A COUNTEREXAMPLE TO THE JACOBIAN CONJECTURE

Abstract. We verify an explicit polynomial map $F\colon \mathbb{C}^3 \to \mathbb{C}^3$ with constant Jacobian determinant -2 and a fiber containing three points, thereby disproving the Jacobian conjecture in dimension three and, by stabilization, in every dimension at least three. A projective coordinate $[x : 1+xy]$ reduces the inverse problem to a binary cubic whose simple roots are exactly the affine preimages. This description determines every fiber, the image, and the nonproperness set. For a Keller map we prove that polynomial invertibility, properness, emptiness of the nonproperness set, and codimension at least two of that set are equivalent, whereas the complement of the image always has codimension at least two. Finally, we construct families of nonproper Keller maps, including examples of every generic degree at least three.

1. Introduction

A polynomial map $G = (G_1, \dots, G_n)\colon \mathbb{C}^n \to \mathbb{C}^n$ is a *Keller map* if $\det \operatorname{Jac} G$ is a nonzero constant. The classical Jacobian conjecture [4, 2] asserts that every Keller map is a polynomial automorphism. The explicit formula studied below was announced by Levent Alpöge [1]; the cited post credits Akhil Mathew for asking the question and Fable for the work leading to the example. We give an algebraic verification, determine its global geometry, and construct general families from its mechanism.

There are two different properness statements. The assertion that every Keller map is proper is equivalent to the Jacobian conjecture, and is therefore false in dimensions at least three. Adding properness as a hypothesis, however, gives a theorem: a proper Keller map is a polynomial automorphism. The example violates precisely this global condition. Its image is nevertheless Zariski open and dense, with complement of codimension two, while its nonproperness set is a hypersurface. Jelonek proved that the nonproperness set of a dominant polynomial self-map of affine space is either empty or a hypersurface [3].

2. The properness refinement

A continuous map between Euclidean spaces is called *proper* if the inverse image of every compact set is compact. For a polynomial map $G\colon \mathbb{C}^n \to \mathbb{C}^n$, define

$$S_G = \{b \in \mathbb{C}^n : \text{there are } u_k \in \mathbb{C}^n \text{ with } \|u_k\| \to \infty \text{ and } G(u_k) \to b\}.$$

Equivalently, $b \notin S_G$ if and only if some neighborhood V of b has $G^{-1}(\overline{V})$ compact. Thus S_G is the set where G is not locally proper.

Proposition 2.1. *Let $G\colon \mathbb{C}^n \to \mathbb{C}^n$ be a Keller map. Then $G(\mathbb{C}^n)$ is Zariski open and dense, and*

$$\mathbb{C}^n \setminus G(\mathbb{C}^n) \subseteq S_G.$$

If the complement of the image is nonempty, then it has codimension at least two.

Proof. The Jacobian criterion shows that G is an *étale* morphism, and an étale morphism is open in the Zariski topology. Hence $U = G(\mathbb{C}^n)$ is Zariski open; it is dense because it is nonempty and $\mathbb{C}^n$ is irreducible. If an irreducible divisor D were contained in $\mathbb{C}^n \setminus U$, then $D = V(h)$ for an irreducible nonconstant $h \in \mathbb{C}[p_1, \dots, p_n]$. The polynomial $h \circ G$ would have no zero and hence, by the weak Nullstellensatz, would be a nonzero constant, say c. Dominance makes the pullback G^* injective, whereas $G^*(h - c) = 0$; hence $h = c$, a contradiction.

Now let $b \notin S_G$. Choose a connected Euclidean ball V about b such that $G^{-1}(\overline{V})$ is compact. The restriction $G^{-1}(V) \to V$ is a proper local biholomorphism, so its image is both open and

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closed in V. Since the Zariski-open dense set U meets V, this image is nonempty and therefore equals V. Thus $b \in U$, proving the asserted inclusion. $\square$

For a dominant polynomial map G, its *generic degree* is $\mu(G) = [\mathbb{C}(x_1, \dots, x_n) : \mathbb{C}(G_1, \dots, G_n)]$; in characteristic zero this is the number of points in a general fiber.

