A POISSON–BESSEL KERNEL BOUND FOR PLANAR SETS AVOIDING INTEGER DISTANCES

Abstract. Let $M(R)$ denote the supremum of the measures of measurable sets $A \subset B_R(0) \subset \mathbb{R}^2$ such that $|a-b|$ is never a positive integer for distinct $a, b \in A$. We prove

$$M(R) \ll R^{1/2} \qquad (R \geq 1).$$

Together with Sárközy's lower bound for robust point sets, this gives $M(R) = R^{1/2+o(1)}$. Thus the paper resolves the order of magnitude of Erdős Problem #953. The proof reduces the measurable problem to a uniform robust point estimate and then proves the latter by a positive Poisson–Bessel kernel. The kernel is

$$K_s(t) = \sum_{k \geq 1} (k + 2sk^2) e^{-sk} J_0(2\pi kt),$$

whose diagonal size is $O(s^{-2})$. A Poisson summation formula converts K_s into a sum of explicitly signed terms; away from s-neighborhoods of the integers all terms are non-positive, and one term contributes $\ll -(1+t)^{-1/2}$.

1. Introduction

Erdős and Sárközy asked how large a measurable subset of a disk in $\mathbb{R}^2$ can be if it avoids all positive integer distances [3]. The problem is listed as Erdős Problem #953 on the Erdős Problems website [1]. For $R > 0$ put

$$M(R) = \sup\big\{|A| : \ A \subset B_R(0) \subset \mathbb{R}^2 \text{ is measurable and } |a-b| \notin \mathbb{Z}_{>0} \text{ for } a \neq b \in A\big\}.$$

The elementary slicing argument gives $M(R) = O(R)$. In the opposite direction, Sárközy's construction of point sets with distances bounded away from the integers gives $M(R) \gg_\varepsilon R^{1/2-\varepsilon}$ for every $\varepsilon > 0$; see [5, 6]. We prove that this lower bound has the correct exponent, thereby settling the order-of-growth form of Erdős Problem #953 and the corresponding question in its discussion thread [2].

Theorem 1.1. *For all $R \geq 1$,*

$$M(R) \ll R^{1/2}.$$

Consequently,

$$M(R) = R^{1/2+o(1)} \qquad (R \to \infty).$$

The proof proceeds through a robust finite analogue. For $0 < \delta < 1/2$, let $N(X, \delta)$ be the largest cardinality of a set $P \subset B_X(0)$ such that

$$\big\||p-p'|\big\|_{\mathbb{Z}} \geq \delta \qquad (p \neq p', \ p, p' \in P),$$

where $\|x\|_{\mathbb{Z}} = \operatorname{dist}(x, \mathbb{Z})$. Konyagin proved that, for each fixed $\delta > 0$, one has $N(X, \delta) \ll_\delta X^{1/2}$ [4]. The uniform dependence on δ needed for the measurable problem is the following estimate.

Theorem 1.2. *There is an absolute constant C such that, for $X \geq 1$ and $0 < \delta < 1/10$,*

$$N(X, \delta) \leq C\delta^{-2} X^{1/2}.$$

The exponent δ^{-2} is the natural scale: even in a bounded disk, one may pack $\asymp \delta^{-2}$ points separated by distance δ in a region of diameter less than 1. Section 2 shows that Theorem 1.2 implies the upper bound in Theorem 1.1. Sections 3–5 prove Theorem 1.2 by a positive-definite Bessel kernel.

Throughout the paper, implicit constants are absolute unless a dependence is explicitly indicated. The disk $B_R(0)$ may be taken open or closed; this has no effect on the asymptotic statements.

2. From measurable sets to robust point sets

We first record the comparison between the measurable problem and the robust point problem.

