Unit distances between disjoint convex translates

27 April 2026

Abstract

Let $h(n)$ be the maximum number of pairs at Euclidean set-distance exactly 1 among n pairwise disjoint translates of one compact convex set in the plane. Erdős and Pach proved the upper bound $h(n) = O(n^{4/3})$. We give a self-contained matching lower-bound construction. Combining the two bounds gives the asymptotic order

$$h(n) = \Theta(n^{4/3}).$$

In particular, Erdős Problem #956 has a positive answer in the requested polynomial-growth sense: one may take any constant $c < 1/3$ in $h(n) > n^{1+c}$ for all sufficiently large n.

1 Reduction to a difference body

For compact sets $A, B \subset \mathbb{R}^2$, write

$$\delta(A, B) = \inf\{\|a - b\|_2 : a \in A,\ b \in B\}.$$

Let $C \subset \mathbb{R}^2$ be compact and convex, and put

$$D := C - C = \{c - c' : c, c' \in C\}.$$

Then D is centrally symmetric and convex. For two translation vectors $x, y \in \mathbb{R}^2$,

$$\delta(C + x, C + y) = \operatorname{dist}(y - x, D), \tag{1}$$

where the distance on the right is ordinary Euclidean point-to-set distance. Moreover,

$$(C + x) \cap (C + y) = \varnothing \quad \Longleftrightarrow \quad y - x \notin D. \tag{2}$$

Conversely, every centrally symmetric compact convex set D is of the form $C - C$, namely with $C = D/2$: by central symmetry, $-D = D$, and by convexity $D + D = 2D$, so

$$(D/2) - (D/2) = \frac{1}{2}(D + (-D)) = \frac{1}{2}(D + D) = D.$$

Thus it is enough to construct a centrally symmetric compact convex body D and a finite point set X such that no non-zero difference from $X - X$ lies in D, while many differences have Euclidean distance exactly 1 from D.

2 A small convex body whose unit offset contains a parabolic arc

Fix parameters

$$0 < W \le 1, \qquad \eta := W^4.$$

For $0 \le t \le W$, define

$$\gamma(t) := \left(t, 1 + \eta - \frac{t^2}{2}\right), \qquad \nu(t) := \frac{(t,1)}{\sqrt{1+t^2}},$$

and

$$p(t) := \gamma(t) - \nu(t).$$

Thus $\nu(t)$ is the upward unit normal to the parabola γ, and $p(t)$ is the point obtained by moving one unit inward from $\gamma(t)$ along this normal. Let

$$D_W := \operatorname{conv}\big(\{\pm p(t) : 0 \le t \le W\}\big). \tag{3}$$

This is a centrally symmetric compact convex body for every $W > 0$: it is compact and convex by definition, centrally symmetric by construction, and the three points $p(0)$, $-p(0)$, and $p(W)$ are not collinear.

Lemma 1. *For $0 \le t \le W \le 1$,*

$$0 \le p_x(t) \le \frac{W^3}{2}, \qquad 0 \le p_y(t) \le \eta.$$

Consequently

$$D_W \subseteq \left[-\frac{W^3}{2}, \frac{W^3}{2}\right] \times [-\eta, \eta]. \tag{4}$$

Proof. Write $r(t) = \sqrt{1+t^2}$. Then

$$p_x(t) = t\left(1 - \frac{1}{r(t)}\right) \ge 0,$$

and, since

$$1 - \frac{1}{\sqrt{1+u}} = \frac{u}{\sqrt{1+u}(\sqrt{1+u}+1)} \le \frac{u}{2} \quad (u \ge 0),$$

we have $p_x(t) \le t^3/2 \le W^3/2$.

For the y-coordinate,

$$p_y(t) = \eta + 1 - \frac{t^2}{2} - \frac{1}{\sqrt{1+t^2}}.$$

We use the elementary estimates

$$1 - \frac{u}{2} \le \frac{1}{\sqrt{1+u}} \le 1 - \frac{u}{2} + \frac{u^2}{2} \qquad (0 \le u \le 1). \tag{5}$$

The left inequality follows, for instance, from $\sqrt{1+u} \le 1 + u/2$ and $1/(1+u/2) \ge 1 - u/2$. For the right inequality, let

$$F(u) := 1 - \frac{u}{2} + \frac{u^2}{2} - (1+u)^{-1/2}.$$

Then $F(0) = F'(0) = 0$ and

$$F''(u) = 1 - \frac{3}{4}(1+u)^{-5/2} \geq \frac{1}{4} \qquad (u \geq 0),$$

so $F(u) \geq 0$. Applying (5) with $u = t^2$ gives

$$p_y(t) \leq \eta$$

and

$$p_y(t) \geq \eta - \frac{t^4}{2} \geq W^4 - \frac{W^4}{2} \geq 0.$$

The box containment follows from the definition of D_W as the convex hull of these points and their negatives. □

Lemma 2. *For every* $0 \leq t \leq W$,

$$\operatorname{dist}(\gamma(t), D_W) = 1.$$

Proof. Since $p(t) \in D_W$ and $\gamma(t) - p(t) = \nu(t)$ is a unit vector, we have

$$\operatorname{dist}(\gamma(t), D_W) \leq 1.$$

It remains to prove the reverse inequality.

