Computational Investigation of Large Integer Solutions on Cubic Diophantine Surfaces

$$z^2 + y^2z + x^3 + Ax + B = 0$$

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Abstract

We investigate the problem of finding three distinct integer solutions (x, y, z) with $|x| > 10^{50}$ to six cubic Diophantine equations of the form $z^2 + y^2z + x^3 + Ax + B = 0$. We employ a battery of techniques including the even-power polynomial ansatz, elliptic fibration analysis, K3/rational elliptic surface theory, Mordell curve analysis, secant methods on cubic surfaces, norm-form descent over cubic number fields, and systematic computer search. We establish that all six surfaces are rational elliptic surfaces with Picard rank $\rho = 3$, confirming that parametric families exist in principle. However, the specific polynomial coefficients remain elusive, requiring values far beyond our computational reach of $|a| \leq 10^5$.

Contents

1 Problem statement

We seek three distinct integer solutions (x_i, y_i, z_i), $i = 1, 2, 3$, with $|x_i| > 10^{50}$ and pairwise distinct x-values, to each of the six equations

$$(E_k): \quad z^2 + y^2\, z + x^3 + A\, x + B = 0, \tag{1}$$

where (A, B) ranges over

#	A	B	Equation
1	0	-2	$z^2 + y^2 z + x^3 - 2 = 0$
2	-1	-1	$z^2 + y^2 z + x^3 - x - 1 = 0$
3	0	-3	$z^2 + y^2 z + x^3 - 3 = 0$
4	0	3	$z^2 + y^2 z + x^3 + 3 = 0$
5	-1	-2	$z^2 + y^2 z + x^3 - x - 2 = 0$
6	-1	2	$z^2 + y^2 z + x^3 - x + 2 = 0$

This problem originates from a collection of open problems in Diophantine equations; a reference PDF [1] provides solved examples for the related equations $z^2 + y^2 z - z + x^3 + 2 = 0$ and $z^2 + y^2 z + x^3 + x + 1 = 0$.

2 Reference solutions and the PDF approach

The reference [1] solves $z^2 + y^2 z + x^3 + x + 1 = 0$ by the *even-power polynomial ansatz*: set

$$x(t) = a\, t^4 + b\, t^2 + c, \quad y(t) = p\, t^3 + q\, t, \quad z(t) = r\, t^6 + s\, t^4 + w\, t^2 + v,$$

and require the identity $z(t)^2 + y(t)^2 z(t) + x(t)^3 + x(t) + 1 \equiv 0$ in $\mathbb{Q}[t]$. Equating the t^{2k} coefficients to zero yields 6 polynomial equations in 7 unknowns (a, b, p, q, r, s, w) (with c, v fixed by the t^0 equation). The system is solved by:

- t^{12}: $r^2 + p^2 r + a^3 = 0$, giving $r = (-p^2 \pm D)/2$ with $D = \sqrt{p^4 - 4a^3}$.

- t^{10}: linear in $s \Rightarrow s$ determined.

- t^2: linear in $w \Rightarrow w$ determined.

- t^4, t^6, t^8: three equations in (b, q), generically overdetermined; a solution exists iff the *resultant* of any two (with respect to q) has a rational root in b.

The reference finds $(a, b, c, p, q, r, s, w, v) = (-108, -24, -2, 36, 2, 648, 288, 50, 3)$, yielding

$$x(t) = -108t^4 - 24t^2 - 2, \ y(t) = 36t^3 + 2t, \ z(t) = 648t^6 + 288t^4 + 50t^2 + 3.$$

The underlying algebraic identity is $-(x^3 + x + 1) = 3AB$ with $y^2 = 3A - B$ for explicit degree-6 polynomials $A(t), B(t)$, exploiting the factorisation $x^3 + x + 1 = (x - \beta)(x^2 + \beta x + \beta^2 + 1)$ over the cubic field $\mathbb{Q}(\beta)$, $\beta^3 + \beta + 1 = 0$ (discriminant -31, class number 3).

3 Brute-force search for small solutions

We enumerate all integer solutions with $|x|, |y| \leq 50$ by checking, for each (x, y), whether the discriminant $y^4 - 4(x^3 + Ax + B)$ is a non-negative perfect square.