Proposition 2.1. *There are absolute constants $c, C > 0$ such that, for $R \geq 1$,*

$$c \sup_{0<\delta<1/10} \delta^2 N(R-1,\delta) \leq M(R) \leq C \sup_{0<\delta<1/10} \delta^2 N(R,\delta).$$

Proof. For the lower bound, let $P \subset B_{R-1}(0)$ be counted by $N(R-1,\delta)$. Replace each point $p \in P$ by the disk $B_{\delta/4}(p)$. These disks are disjoint because the robust condition implies $|p - p'| \geq \delta$ for $p \neq p'$. The union lies in $B_R(0)$. Distances within a single small disk are less than $\delta/2 < 1$, while distances between points in two different small disks differ from the corresponding center distance by at most $\delta/2$. Hence no positive integer distance is created. The union therefore has measure $\gg \delta^2 |P|$.

For the upper bound, let $A \subset B_R(0)$ be admissible and choose compact $K \subset A$ with $|K| > |A| - \varepsilon$. The distance set

$$D_K = \{|x-y| : \ x, y \in K\}$$

is compact and is disjoint from the finite set $\{1, 2, \ldots, \lfloor 2R \rfloor\}$. Therefore there exists $\eta > 0$ such that every distance in D_K is at least η away from every positive integer. Choose $0 < \delta < \min(\eta, 1/10)$. Let $P \subset K$ be a maximal δ-separated subset of K. Then P is counted by $N(R,\delta)$: separation gives distance at least δ from 0, and the choice of $\delta \leq \eta$ gives distance at least δ from every positive integer. By maximality, the disks $B_\delta(p)$, $p \in P$, cover K. Thus

$$|K| \leq \pi\delta^2 |P| \leq \pi\delta^2 N(R,\delta) \leq \pi \sup_{0<u<1/10} u^2 N(R,u).$$

Letting $\varepsilon \downarrow 0$ proves the upper bound. □

Remark 2.2. *The exact value for small radii is immediate: if $0 < R \leq 1/2$, then $M(R) = \pi R^2$, because all distances inside $B_R(0)$ are less than 1. The elementary slicing argument gives, for all $R > 1/2$,*

$$M(R) \leq \int_{-R}^{R} \min\big(2\sqrt{R^2 - x^2}, 1\big)\, dx = \sqrt{R^2 - \frac{1}{4}} + 2R^2 \arcsin \frac{1}{2R} = 2R + O(R^{-1}).$$

Indeed, almost every vertical fiber has one-dimensional measure at most 1, since it contains no two points whose difference is a positive integer.

3. The Poisson–Bessel kernel

For $0 < s < 1$ define

$$K_s(t) = \sum_{k=1}^{\infty} (k + 2sk^2) e^{-sk} J_0(2\pi k t), \qquad t \geq 0, \tag{3.1}$$

where J_0 is the Bessel function of the first kind. The two features we need are positive definiteness and a diagonal bound.

Lemma 3.1. *The function $x \mapsto K_s(|x|)$ is positive definite on $\mathbb{R}^2$, and*

$$K_s(0) \ll s^{-2}.$$

Proof. For each $k \geq 1$,

$$J_0(2\pi k|x|) = \int_{S^1} e^{2\pi i k \omega \cdot x}\, d\sigma(\omega),$$

where σ is normalized arclength measure on the unit circle. Hence $x \mapsto J_0(2\pi k|x|)$ is positive definite. Since all coefficients in (3.1) are non-negative, K_s is positive definite. Also

$$K_s(0) = \sum_{k \geq 1} (k + 2sk^2) e^{-sk} \ll s^{-2} + s\, s^{-3} \ll s^{-2}.$$

□

We next rewrite K_s by Poisson summation. Put

$$\lambda_m = s + 2\pi i m, \qquad q_m(t) = \lambda_m^2 + (2\pi t)^2,$$

with principal branches used for fractional powers, and define

$$T_m(s,t) = \Re\left[\lambda_m q_m(t)^{-3/2} + 2s\left(-q_m(t)^{-3/2} + 3\lambda_m^2 q_m(t)^{-5/2}\right)\right]. \tag{3.2}$$

Proposition 3.2 (Poisson expansion)**.** *For every $s > 0$ and $t \geq 0$,*

$$K_s(t) = \sum_{m\in\mathbb{Z}} T_m(s,t),$$

and the series is absolutely convergent. Moreover $T_{-m}(s,t) = T_m(s,t)$.