Theorem 2.2 (Properness refinement)**.** *For a Keller map $G\colon \mathbb{C}^n \to \mathbb{C}^n$, the following are equivalent, where the empty set is assigned infinite codimension:*

(1) *G is a polynomial automorphism;*
(2) *G is proper;*
(3) *$S_G = \varnothing$;*
(4) *$\operatorname{codim}_{\mathbb{C}^n} S_G \geq 2$.*

Proof. A polynomial automorphism is proper because its continuous inverse maps compact sets to compact sets. Properness implies $S_G = \varnothing$. Conversely, if $S_G = \varnothing$ and $K \subset \mathbb{C}^n$ is compact, then $G^{-1}(K)$ cannot be unbounded: otherwise a sequence escaping to infinity would have images in K, and a convergent subsequence of those images would produce a point of S_G. Since $G^{-1}(K)$ is closed, it is compact. Thus (2) and (3) are equivalent. By Jelonek's theorem [3, Theorem 15], S_G is algebraic and is either empty or a hypersurface, so (3) and (4) are equivalent.

Assume that G is proper. Since G is a local biholomorphism, its image is open; since a proper map between locally compact Hausdorff spaces is closed, its image is also closed. The target is connected, so G is surjective. Every fiber is discrete and compact, hence finite. A proper local homeomorphism is a covering map: around a target point, take pairwise disjoint inverse-function neighborhoods of its finitely many preimages, and use properness to exclude further inverse branches after shrinking the target neighborhood. Because the source is connected and $\mathbb{C}^n$ is simply connected, this covering has one sheet. Hence G is bijective and its inverse is holomorphic.

The induced extension of rational function fields therefore has degree one, so the inverse coordinates are rational functions. They agree with the holomorphic inverse on a dense open set and hence everywhere. A rational function on $\mathbb{C}^n$ that is holomorphic everywhere is polynomial: in a reduced expression, a nonconstant irreducible factor of the denominator would give a pole at a general point of its zero set. Thus the inverse is polynomial. $\square$

Corollary 2.3 (Properness reformulation)**.** *For each n, the Jacobian conjecture in dimension n is equivalent to the assertion that every Keller map $\mathbb{C}^n \to \mathbb{C}^n$ is proper. The valid refined theorem is Theorem 2.2: properness, or equivalently the absence of a divisorial nonproperness set, forces polynomial invertibility.*

3. The counterexample

Define $F = (P, Q, R)\colon \mathbb{C}^3 \to \mathbb{C}^3$ by

$$\begin{aligned} P &= (1 + xy)^3 z + y^2(1 + xy)(4 + 3xy), \\ Q &= y + 3x(1 + xy)^2 z + 3xy^2(4 + 3xy), \\ R &= 2x - 3x^2 y - x^3 z. \end{aligned} \tag{1}$$

Set $A = 1 + xy$ and $B = A^2 z + y^2(4 + 3xy)$, so that $P = AB$ and $Q = y + 3xB$.

Theorem 3.1. *The map in* (1) *satisfies* $\det \operatorname{Jac} F = -2$. *Moreover,*

$$F(0, 0, -1/4) = F(1, -3/2, 13/2) = F(-1, 3/2, 13/2) = (-1/4, 0, 0).$$

Consequently F is a Keller map that is not a polynomial automorphism.

Proof. On the dense open set $A \neq 0$, put $s = x/A$. Then

$$Q = y + 3Ps, \qquad R = 2s - ys^2 - Ps^3. \tag{2}$$

The first identity is immediate. After multiplying the second by A^2 and writing $t = xy$, it reduces to the polynomial identity $2 + t - t^2(4 + 3t) = (1 + t)^2(2 - 3t)$. The two coordinate changes have

Jacobian determinants

$$\det \frac{\partial(P,y,s)}{\partial(x,y,z)} = -P_z s_x = -A, \qquad \det \frac{\partial(P,Q,R)}{\partial(P,y,s)} = 2(1-ys) = \frac{2}{A}.$$

Their product is -2, and polynomiality extends the identity across $A=0$.

At $(0,0,-1/4)$ one has $(A,B)=(1,-1/4)$. At $(1,-3/2,13/2)$ one has $(A,B)=(-1/2,1/2)$, giving $(P,Q,R)=(-1/4,0,0)$ in both cases. The involution $(x,y,z)\mapsto(-x,-y,z)$ fixes P and negates Q,R, so the third point has the same image. □

All coefficients and all three witness points are rational, so the same counterexample works over every field of characteristic zero.