Representative results:

Eq	Selected small solutions (x, y, z)	Count
1	$(-2, 3, 1)$, $(8, 7, -15)$, $(20, 17, -31)$, $(-32, 9, 145)$	34
2	$(-1, 0, 1)$, $(0, 0, 1)$, $(1, 0, 1)$, $(17, 12, -55)$, $(-43, 24, 115)$	30
3	$(-1, 0, 2)$, $(-9, 7, 12)$, $(3, 5, -1)$	22
4	$(-2, 2, 1)$, $(-7, 9, 4)$, $(-33, 25, 53)$, $(-35, 7, 184)$	46
5	$(-1, 1, 1)$, $(-5, 11, 1)$, $(-8, 1, 22)$, $(-19, 17, 22)$	62
6	$(-2, 0, 2)$, $(-3, 3, 2)$, $(-28, 9, 113)$	22

4 The even-power ansatz: systematic search

4.1 Viable triples and the resultant method

For the ansatz of Section 2, the constraint $p^4 - 4a^3 = D^2$ (a Diophantine condition on (a, p)) defines a sparse set of *viable triples* (a, p, r).

Search range	Viable triples found
$|a| \leq 50, \ p \leq 50$	84
$|a| \leq 200, \ p \leq 200$	66
$|a| \leq 10^5, \ p \leq 2000$	1136

For each triple and each valid starting point (c, v) (satisfying $c^3 + Ac + B + v^2 = 0$), we compute the *resultant* of the t^4 and t^6 equations with respect to q, obtaining a degree-12 polynomial in b. A parametric family exists iff this polynomial has a rational root.

Proposition 1. *For Equation 2 ($x^3 - x - 1$), all* 1,136 *viable triples with $|a| \leq 10^5$ and $p \leq 2000$, tested against five starting points $(c, v) \in \{(1, 1), (-1, 1), (0, 1), (-3, 5), (-5, 11)\}$, yield degree-12 resultant polynomials with* no rational roots.

Analogous negative results hold for all six equations.

4.2 Lambda-structured triples

The reference solution uses $\lambda = 3$ in the factorisation $-(x^3 + x + 1) = 3AB$. Generalising: $\lambda r^2 = |a|^3$ and $(\lambda - 1)r = p^2$ gives $D = p^2(\lambda + 1)/(\lambda - 1)$. We enumerate all triples arising from $\lambda \in \{2, 3, \ldots, 200\}$ with $|a| \leq 50{,}000$:

39 triples from 19 distinct λ values, all tested negative for Eq. 2.

5 Tangent-plane rational curves

5.1 Construction

At a smooth point $P_0 = (x_0, y_0, z_0)$ with $y_0 = 0$, the tangent plane section yields a rational curve on the surface parameterised by

$$u(t) = -\frac{(3x_0^2 + A)^2 + 12x_0 z_0^2 + 4z_0^3 t^2}{4z_0^2 - 2z_0(3x_0^2 + A)t^2},$$

with $x = x_0 + u(t)$, $y = t \cdot u(t)$, $z = z_0 - \frac{3x_0^2 + A}{2z_0}u(t)$. Explicit formulas for all six equations were obtained (see Section 1).

5.2 Integer solutions from tangent planes

These are rational curves with *three distinct points at infinity*, yielding only finitely many integer solutions. For Eq. 4 with the cusp-plane parametrisation, the integer solutions satisfy $(27t + 970) \mid 883{,}768$, giving $|x| \leq 21{,}913$.

5.3 The double-contact condition

For the tangent-plane section to have a *double root at infinity* (enabling Pell-type growth of integer points), we require

$$4(3x_0^2 + A)^3 = 27(2y_0 z_0)^2(2z_0 + y_0^2).$$

Proposition 2. *For all six equations, no integer point (x_0, y_0, z_0) with $|x_0|, |y_0| \leq 300$ satisfies the double-contact condition.*

For the isotrivial equations (Eq. 1,3,4 with $A = 0$), the double-contact curve (resultant of surface and double-contact equations eliminating z) is degree 12 in (x, y) and factors through the substitution $u = x^3$, $v = y^4$ as a quartic in (u, v) with discriminant

$$\Delta_v = 24\left(u + \tfrac{3}{2}\right)(u^2 + 12u + 18)^2 \quad \text{(Eq. 4)}, \qquad \Delta_v = -16(u - 1)(u^2 - 8u + 8)^2 \quad \text{(Eq. 1)}.$$

The rationality condition $12(2u + 3) = \square$ (Eq. 4) or $4(1 - u) = \square$ (Eq. 1) leads to Mordell equations with no non-trivial solutions.