Proof. For $\Re\lambda > 0$ one has the standard Laplace transform identity (see, for example, [7])

$$\int_0^\infty e^{-\lambda x} J_0(2\pi t x)\, dx = \left(\lambda^2 + (2\pi t)^2\right)^{-1/2}. \tag{3.3}$$

Applying Poisson summation to the even function $x \mapsto e^{-s|x|} J_0(2\pi t|x|)$ gives

$$\sum_{k\geq 1} e^{-sk} J_0(2\pi k t) = \sum_{m\in\mathbb{Z}} \Re q_m(t)^{-1/2} - \frac{1}{2}.$$

One may justify this directly by inserting a Gaussian factor $e^{-\alpha x^2}$, applying the Schwartz-class Poisson formula, and then letting $\alpha \downarrow 0$. The differentiated identities follow in the same regularized limit: after one s-derivative the summands are $O_s(|m|^{-2})$, and after two derivatives they are again $O_s(|m|^{-2})$, uniformly for s in compact subintervals of $(0,\infty)$. Thus

$$\sum_{k\geq 1} k e^{-sk} J_0(2\pi k t) = \sum_{m\in\mathbb{Z}} \Re\big(\lambda_m q_m(t)^{-3/2}\big)$$

and

$$\sum_{k\geq 1} k^2 e^{-sk} J_0(2\pi k t) = \sum_{m\in\mathbb{Z}} \Re\big(-q_m(t)^{-3/2} + 3\lambda_m^2 q_m(t)^{-5/2}\big).$$

Combining these identities gives the formula for K_s. For fixed $s > 0$ and $t \geq 0$, the summand in (3.2) is $O_s(|m|^{-2})$ as $|m| \to \infty$, so the series is absolutely convergent. Finally, replacing m by $-m$ conjugates both λ_m and $q_m(t)$, and hence leaves the real part unchanged. □

4. Sign of the Poisson terms

The central estimate is that each Poisson term is non-positive away from an s-neighborhood of the corresponding integer, and is quantitatively negative on one side of it.

Lemma 4.1 (One-term estimate)**.** *There exist absolute constants $L \geq 1$ and $s_0 > 0$ with the following property. Let $0 < s < s_0$, let $m \geq 1$, and set*

$$a = 2\pi m, \qquad b = 2\pi t.$$

Assume $|a - b| \geq Ls$. If $b < a$, then

$$T_m(s,t) \leq -\frac{1}{4}\, a(a^2 - b^2)^{-3/2}. \tag{4.1}$$

If $b > a$, then

$$T_m(s,t) \leq -\frac{1}{4}\, s\big((b^2 - a^2)^{-3/2} + a^2(b^2 - a^2)^{-5/2}\big) \leq 0. \tag{4.2}$$

For $m = 0$ one has $T_0(s,t) \leq 0$ whenever $2\pi t \geq Ls$, after increasing L if necessary.

Proof. The case $m = 0$ is explicit. Since $q_0 = s^2 + (2\pi t)^2$,

$$T_0(s,t) = s\frac{5s^2 - (2\pi t)^2}{(s^2 + (2\pi t)^2)^{5/2}},$$

which is non-positive when $2\pi t \geq \sqrt{5}\,s$.

Assume now that $m \geq 1$, so $a \geq 2\pi$. Write

$$\zeta = s^2 + 2ias.$$

In both cases below we shall use

$$|\zeta| \leq 3as \tag{4.3}$$

for $s_0 \leq 1$. If $D = |a^2 - b^2|$, then

$$D = |a-b|(a+b) \geq aLs, \qquad \frac{|\zeta|}{D} \leq \frac{3}{L}. \tag{4.4}$$

We choose L large enough that the Taylor expansions used below are valid for $|z| \leq 3/L$.