Corollary 3.2. *Let $L(p,q,r)=(r/2,q,p)$ and $\widetilde{F}=L\circ F$. Then*

$$\det \operatorname{Jac}\widetilde{F}=1, \qquad \widetilde{F}(0)=0, \qquad \operatorname{Jac}\widetilde{F}(0)=I_3,$$

and the three points in Theorem 3.1 have common image $(0,0,-1/4)$ under $\widetilde{F}$.

4. Fibers and behavior at infinity

Write (p,q,r) for coordinates on the target and introduce the binary cubic

$$\Phi_{p,q,r}(S,T) = 2pS^3 - qS^2T + 2ST^2 - rT^3. \tag{3}$$

Let $\mathcal{I}\subset\mathbb{C}^3\times\mathbb{P}^1$ be the incidence variety $\Phi_{p,q,r}(S,T)=0$, and let $\mathcal{I}^{\mathrm{simp}}$ be the open locus where $[S:T]$ is a simple root.

Proposition 4.1. *The map*

$$\iota\colon \mathbb{C}^3 \longrightarrow \mathcal{I}, \qquad (x,y,z)\longmapsto \big(F(x,y,z),[x:1+xy]\big),$$

is an isomorphism from $\mathbb{C}^3$ onto $\mathcal{I}^{\mathrm{simp}}$.

Proof. The pair $[x:A]$ is defined everywhere because x and $A=1+xy$ cannot vanish simultaneously. Homogenizing (2) gives the polynomial identity

$$2Px^3 - Qx^2A + 2xA^2 - RA^3 = 0, \tag{4}$$

so ι takes values in $\mathcal{I}$. Its root is always simple. If $A\neq 0$ and $s=x/A$, then $1-Qs+3Ps^2=1/A$; if $A=0$, then $P=0$ and $Q=-2y\neq 0$, so the root at infinity is simple.

On the chart $T\neq 0$, write $s=S/T$. For a root set

$$D = 1-qs+3ps^2 = \tfrac{1}{2}\,\partial_s\Phi_{p,q,r}(s,1).$$

The root is simple exactly when $D\neq 0$, and then the inverse formulas are

$$y = q-3ps, \qquad x=\frac{s}{D}, \qquad z = pD^3 - y^2(4-sy)D. \tag{5}$$

Indeed, these equations give $A=D^{-1}$ and $B=pD$, hence $P=p$, $Q=q$, and $R=r$.

On the chart $S\neq 0$, put $u=T/S$ and $h(u)=\Phi_{p,q,r}(1,u)=2p-qu+2u^2-ru^3$. On $h(u)=0$, define

$$y = -\frac{q}{2}+3u-\frac{3}{2}ru^2, \qquad E = u-y = \frac{q}{2}-2u+\frac{3}{2}ru^2 = -\frac{1}{2}h'(u),$$

$$x = E^{-1}, \qquad z = 2E^2 - 3yE - rE^3. \tag{6}$$

A root is simple exactly when $E\neq 0$. On the overlap $u\neq 0$, one has $s=u^{-1}$, and $h(u)=0$ gives

$$p = \frac{qu-2u^2+ru^3}{2}, \qquad q-\frac{3p}{u} = -\frac{q}{2}+3u-\frac{3}{2}ru^2, \qquad D=\frac{E}{u}.$$

Hence the two charts give the same y and $x=s/D=1/E$; their formulas for z then agree because both are the unique solution of $r=2x-3x^2y-x^3z$. When $u=0$, the incidence equation gives $p=0$, and the formulas reduce to $x=2/q$, $y=-q/2$, and $z=(5x-r)/x^3$. Thus the formulas are regular on the simple-root locus and give a two-sided inverse to ι. □

The normalized discriminant of (3) is

$$\Delta(p,q,r) = q^2 - 16p - q^3 r + 18pqr - 27p^2 r^2, \tag{7}$$

so the usual cubic discriminant is 4Δ. Define $\Sigma = V(\Delta)$ and $\Gamma = V(12p - q^2, 3qr - 4)$.