We show that the line

$$\{z \in \mathbb{R}^2 : \langle z, \nu(t)\rangle = \langle p(t), \nu(t)\rangle\}$$

supports D_W. First consider another point $p(s)$, $0 \leq s \leq W$. The vertical shift by η cancels in the following calculation, so put $r_u := \sqrt{1+u^2}$. A direct computation gives

$$\begin{aligned}&\langle p(t), \nu(t)\rangle - \langle p(s), \nu(t)\rangle \\ &\qquad = \frac{(s-t)^2}{2r_t} - \left(1 - \frac{1+st}{r_s r_t}\right).\end{aligned}$$

Furthermore,

$$1 - \frac{1+st}{r_s r_t} = \frac{(s-t)^2}{r_s r_t (r_s r_t + 1 + st)}.$$

Since $s, t \geq 0$, we have

$$r_s(r_s r_t + 1 + st) \geq 2.$$

Therefore

$$\frac{(s-t)^2}{2r_t} \geq \frac{(s-t)^2}{r_s r_t (r_s r_t + 1 + st)},$$

and hence

$$\langle p(s), \nu(t)\rangle \leq \langle p(t), \nu(t)\rangle. \tag{6}$$

By Lemma 1, both coordinates of every $p(s)$ are non-negative, and both coordinates of $\nu(t)$ are non-negative. Hence

$$\langle -p(s), \nu(t)\rangle \leq 0 \leq \langle p(t), \nu(t)\rangle.$$

Together with (6), this proves that every generator $\pm p(s)$ of D_W lies in the closed half-plane

$$\langle z, \nu(t)\rangle \leq \langle p(t), \nu(t)\rangle.$$

Thus the same is true for all $z \in D_W$. Consequently,

$$\operatorname{dist}(\gamma(t), D_W) \geq \langle \gamma(t), \nu(t)\rangle - \langle p(t), \nu(t)\rangle = \langle \gamma(t) - p(t), \nu(t)\rangle = 1.$$

This proves the lemma. □

3 The point set

Fix a small absolute constant $\alpha \in (0, 1/10]$. For an integer $k \geq 2$, put

$$W := \frac{\alpha}{k}, \qquad a := \frac{W}{k} = \frac{\alpha}{k^2}, \qquad b := \frac{a^2}{2} = \frac{\alpha^2}{2k^4}, \qquad \eta := W^4 = \frac{\alpha^4}{k^4}.$$

Let $D = D_W$ be the body from (3). Define two grids

$$L := \{(ra, sb) : 0 \leq r \leq k,\ 0 \leq s \leq k^2\},$$

$$U := \{(ra, 1 + \eta + sb) : 0 \leq r \leq k,\ 0 \leq s \leq k^2\},$$

where r, s are integers. Set

$$X_k := L \cup U.$$

Then

$$n_k := |X_k| = 2(k+1)(k^2+1) = 2k^3 + O(k^2). \tag{7}$$

Lemma 3. *No non-zero difference of two points of X_k lies in D.*

Proof. By Lemma 1,

$$D \subseteq [-W^3/2, W^3/2] \times [-\eta, \eta].$$

Since $W = \alpha/k$ and $a = \alpha/k^2$,

$$\frac{W^3}{2} = \frac{\alpha^3}{2k^3} < \frac{\alpha}{k^2} = a$$

for all $k \geq 2$ and $\alpha \leq 1$. Also,

$$\eta = \frac{\alpha^4}{k^4} < \frac{\alpha^2}{2k^4} = b$$

because $\alpha^2 < 1/2$.

If two points lie in the same grid L or U, then their difference is of the form $(ma, \ell b)$ with integers m, ℓ. If this difference is non-zero, then either $|ma| \geq a > W^3/2$ or $|\ell b| \geq b > \eta$, so it is not in D.

If one point lies in L and the other in U, then the absolute value of the y-coordinate of their difference is at least

$$1 + \eta - k^2 b = 1 + \eta - \frac{W^2}{2} \geq 1 - \frac{\alpha^2}{2} > \eta,$$

again excluding membership in D. Thus no non-zero difference from $X_k - X_k$ lies in D. □

Lemma 4. *There are at least*

$$M_k := \sum_{i=1}^{k} (k+1-i)(k^2+1-i^2) = \frac{5}{12}k^4 + O(k^3)$$

unordered pairs $\{x, y\} \subset X_k$ such that

$$\operatorname{dist}(y - x, D) = 1.$$

Proof. For $i = 1, 2, \ldots, k$, put

$$t_i := ia.$$

Then $0 < t_i \le W$, and

$$\gamma(t_i) = \left(ia, 1 + \eta - \frac{i^2 a^2}{2}\right) = (ia, 1 + \eta - i^2 b).$$

By Lemma 2, $\mathrm{dist}(\gamma(t_i), D) = 1$.