First suppose $b < a$, and set $D = a^2 - b^2$. Then $q_m = -D + \zeta = -D(1 - \zeta/D)$. On the principal branch,

$$q_m^{-3/2} = iD^{-3/2}\big(1 + O(L^{-1})\big), \qquad q_m^{-5/2} = -iD^{-5/2}\big(1 + O(L^{-1})\big).$$

Substitution into (3.2) gives

$$T_m(s,t) = -aD^{-3/2} + O\big(L^{-1}aD^{-3/2}\big) + O\big(sD^{-3/2}\big) + O\big(s(a^2+s^2)D^{-5/2}\big).$$

The third term is at most $O(s_0)aD^{-3/2}$, since $a \geq 2\pi$. The last term is at most

$$O\left(\frac{as}{D}\right)aD^{-3/2} + O(s_0^2)aD^{-3/2} = O(L^{-1})aD^{-3/2} + O(s_0^2)aD^{-3/2},$$

using (4.4). Taking s_0 sufficiently small and then L sufficiently large proves (4.1).

Now suppose $b > a$, and set $D = b^2 - a^2$. Then $q_m = D + \zeta = D(1+z)$ with $z = \zeta/D$ and $|z| \leq 3/L$. We use

$$(1+z)^{-3/2} = 1 - \frac{3}{2}z + E_3(z), \qquad |E_3(z)| \leq C|z|^2,$$

and

$$(1+z)^{-5/2} = 1 + E_5(z), \qquad |E_5(z)| \leq C|z|.$$

Keeping the real part through first order in the $q_m^{-3/2}$ terms and using zeroth order in the $q_m^{-5/2}$ term gives the main contribution

$$\begin{aligned} M_0 &= \Re\left[(s+ia)D^{-3/2}\left(1 - \frac{3}{2}\frac{\zeta}{D}\right)\right] \\ &\quad + 2s\Re\left[-D^{-3/2}\left(1 - \frac{3}{2}\frac{\zeta}{D}\right) + 3(s+ia)^2D^{-5/2}\right] \\ &= -sD^{-3/2} - 3a^2sD^{-5/2} + \frac{15}{2}s^3D^{-5/2}. \end{aligned}$$

Decreasing s_0 if necessary, the last term is at most $\frac{1}{2}a^2sD^{-5/2}$, and hence

$$M_0 \leq -sD^{-3/2} - \frac{5}{2}a^2sD^{-5/2}. \tag{4.5}$$

The remainder $R = T_m - M_0$ satisfies

$$\begin{aligned} |R| &\leq C(a+s)|z|^2D^{-3/2} + Cs|z|^2D^{-3/2} + Cs(a^2+s^2)|z|D^{-5/2} \\ &\leq CL^{-2}sD^{-3/2} + CL^{-1}a^2sD^{-5/2}. \end{aligned}$$

Indeed, $|z| \leq Cas/D$ and $as/D \leq L^{-1}$; the first term is bounded by $CL^{-1}a^2sD^{-5/2}$, the second by $CL^{-2}sD^{-3/2}$, and the third by $CL^{-1}a^2sD^{-5/2}$, after decreasing s_0 so that $s \leq a$. Choosing L large in comparison with the absolute constant C and using (4.5) gives (4.2). □

Proposition 4.2 (Kernel negativity). *There exist absolute constants $A, c > 0$ and $s_0 > 0$ such that, whenever $0 < s < s_0$ and*

$$\|t\|_{\mathbb{Z}} \geq As,$$

one has

$$K_s(t) \leq -c(1+t)^{-1/2}.$$

Proof. Let L and s_0 be as in Lemma 4.1, and put $A = L/(2\pi)$. Decrease s_0 so that $As_0 < 1/2$. If $\|t\|_{\mathbb{Z}} \geq As$, then

$$|2\pi t - 2\pi m| \geq Ls \qquad (m = 0, 1, 2, \ldots).$$

By Lemma 4.1, $T_m(s,t) \leq 0$ for every $m \geq 0$. By Proposition 3.2, $T_{-m} = T_m$, so every term in the Poisson expansion of $K_s(t)$ is non-positive.