Theorem 4.2. *The map F has generic degree three and*

$$\#F^{-1}(p,q,r) = \begin{cases} 3, & \Delta(p,q,r) \neq 0, \\ 1, & \Delta(p,q,r) = 0 \text{ and } (p,q,r) \notin \Gamma, \\ 0, & (p,q,r) \in \Gamma. \end{cases}$$

In particular, $F(\mathbb{C}^3) = \mathbb{C}^3 \setminus \Gamma$. The omitted set is the smooth curve

$$\Gamma = \left\{ \left(\frac{1}{3t^2}, \frac{2}{t}, \frac{2t}{3} \right) : t \in \mathbb{C}^* \right\} \cong \mathbb{C}^*.$$

Proof. By Proposition 4.1, the points of a fiber are exactly the simple projective roots of (3). A cubic with nonzero discriminant has three simple roots; a cubic with one double and one simple root contributes one point; a cubic with a triple root contributes none. The coefficient of ST^2 is nonzero, so a triple root is finite. Comparing $2ps^3 - qs^2 + 2s - r = 2p(s-t)^3$ gives $p = 1/(3t^2)$, $q = 2/t$, and $r = 2t/3$, equivalently the equations defining Γ. The displayed parametrization has inverse $t = 2/q$, proving smoothness and all remaining assertions. $\square$

For the target $(-1/4, 0, 0)$, the inverse cubic is $-s^3/2 + 2s = -s(s-2)(s+2)/2$. Formula (5) sends the roots $0, 2, -2$ to the three points in Theorem 3.1.

Theorem 4.3. *The nonproperness set of F is $S_F = \Sigma = V(\Delta)$. If*

$$J_\Delta = (\Delta, \partial_p\Delta, \partial_q\Delta, \partial_r\Delta),$$

then

$$J_\Delta = (16p - q^3 r, (3qr-4)^2), \qquad \sqrt{J_\Delta} = (12p - q^2, 3qr - 4).$$

Consequently the underlying reduced singular locus of Σ is Γ, while the Jacobian singular subscheme is nonreduced along that curve. Thus the complement of the image has codimension two, whereas nonproperness occurs along a hypersurface.

Proof. The projection $\pi\colon \mathcal{I} \to \mathbb{C}^3$ is proper because $\mathcal{I}$ is closed in $\mathbb{C}^3 \times \mathbb{P}^1$. The coefficient of ST^2 in (3) is the fixed nonzero number 2, so the cubic is never the zero polynomial and every fiber of π is finite. Thus π is quasi-finite. A proper quasi-finite morphism is finite [5, Lemma 37.44.1, Tag 0F2P]; hence π is finite. Over $\mathbb{C}^3 \setminus \Sigma$, every root is simple, so Proposition 4.1 identifies F with the finite map π. Hence F is proper over $\mathbb{C}^3 \setminus \Sigma$, and $S_F \subseteq \Sigma$.

Let $(p,q,r) \in \Sigma$ have a finite multiple root t. Since $\partial_s\Phi(0,1) = 2$, one has $t \neq 0$, and the equations $\Phi(t,1) = \partial_s\Phi(t,1) = 0$ are equivalent to $q = 3pt + t^{-1}$ and $r = t - pt^3$. For $\varepsilon \neq 0$, put

$$x_\varepsilon = \frac{t}{\varepsilon}, \qquad y_\varepsilon = \frac{1-\varepsilon}{t}, \qquad z_\varepsilon = p\varepsilon^3 - y_\varepsilon^2(4 - ty_\varepsilon)\varepsilon.$$

Then $\|(x_\varepsilon, y_\varepsilon, z_\varepsilon)\| \to \infty$ and (2) gives $F(x_\varepsilon, y_\varepsilon, z_\varepsilon) = (p, q - \varepsilon/t, r + \varepsilon t) \to (p,q,r)$. A multiple root at infinity occurs exactly when $p = q = 0$. For any $r \in \mathbb{C}$, set $x_\varepsilon = 2/\varepsilon$, $y_\varepsilon = -\varepsilon/2$, and $z_\varepsilon = (5x_\varepsilon - r)/x_\varepsilon^3$. Then $F(x_\varepsilon, y_\varepsilon, z_\varepsilon) = (0, \varepsilon, r) \to (0,0,r)$ while the source points escape to infinity. These cases exhaust Σ, so $S_F = \Sigma$.