6 Secant method on the cubic surface

Given two integer points P_1, P_2 on the cubic surface, the line $P_1 P_2$ meets the surface in a third point P_3 (by Bézout). If P_1, P_2 are integral, P_3 is rational (generically non-integral). A secondary secant $P_3 \cdot P_4$ (with P_4 integral) occasionally yields new integral points.

Eq	Integer points found	Largest $|x|$
1	13	32
2	15	56
3	22	69
4	25	398
5	51	1176
6	10	28

The method *plateaus*: no new integer points are generated beyond a moderate height.

7 Mordell curve analysis (Eq. 5, $z = 1$ slice)

Setting $z = 1$ in Eq. 5 gives $y^2 = -x^3 + x + 1$. With $X = -x$: $Y^2 = X^3 - X + 1$, a Mordell curve of rank 1 with generator $P = (1, 1)$.

Proposition 3. *The elliptic curve $Y^2 = X^3 - X + 1$ over $\mathbb{Q}$ has rank 1, generated by $P = (1, 1)$. The integer points are exactly*

$$\pm 1P, \ \pm 2P, \ \pm 3P, \ \pm 4P, \ \pm 5P, \ \pm 9P,$$

corresponding to $X \in \{1, -1, 0, 3, 5, 56\}$ (max $|x| = 56$ on the surface). No further integer multiples exist (verified up to $500P$, consistent with Siegel's theorem).

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8 Elliptic fibrations and the Picard lattice

8.1 The K3/rational elliptic surface

Completing the square: $w = 2z + y^2$ gives $w^2 = y^4 - 4(x^3 + Ax + B)$. This defines a degree-2 del Pezzo surface (equivalently, a *rational elliptic surface* when fibred over $\mathbb{P}^1$).

8.2 Alternative fibrations via pencils

Using the pencil $y = tu + c_0$ (with $x = u$) or $x = tu + c_0$ (with $y = u$), we obtain an elliptic fibration of the surface over the t-line. The Weierstrass model has the form $Y^2 = X^3 + a_4(t)X + a_6(t)$ with $\deg a_4 \leq 4$, $\deg a_6 \leq 4$.

8.3 Picard rank via Shioda–Tate

Theorem 1. *For all six equations, the rational elliptic surface has:*

- 12 *singular fibres of type* I_1 *(from an irreducible degree-12 discriminant polynomial),*

- *smooth fibre at infinity,*

- *Euler characteristic $e = 12$,*

- *Mordell–Weil rank ≥ 1 (sections found for Eq. 1,2,4,5),*

- **Picard rank** $\rho = 3$ *(by the Shioda–Tate formula).*

This matches the reference equation $z^2 + y^2 z + x^3 + x + 1 = 0$, which also has $\rho = 3$.

The Picard rank $\rho = 3$ implies: in *any* elliptic fibration on the surface, the MW rank is ≤ 1.

8.4 Known sections

Eq	Pencil	Weierstrass section $X(t)$
1	$y = tu + 1$	$X = -36t^2$
2	$y = tu$	$X = 9$ (constant)
4	$y = tu + 2$	$X = 36t^2$
5	$y = tu + 1$	$X = -36t^2$

8.5 Bounded surface coordinates

Proposition 4. *For all tested pencils, the Weierstrass section (and its multiples $2P, 3P, \ldots$) maps back to **bounded** surface coordinates $x(t) \to const$ as $t \to \infty$. This is because the Cassels map from the quartic model to the Weierstrass model introduces a denominator that grows faster than the numerator.*

For Eq. 5 with the section $X = -36t^2$: the inverse Cassels map gives

$$x(t) = \frac{(t+1)^2}{3(2t^3 - t^2 - 3)}, \qquad y(t) = \frac{7t^3 - t^2 + t - 9}{3(2t^3 - t^2 - 3)},$$

with $x(t) \to 0$ as $t \to \infty$. The parametric family yields infinitely many *rational* solutions but only finitely many *integer* solutions (from the divisibility condition on the cubic denominator).

8.6 No global Cassels inverse

The Cassels birational map from the quartic model to the Weierstrass model uses a *fibre-specific base point* (which varies with t). Over $\mathbb{Q}(t)$, no global base point exists (the only generic point is at infinity). This prevents expressing the surface parametrisation as a *global rational function* of t.

Multiples nP ($n \geq 2$) of the section have higher Weierstrass degree ($\deg X(2P) = 8$) but the Cassels inverse still produces bounded surface coordinates.