Let $n = \lceil t \rceil$. Since $As < 1/2$ and $\|t\|_{\mathbb{Z}} \geq As$, we have $n \geq 1$ and $n - t \geq As$. Applying the $b < a$ case of Lemma 4.1 to $m = n$ gives

$$K_s(t) \leq T_n(s,t) \leq -c_1(2\pi n)\left((2\pi n)^2 - (2\pi t)^2\right)^{-3/2}.$$

Because $0 < n - t \leq 1$ and $n \leq t + 1$,

$$(2\pi n)^2 - (2\pi t)^2 = (2\pi)^2(n-t)(n+t) \leq C(1+t),$$

while $2\pi n \geq c(1+t)$. Therefore

$$(2\pi n)\left((2\pi n)^2 - (2\pi t)^2\right)^{-3/2} \geq c_2(1+t)^{-1/2},$$

and the claim follows. □

5. The robust upper bound

We now prove Theorem 1.2.

Proof of Theorem 1.2. Let $P = \{p_1, \ldots, p_n\} \subset B_X(0)$ satisfy $\||p_i - p_j|\|_{\mathbb{Z}} \geq \delta$ for $i \neq j$. Let A_0 and s_0 be the constants in Proposition 4.2, and put

$$A = \max\{A_0, (10s_0)^{-1}\}, \qquad s = \delta/A.$$

Then $0 < s < s_0$ and $\delta = As \geq A_0 s$.

By positive definiteness of $x \mapsto K_s(|x|)$,

$$0 \leq \sum_{i,j=1}^{n} K_s(|p_i - p_j|).$$

The diagonal terms contribute at most Cns^{-2} by Lemma 3.1. For $i \neq j$, the robust condition gives $\||p_i - p_j|\|_{\mathbb{Z}} \geq \delta \geq A_0 s$, and Proposition 4.2 gives

$$K_s(|p_i - p_j|) \leq -c(1 + |p_i - p_j|)^{-1/2} \leq -c(1+2X)^{-1/2}.$$

Hence

$$0 \leq Cns^{-2} - cn(n-1)(1+2X)^{-1/2}.$$

If $n \geq 2$, this implies

$$n \ll s^{-2}X^{1/2} \ll \delta^{-2}X^{1/2}.$$

The same bound is trivial when $n = 0$ or 1, after enlarging the implicit constant. □

6. The measurable problem

Proof of Theorem 1.1. By Proposition 2.1 and Theorem 1.2, for $R \geq 1$,

$$M(R) \leq C \sup_{0<\delta<1/10} \delta^2 N(R,\delta) \leq C \sup_{0<\delta<1/10} \delta^2 \delta^{-2} R^{1/2} \ll R^{1/2}.$$

For the lower bound, Sárközy proved [5, 6] that for all sufficiently small fixed $\delta > 0$ and all sufficiently large Y one has

$$N(Y,\delta) > Y^{1/2-\delta^{1/7}}.$$

Apply this with $Y = R-1$ for large R. Thickening each point to a disk of radius $\delta/3$ gives an admissible measurable subset of $B_R(0)$ of area $\gg_\delta R^{1/2-\delta^{1/7}}$. Taking δ depending only on ε gives $M(R) \gg_\varepsilon R^{1/2-\varepsilon}$ for every $\varepsilon > 0$. Together with the upper bound, this is equivalent to $M(R) = R^{1/2+o(1)}$. □

References

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