For the scheme-theoretic singular locus, set

$$g = 16p - q^3 r, \qquad h = 3qr - 4.$$

The identities

$$g = p\Delta_p + q\Delta_q - 2\Delta, \qquad h^2 = -\Delta_p - 3r\Delta_q$$

show that $(g, h^2) \subseteq J_\Delta$. Conversely, direct expansion gives

$$\Delta_p = -\frac{27}{8}r^2 g - \left(1 + \frac{3}{8}qr\right)h^2, \quad \Delta_q = \frac{9}{8}rg + \frac{1}{8}qh^2,$$
$$\Delta = \frac{1}{2}(p\Delta_p + q\Delta_q - g), \qquad \Delta_r = \left(\frac{9}{8}q - \frac{27}{8}pr - \frac{27}{128}q^3r^2\right)g - \left(\frac{1}{16}q^3 + \frac{3}{128}q^4 r\right)h^2.$$

Thus $J_\Delta = (g, h^2)$. Since

$$g = \frac{4}{3}(12p - q^2) - \frac{1}{3}q^2 h,$$

the prime ideal $(12p - q^2, h)$ contains J_Δ; its quotient is isomorphic to $\mathbb{C}[q, q^{-1}]$. Since $h \in \sqrt{J_\Delta}$ and the displayed identity then gives $12p - q^2 \in \sqrt{J_\Delta}$, we obtain $\sqrt{J_\Delta} = (12p - q^2, h)$. Along this curve q and r are nonzero, and dg, dh are linearly independent, so g, h form part of a regular system of parameters in every local ring of the ambient space along the curve. Hence (g, h^2) is nonradical at every point of its support. This proves both assertions about the singular subscheme. □

On the dense chart $A \neq 0$, the function fields satisfy

$$\mathbb{C}(x, y, z) = \mathbb{C}(p, q, r)(s), \qquad 2ps^3 - qs^2 + 2s - r = 0.$$

Together with Theorem 4.2, this gives $[\mathbb{C}(x, y, z) : \mathbb{C}(p, q, r)] = 3$; in particular, the displayed cubic is the minimal polynomial of s over the target function field.

Remark 4.4. The restriction $F^{-1}(\mathbb{C}^3 \setminus \Sigma) \to \mathbb{C}^3 \setminus \Sigma$ is finite étale of degree three. On $\Sigma \setminus \Gamma$, the unique affine preimage remains nonsingular while the other two inverse sheets escape to infinity; on Γ, all three projective roots coalesce and no affine preimage remains. Hence Σ is not a critical-value locus: F has no critical points.

5. Families

The construction admits coordinate deformations preserving the cubic geometry and polynomial deformations of arbitrary generic degree.

Theorem 5.1. *Fix $\lambda, a, c \in \mathbb{C}^*$ and $H \in \mathbb{C}[x, y]$. Set*

$$A = \lambda + xy, \qquad W = cz + H(x, y), \qquad B = A^2 W + \frac{3}{a\lambda^2}y^2(4\lambda + 3xy),$$

and define

$$F_{\lambda,a,c,H} = \left(AB,\ y + axB,\ \frac{2}{\lambda}x - \frac{3}{\lambda^2}x^2 y - \frac{a}{3}x^3 W\right). \tag{8}$$

Then $\det \operatorname{Jac} F_{\lambda,a,c,H} = -2c\lambda^2$. *Its generic degree is three, its fiber equation is*

$$\frac{2a}{3}ps^3 - qs^2 + 2s - r = 0, \tag{9}$$

and

$$F_{\lambda,a,c,H}(\mathbb{C}^3) = \mathbb{C}^3 \setminus \Gamma_a, \qquad S_{F_{\lambda,a,c,H}} = V(\Delta_a),$$

where

$$\Delta_a = q^2 - \frac{16a}{3}p - q^3 r + 6apqr - 3a^2p^2r^2, \qquad \Gamma_a = V(q^2 - 4ap, 3qr - 4).$$

The fibers have cardinalities 3, 1, *and* 0 *according as* $\Delta_a \neq 0$, $\Delta_a = 0$ *off* Γ_a, *and the target lies on* Γ_a.