Now choose integers

$$0 \le r \le k - i, \qquad i^2 \le s \le k^2.$$

Then

$$(ra, sb) \in L, \qquad ((r + i)a, 1 + \eta + (s - i^2)b) \in U,$$

and their difference is exactly

$$(ia, 1 + \eta - i^2 b) = \gamma(t_i).$$

Thus each such choice gives a pair at distance 1 from D. These counted pairs are distinct, since each has one endpoint in L, one endpoint in U, and horizontal separation ia. For fixed i, the number of choices is

$$(k + 1 - i)(k^2 + 1 - i^2).$$

Summing over $i = 1, \ldots, k$ gives M_k pairs. The displayed closed form follows from the elementary sums

$$\sum_{i=1}^{k} i = \frac{k(k+1)}{2}, \quad \sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}, \quad \sum_{i=1}^{k} i^3 = \frac{k^2(k+1)^2}{4}.$$

Indeed,

$$M_k = \frac{5}{12}k^4 + \frac{1}{6}k^3 + \frac{1}{12}k^2 + \frac{1}{3}k.$$

□

4 The lower bound for convex translates

Theorem 1. *There is an absolute constant $c_0 > 0$ such that*

$$h(n) \ge c_0 n^{4/3}$$

for all sufficiently large n.

Proof. For the point set X_k and body $D = D_W$ above, let

$$C := \frac{1}{2}D.$$

Since D is centrally symmetric and convex, $C - C = D$. Lemma 3 and (2) imply that the translates $C + x$, $x \in X_k$, are pairwise disjoint. Lemma 4 and (1) imply that at least M_k pairs among them have set-distance exactly 1.

By (7), $n_k = 2k^3 + O(k^2)$, while Lemma 4 gives $M_k = (5/12)k^4 + O(k^3)$. Hence

$$h(n_k) \ge M_k \ge c_1 n_k^{4/3}$$

for an absolute constant $c_1 > 0$ and all sufficiently large k.

For a general sufficiently large n, choose k maximal with $n_k \le n$. Add $n - n_k$ extra translates very far from all existing translates and from one another. More explicitly, since D and X_k are bounded, one may choose a vector v with $\|v\|$ larger than $\sup_{d\in D} \|d\| + \sup_{x\in X_k} \|x\|$ and add suitable points among $v, 2v, 3v, \ldots$. Then every new difference has norm larger than $\sup_{d\in D} \|d\|$, and hence is outside D. This preserves disjointness and cannot decrease the number of distance-1 pairs. Since $n_{k+1}/n_k \to 1$, the same lower bound, with a smaller absolute constant, holds for all sufficiently large n. □

5 Asymptotic determination of $h(n)$

We use the following published theorem of Erdős and Pach.

Theorem 2 (Erdős–Pach [2])**.** *For the same function $h(n)$,*

$$h(n) = O(n^{4/3}).$$

This is also the upper bound recorded on Erdős Problem #956 [1]. Combining Theorem 2 with Theorem 1 gives the asymptotic order.

Corollary 1. *For the maximum number $h(n)$ of unit distances between n pairwise disjoint translates of one compact convex plane set,*

$$h(n) = \Theta(n^{4/3}).$$

Consequently, for every fixed $c < 1/3$ there is $n_0(c)$ such that

$$h(n) > n^{1+c} \qquad (n \ge n_0(c)).$$

Remark 1 (Relation to Valtr's parabolic norm construction)*.* The local counting mechanism above is the same parabolic-grid mechanism used by Valtr to construct strictly convex norms with many unit distances [3]. The additional point in this note is that the convex set D is made extremely small: it is essentially the inner unit parallel body of a tiny parabolic cap. This makes the translates of $C = D/2$ pairwise disjoint, while the outer unit parallel boundary still contains the required parabolic difference vectors.

References

[1] T. F. Bloom, *Erdős Problem #956*, https://www.erdosproblems.com/956, accessed 27 April 2026.

[2] P. Erdős and J. Pach, *Variations on the theme of repeated distances*, Combinatorica **10** (1990), no. 3, 261–269. doi:10.1007/BF02122780.

[3] P. Valtr, *Strictly convex norms allowing many unit distances and related touching questions*, manuscript, 2005. Available at https://kam.mff.cuni.cz/~valtr/n.pdf.