$

The decisive point is the following negativity statement.

Proposition 4. *There are absolute constants $A, c, s_0 > 0$ such that, if $0 < s < s_0$ and $\|t\|_{\mathbb{Z}} \geq As$, then*

$$K_s(t) \leq -c(1+t)^{-1/2}.$$

Proof. Put

$$\lambda_m = s + 2\pi i m, \qquad q_m(t) = \lambda_m^2 + (2\pi t)^2,$$

and use principal branches for fractional powers. Applying Poisson summation to $x \mapsto e^{-s|x|} J_0(2\pi t|x|)$, using

$$\int_0^{\infty} e^{-\lambda x} J_0(2\pi t x)\, dx = \left(\lambda^2 + (2\pi t)^2\right)^{-1/2} \qquad (\Re\lambda > 0),$$

and then applying $-\partial_s + 2s\partial_s^2$, gives

$$K_s(t) = \sum_{m\in\mathbb{Z}} T_m(s,t), \tag{2.3}$$

where $T_{-m} = T_m$ and

$$T_m = \Re\left[\lambda_m q_m^{-3/2} + 2s\{-q_m^{-3/2} + 3\lambda_m^2 q_m^{-5/2}\}\right]. \tag{2.4}$$

The formula is justified by inserting $e^{-\alpha x^2}$, applying the Schwartz-class Poisson formula, differentiating, and then letting $\alpha \downarrow 0$; after differentiation the summands are $O_s(|m|^{-2})$.

We prove that the terms in (2.3) are non-positive away from an $O(s)$-neighbourhood of their singular radii. Fix a large absolute constant L and a small absolute constant s_0. Let $m \geq 1$, $a = 2\pi m$, $b = 2\pi t$, and assume $|a - b| \geq Ls$. Write $\zeta = s^2 + 2ias$. Since $a \geq 2\pi$ and $s < s_0 \leq 1$, $|\zeta| \leq 3as$.

If $b < a$, put $D = a^2 - b^2$. Then $D \geq aLs$ and $|\zeta|/D \leq 3/L$. Since $q_m = -D + \zeta$, the principal-branch expansions

$$q_m^{-3/2} = iD^{-3/2}(1 + O(L^{-1})), \qquad q_m^{-5/2} = -iD^{-5/2}(1 + O(L^{-1}))$$

give

$$T_m = -aD^{-3/2} + O\big((L^{-1} + s_0)aD^{-3/2}\big) \leq -\tfrac{1}{2}aD^{-3/2}$$

if L is large and s_0 is small.

If $b > a$, put $D = b^2 - a^2$ and $q_m = D + \zeta = D(1+z)$, where $|z| \leq 3/L$. Using

$$(1+z)^{-3/2} = 1 - \frac{3}{2}z + O(|z|^2), \qquad (1+z)^{-5/2} = 1 + O(|z|),$$

and retaining the real part through first order, one obtains

$$T_m = -sD^{-3/2} - 3a^2 sD^{-5/2} + \frac{15}{2}s^3 D^{-5/2} + R,$$

with

$$|R| \leq CL^{-2}sD^{-3/2} + CL^{-1}a^2 sD^{-5/2}.$$

Here we used $as/D \le L^{-1}$; when $D \ge a^2$ the quadratic remainder is absorbed by $sD^{-3/2}$, and when $D < a^2$ it is absorbed by $a^2 s D^{-5/2}$. For L large and s_0 small this is at most

$$-\tfrac{1}{2} s D^{-3/2} - a^2 s D^{-5/2} \le 0.$$

Finally, for $m = 0$ one computes directly

$$T_0 = s \frac{5s^2 - (2\pi t)^2}{(s^2 + (2\pi t)^2)^{5/2}},$$

which is non-positive when $2\pi t \ge Ls$, after increasing L.

Let $A = L/(2\pi)$ and decrease s_0 so that $As_0 < 1/2$. If $\|t\|_{\mathbb{Z}} \ge As$, then every term in (2.3) is non-positive. Let $n = \lceil t \rceil$. Then $n \ge 1$ and $0 < n - t \le 1$. Applying the case $b < a$ to $m = n$ gives

$$K_s(t) \le T_n(s,t) \le -c_1(2\pi n)\big((2\pi n)^2 - (2\pi t)^2\big)^{-3/2}.$$

Since

$$(2\pi n)^2 - (2\pi t)^2 \le C(1+t), \qquad 2\pi n \ge c(1+t),$$

the right-hand side is at most $-c(1+t)^{-1/2}$. □

3. Completion of the proof

Proof of Theorem 3. Let $P = \{p_1, \dots, p_n\} \subset B_X(0)$ be counted by $N(X, \delta)$. Take $s = \delta/A'$ with A' large enough, in terms of the constants in Proposition 4, that $s < s_0$ and $\delta \ge As$. Positive definiteness gives

$$0 \le \sum_{i,j=1}^{n} K_s(|p_i - p_j|).$$

By (2.2), the diagonal contribution is $O(ns^{-2})$. For $i \ne j$, Proposition 4 and $|p_i - p_j| \le 2X$ give

$$K_s(|p_i - p_j|) \le -c(1+2X)^{-1/2}.$$

Thus

$$0 \le Cns^{-2} - cn(n-1)(1+2X)^{-1/2},$$

and hence $n \ll s^{-2} X^{1/2} \ll \delta^{-2} X^{1/2}$, after enlarging the constant to cover $n \le 1$. □

Proof of Theorem 1. The upper bound follows immediately from Lemma 2 and Theorem 3. For the lower bound, Sárközy proved that, for each sufficiently small fixed $\delta > 0$ and all sufficiently large Y, there are $> Y^{1/2 - \delta^{1/7}}$ points in $B_Y(0)$ whose mutual distances are δ-away from the integers [5, 6]. Thickening these points by disks of radius $\delta/3$ and taking $\delta = \delta(\varepsilon)$ gives $M(R) \gg_\varepsilon R^{1/2-\varepsilon}$ for every $\varepsilon > 0$. Together with the upper bound this is $M(R) = R^{1/2+o(1)}$. □

References

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[3] P. Erdős, *Problems and results on combinatorial number theory. III*, Number Theory Day, Proc. Conf., Rockefeller Univ., New York, 1976, Lecture Notes in Math. **626**, Springer, 1977, 43–72.

[4] S. V. Konyagin, *About distances between points on the plane*, Mathematical Notes **69** (2001), 578–581.

[5] A. Sárközy, *On distances near integers. I*, Studia Sci. Math. Hungar. **11** (1976), 37–50.

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[7] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, 2nd ed., Cambridge University Press, 1944.