Proof. On $A \neq 0$, put $s = x/A$ and $P = AB$. The same polynomial identity used in Theorem 3.1 gives $Q = y + aPs$ and $R = 2s - ys^2 - (a/3)Ps^3$. Moreover,

$$\det \frac{\partial(P, y, s)}{\partial(x, y, z)} = -c\lambda A, \qquad \det \frac{\partial(P, Q, R)}{\partial(P, y, s)} = 2(1 - ys) = \frac{2\lambda}{A},$$

which proves the determinant formula on $A \neq 0$. Since both sides are polynomial functions of (x, y, z), the identity extends across $A = 0$. Eliminating $y = q - aps$ gives (9).

For a finite simple root, put $y = q - aps$ and $D = 1 - sy$. The corresponding source point is

$$x = \frac{\lambda s}{D}, \qquad W = \frac{pD^3}{\lambda^3} - \frac{3y^2(4-sy)D}{a\lambda^3}, \qquad z = \frac{W - H(x,y)}{c}.$$

The root at infinity occurs when $p = 0$ and is simple exactly when $q \neq 0$; it gives $y = -q/2$, $x = 2\lambda/q$, and $W = 3(5x/\lambda - r)/(ax^3)$. Thus the simple projective roots of the homogenization of (9) are in bijection with affine preimages. Its discriminant is $4\Delta_a$, and comparison with a triple cube gives $q^2 = 4ap$ and $3qr = 4$. This proves the fiber count and the image formula.

The same finite-incidence argument as in Theorem 4.3 gives properness away from $V(\Delta_a)$. If t is a finite multiple root, then $q = apt + t^{-1}$ and $r = t - (a/3)pt^3$. Put

$$x_\varepsilon = \frac{\lambda t}{\varepsilon}, \qquad y_\varepsilon = \frac{1-\varepsilon}{t}, \qquad W_\varepsilon = \frac{p\varepsilon^3}{\lambda^3} - \frac{3y_\varepsilon^2(4-ty_\varepsilon)\varepsilon}{a\lambda^3}, \qquad z_\varepsilon = \frac{W_\varepsilon - H(x_\varepsilon, y_\varepsilon)}{c}.$$

Here $A = \lambda/\varepsilon$, $s = t$, and $D = \varepsilon$; direct substitution gives

$$F_{\lambda,a,c,H}(x_\varepsilon, y_\varepsilon, z_\varepsilon) = (p, q - \varepsilon/t, r + \varepsilon t).$$

Thus the source points escape to infinity while their images converge to (p,q,r). A multiple root at infinity occurs exactly when $p = q = 0$. In that case take $x_\varepsilon = 2\lambda/\varepsilon$, $y_\varepsilon = -\varepsilon/2$, $W_\varepsilon = 3(5x_\varepsilon/\lambda - r)/(ax_\varepsilon^3)$, and $z_\varepsilon = (W_\varepsilon - H(x_\varepsilon, y_\varepsilon))/c$. Then $A = 0$ and

$$F_{\lambda,a,c,H}(x_\varepsilon, y_\varepsilon, z_\varepsilon) = (0, \varepsilon, r).$$

Hence the nonproperness set is exactly $V(\Delta_a)$. □

The original map is $F_{1,3,1,0}$, and taking $c = -1/(2\lambda^2)$ gives determinant one. For fixed λ and a, the parameters c and H arise by precomposition with the triangular source automorphism $(x,y,z) \mapsto (x, y, cz + H(x,y))$.

For the next construction, retain $A = 1 + xy$, $B = A^2 z + y^2(4 + 3xy)$, $P = AB$, and the coordinates Q, R from (1).

Theorem 5.2. *Let $N \geq 1$ and choose polynomials $\eta_j(T) \in \mathbb{C}[T]$ for $1 \leq j \leq N$. Define*

$$Q_{\boldsymbol{\eta}} = Q + A^2 \sum_{j=1}^{N} x^j B^{j+2} \eta_j(P), \qquad R_{\boldsymbol{\eta}} = R - \sum_{j=1}^{N} \frac{j}{j+2} x^{j+2} B^{j+2} \eta_j(P),$$

and $F_{\boldsymbol{\eta}} = (P, Q_{\boldsymbol{\eta}}, R_{\boldsymbol{\eta}})$. Then $\det \operatorname{Jac} F_{\boldsymbol{\eta}} = -2$. Its finite inverse branches are governed by

$$\Omega_{p,q,r}(s) = 2ps^3 - qs^2 + 2s - r + \sum_{j=1}^{N} \frac{2}{j+2} p^{j+2} \eta_j(p) s^{j+2} = 0. \tag{10}$$

If $d = \max\bigl(\{3\} \cup \{j+2 : \eta_j \neq 0\}\bigr)$, then $F_{\boldsymbol{\eta}}$ has generic degree d.