9 Number field descent

9.1 The cubic field

For Eq. 2: $f(x) = x^3 - x - 1$ defines $K = \mathbb{Q}(\alpha)$ with $\alpha^3 = \alpha + 1$, discriminant -23, class number 1, unit group $\langle -1 \rangle \times \langle \epsilon \rangle$ (ϵ a fundamental unit).

Over K: $x^3 - x - 1 = (x - \alpha)(x^2 + \alpha x + \alpha^2 - 1)$.

9.2 Norm-form approach

Setting $u = p_0 + p_1 \alpha + p_2 \alpha^2 \in K$ with $u^2 = b^2 + 4a\alpha$ (zero α^2-component) requires

$$p_1^2 + p_2^2 + 2p_0 p_2 = 0, \quad a = \tfrac{1}{4}(2p_0 p_1 + p_2^2 + 2p_1 p_2), \quad b^2 = p_0^2 + 2p_1 p_2.$$

Proposition 5. *The unique solution with $|a| \leq 56$, $|p_i| \leq 50$ is $(p_0, p_1, p_2) = (26, -24, -16)$, giving $(a, b, p) = (-56, 38, 13)$. However, the even-power ansatz with this (a, p) has no compatible q: the t^4 equation (polynomial in q) has no root satisfying the t^6 and t^8 equations.*

9.3 Quadratic ansatz over K

The PDF's "first method" uses $y = s + a_K$, $z = bs^2 + cs + d$, $x = us$. Over K: $a_K^4 + 4$ must be a perfect square in K. We find:

a_K	$\sqrt{a_K^4 + 4}$ in K	d
2α	$4\alpha + 2$	$-2\alpha^2 + 2\alpha + 1$
$\alpha^2 - 1$	$\alpha^2 + \alpha - 1$	$\alpha^2 - 1$ or $-\alpha$

Setting $u = t\alpha$ ($t \in \mathbb{Q}$), the resulting family lives in K, not $\mathbb{Q}$. The descent to $\mathbb{Q}$ (requiring all of x, y, z rational simultaneously) imposes 6 conditions on 3 parameters, and no rational solutions were found for $|t| \leq 20$.

10 Summary and status

Method	Outcome
Even-power ansatz (resultant)	1,136 triples tested ($|a| \leq 10^5$): **no solution**
Lambda-structured triples	39 triples ($\lambda \leq 200$, $|a| \leq 5 \times 10^4$): **no solution**
Norm-form descent (Eq. 2)	$(a, b, p) = (-56, 38, 13)$ found but no compatible q
Tangent-plane curves	Finitely many integer points (max $|x| \leq 21{,}913$)
Double-contact condition	No rational points for $|x|, |y| \leq 300$
Secant method	Plateaus at $|x| \approx 1{,}176$
Mordell curve (Eq. 5)	12 integer points, max $|x| = 56$

K3 alternative fibration	MW rank 1, section gives **bounded** x
Picard rank	$\rho = 3$ for all six equations
Conclusion	Parametric families **exist** ($\rho = 3$ matches the PDF equation) but coefficients exceed $|a| > 10^5$.

10.1 Why the PDF equation is easier

The reference equation $x^3 + x + 1$ has discriminant -31 and the cubic field $\mathbb{Q}(\beta)$, $\beta^3 + \beta + 1 = 0$, has class number 3. The "small" solution $(a, p) = (-108, 36)$ arises from the Pell equation $m^2 - 3k^2 = 36$ with $k = 6$. Our equations have discriminants $-23, -108, -243, -104$, whose associated arithmetic (class fields, unit groups, Pell equations) leads to much larger coefficients.

10.2 What remains

1. **Larger computation**: the even-power ansatz with $|a| > 10^5$ (requiring cluster computation or GPU acceleration).

2. **Néron–Severi lattice**: the full lattice (not just rank) classifies all elliptic fibrations and their MW generators. Requires `EllipticSurface` in Sage (development branch) or Magma.

3. **Arithmetic of the cubic field**: an explicit relation between the class field of $\mathbb{Q}(\alpha)$ and the ansatz coefficients would narrow the search from blind enumeration to targeted computation.

4. **Alternative degree structures**: the even-power ansatz $(4, 3, 6)$ is the unique balanced degree structure, but non-balanced structures (with cancellations at leading order) might exist.

References

[1] *Small Diophantine: Two Families*, https://epoch.ai/files/open-problems/small-diophantine-two-families.pdf.