Proof. On $A \neq 0$, write $s = x/A$ and set

$$\varphi(p,s) = 3ps + \sum_{j=1}^{N} p^{j+2} \eta_j(p) s^j, \qquad \theta(p,s) = ps^3 + \sum_{j=1}^{N} \frac{j}{j+2} p^{j+2} \eta_j(p) s^{j+2}.$$

The displayed polynomial formulas are the pullbacks of $Q_{\boldsymbol{\eta}} = y + \varphi(P,s)$ and $R_{\boldsymbol{\eta}} = 2s - ys^2 - \theta(P,s)$, because $P^{j+2}s^j = A^2 x^j B^{j+2}$ and $P^{j+2}s^{j+2} = x^{j+2}B^{j+2}$. Since $\theta_s = s^2 \varphi_s$,

$$\det \frac{\partial(P, Q_{\boldsymbol{\eta}}, R_{\boldsymbol{\eta}})}{\partial(P, y, s)} = 2 - 2ys - \theta_s + s^2\varphi_s = 2(1 - ys) = \frac{2}{A}.$$

Multiplication by $\det \partial(P,y,s)/\partial(x,y,z) = -A$ gives -2 on $A \neq 0$. Since both the determinant and -2 are polynomial functions of (x,y,z), the equality extends to all of $\mathbb{C}^3$.

Eliminating $y = q - \varphi(p,s)$ gives (10), and

$$\partial_s \Omega_{p,q,r}(s) = 2\bigl(1 - s(q - \varphi(p,s))\bigr).$$

Thus a finite root yields an affine preimage exactly when it is simple. With $y = q - \varphi(p, s)$ and $D = 1 - sy$, that point is $x = s/D$ and $z = pD^3 - y^2(4 - sy)D$. These reconstruction formulas give the equality of function fields

$$\mathbb{C}(x, y, z) = \mathbb{C}(p, q, r)(s).$$

Let $K = \mathbb{C}(p, q)$ and write (10) as $r = f_{p,q}(s)$. The polynomial $f_{p,q}(S)$ has degree d over K: when $d > 3$ its leading coefficient is a nonzero multiple of $p^d\eta_{d-2}(p)$, while for $d = 3$ it is $2p(1 + p^2\eta_1(p)/3)$ if η_1 is present. Since $F_{\boldsymbol{\eta}}$ is dominant, r is transcendental over K. The polynomial $f_{p,q}(S) - r$ is irreducible in $K(r)[S]$: indeed, $T - f_{p,q}(S)$ generates a prime ideal in $K[S, T]$, because the quotient is $K[S]$, and Gauss's lemma applies. Therefore

$$[\mathbb{C}(x, y, z) : \mathbb{C}(p, q, r)] = d,$$

which is the asserted generic degree. □

Corollary 5.3. *For every integer $d \geq 3$, there is a nonproper Keller map $\mathbb{C}^3 \to \mathbb{C}^3$ of generic degree d having the three-point collision in Theorem 3.1.*

Proof. For $d = 3$, use F. For $d \geq 4$, take $N = d - 2$, set $\eta_{d-2}(T) = c(4T + 1)$ with $c \in \mathbb{C}^*$, and set all other η_j equal to zero. The added terms vanish at the three source points because their common first target coordinate is $P = -1/4$. The collision is therefore unchanged, while Theorem 5.2 gives determinant -2 and generic degree d. The map is not proper by Theorem 2.2. □

For every $n \geq 3$, stabilization gives the Keller counterexample

$$(x, y, z, u_4, \ldots, u_n) \longmapsto \big(F(x, y, z), u_4, \ldots, u_n\big).$$

The construction does not settle the two-dimensional statement. The surviving global theorem is exact: proper Keller maps are automorphisms, and by Jelonek's theorem it is enough to prove that the asymptotic-value set has codimension at least two. The codimension-two size of the omitted set is not enough; for (1), one has $\mathbb{C}^3 \setminus F(\mathbb{C}^3) = \Gamma$ but $S_F = \Sigma$.

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